

Statistical mechanics of non-extensive systems/Mécanique statistique des systèmes non-extensifs

Newtonian gravity in d dimensions

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Abstract

We study the influence of the dimension of space on the thermodynamics of the classical and quantum self-gravitating gas. We consider Hamiltonian systems of self-gravitating particles described by the microcanonical ensemble and self-gravitating Brownian particles described by the canonical ensemble. We present a gallery of caloric curves in different dimensions of space and discuss the nature of phase transitions as a function of the dimension d . We also provide the general form of the Virial theorem in d dimensions and discuss the particularity of the dimension $d = 4$ for Hamiltonian systems and the dimension $d = 2$ for Brownian systems. **To cite this article:** *P.-H. Chavanis, C. R. Physique 7 (2006).*

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Résumé

Quelques aspects de la gravité newtonienne en dimension d . J'étudie l'influence de la dimension de l'espace sur la thermodynamique du gaz auto-gravitant classique et quantique. Je considère des systèmes hamiltoniens de particules auto-gravitantes décrits par l'ensemble microcanonique et des systèmes de particules browniennes auto-gravitantes décrits par l'ensemble canonique. Je présente une galerie de courbes caloriques pour différentes dimensions de l'espace et discute la nature des transitions de phase en fonction de la dimension d . Je fournis aussi la forme générale du théorème du Viriel en d dimensions et discute la particularité de la dimension $d = 4$ pour les systèmes hamiltoniens et de la dimension $d = 2$ pour les systèmes browniens. **Pour citer cet article :** *P.-H. Chavanis, C. R. Physique 7 (2006).*

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1. Introduction

Self-gravitating systems have a strange thermodynamics due to the attractive long-range nature of the gravitational interaction [1–4]. This gives rise to inequivalence of statistical ensembles associated with regions of negative specific heats in the microcanonical ensemble and to phase transitions associated with gravitational collapse (see, e.g., [5]). The form of the caloric curve and the nature of the phase transitions crucially depend on the dimension of space d [6,7]. In this paper, we study the influence of the dimension of space on the thermodynamics of self-gravitating systems. We distinguish between Hamiltonian and Brownian systems [8]. A Hamiltonian system of self-gravitating particles evolves at fixed mass and energy and is described by the microcanonical ensemble [1]. In the mean-field

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approximation valid in a proper thermodynamic limit $N \rightarrow +\infty$ with fixed $\Lambda = -ER/GM^2$ (where R is the size of the system), the distribution function maximizes, at statistical equilibrium, the Boltzmann entropy at fixed mass and energy. The relaxation towards statistical equilibrium is governed by the mean-field Landau–Poisson system which conserves mass and energy and monotonically increases the Boltzmann entropy (H -theorem). A system of self-gravitating Brownian particles [9] evolves at fixed mass and temperature and is described by the canonical ensemble. In the mean-field approximation valid in a proper thermodynamic limit $N \rightarrow +\infty$ with fixed $\eta = \beta GMm/R$, the distribution function minimizes, at statistical equilibrium, the Boltzmann free energy at fixed mass. The relaxation towards statistical equilibrium is governed by the mean-field Kramers–Poisson system which conserves mass and monotonically decreases the Boltzmann free energy.

These two systems (Hamiltonian and Brownian) are defined in Section 2 starting from N -body equations. In Section 3, we derive the general form of Virial theorem in d dimensions and we discuss the particularities of the dimension $d = 4$ for Hamiltonian systems and the dimension $d = 2$ for Brownian systems. In the first case, we show that the system evaporates for $E > 0$ and collapses for $E < 0$. In the second case, the system evaporates for $T > T_c$ and collapses for $T < T_c$ where $T_c = (N - 1)Gm^2/4k_B$ is a critical temperature. We also derive a generalization of the Einstein relation including self-gravity in $d = 2$. Finally, in Section 4, we present a gallery of caloric curves in various dimensions of space for self-gravitating classical and quantum particles (fermions) and discuss the nature of phase transitions as a function of d .

2. Self-gravitating Hamiltonian and Brownian systems

In a space of dimension d , we consider a system of N particles with mass m_α in gravitational interaction whose dynamics is described by the equations of motion

$$\ddot{x}_i^\alpha = \sum_{\beta \neq \alpha} \frac{Gm_\beta(x_i^\beta - x_i^\alpha)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^d} - \xi_\alpha \dot{x}_i^\alpha + \sqrt{2D_\alpha} R_i^\alpha(t) \quad (1)$$

Here, the Greek letters refer to the particles and the Latin letters to the coordinates of space. When the last two terms are set equal to zero, we recover the usual Hamiltonian model of self-gravitating systems. However, we consider here a more general situation where the particles are subject, in addition to self-gravity, to a friction force and a stochastic force. These terms can mimic the influence of a thermal bath of non-gravitational origin. We thus obtain a model of self-gravitating Brownian particles [9] extending the usual Brownian model introduced by Einstein and Smoluchowski. We consider the case of a multicomponent system [10]. Here, ξ_α is the friction coefficient, D_α is the diffusion coefficient and $\mathbf{R}^\alpha(t)$ is a white noise acting independently on the particles and satisfying $\langle R_i^\alpha(t) \rangle = 0$ and $\langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$. The diffusion coefficient and the friction force satisfy the Einstein relation $D_\alpha = \xi_\alpha k_B T / m_\alpha$ where T is the temperature of the bath (see below). Since $D \sim T$, the temperature measures the strength of the stochastic force.

The evolution of the N -body distribution function $P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N, t)$ is governed by the N -body Fokker–Planck equation [8]:

$$\frac{\partial P_N}{\partial t} + \sum_{\alpha=1}^N \left(\mathbf{v}_\alpha \cdot \frac{\partial P_N}{\partial \mathbf{r}_\alpha} + \frac{\mathbf{F}_\alpha}{m_\alpha} \cdot \frac{\partial P_N}{\partial \mathbf{v}_\alpha} \right) = \sum_{\alpha=1}^N \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[D_\alpha \frac{\partial P_N}{\partial \mathbf{v}_\alpha} + \xi_\alpha P_N \mathbf{v}_\alpha \right] \quad (2)$$

where $\mathbf{F}_\alpha = -\nabla_\alpha U(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is the force acting on the α th particle (U is the potential of interaction). If we enclose the system within a box and replace the bare gravitational potential by a softened potential that is regularized at short distances (so that a statistical equilibrium state exists), we find that the stationary solution of Eq. (2) is the canonical distribution

$$P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N) = \frac{1}{Z(\beta)} e^{-\beta H(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N)} \quad (3)$$

where $H = \sum_\alpha m_\alpha \frac{v_\alpha^2}{2} + U(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is the Hamiltonian, provided that the diffusion and friction coefficients are related to each other by the Einstein relation. When $\xi = D = 0$, Eq. (2) becomes the Liouville equation appropriate to

Hamiltonian systems. In that case, Eq. (3) is not valid anymore. Since energy is now conserved during the evolution, the system is expected to reach, for $t \rightarrow +\infty$, the microcanonical distribution

$$P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N) = \frac{1}{g(E)} \delta(E - H(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N)) \quad (4)$$

expressing the equiprobability of accessible microstates (with the right value of energy).

In the mean-field approximation valid in a proper thermodynamic limit with $N \rightarrow +\infty$ [8,10], the evolution of the distribution function $f_a(\mathbf{r}, \mathbf{v}, t)$ of each species of the self-gravitating Brownian gas is governed by the multicomponent Kramers–Poisson (KP) system

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(D_a \frac{\partial f_a}{\partial \mathbf{v}} + \xi_a f_a \mathbf{v} \right) \quad (5)$$

$$\Delta \Phi = S_d G \int f \, d\mathbf{v} \quad (6)$$

where $f = \sum_a f_a$ is the total distribution function. The multicomponent KP system conserves the mass $M_a = \int f_a \, d\mathbf{r} \, d\mathbf{v}$ of each species and monotonically decreases the free energy $F = E - TS$ where

$$E = \frac{1}{2} \int f v^2 \, d\mathbf{r} \, d\mathbf{v} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r} = K + W \quad (7)$$

is the total energy and

$$S = -k_B \sum_{a=1}^X \int \frac{f_a}{m_a} \ln \left(\frac{f_a}{m_a} \right) \, d\mathbf{r} \, d\mathbf{v} \quad (8)$$

is the Boltzmann entropy. The linearly dynamically stable stationary solution of the KP system is the mean-field Maxwell–Boltzmann distribution

$$f_a(\mathbf{r}, \mathbf{v}) = A_a e^{-\beta m_a [v^2/2 + \Phi(\mathbf{r})]} \quad (9)$$

that is a local minimum of free energy at fixed mass (canonical description). In the strong friction limit $\xi \rightarrow +\infty$, the velocity distribution of the particles is close to the Maxwellian and the evolution of the spatial distribution is governed by the multicomponent Smoluchowski–Poisson (SP) system [8,10]. Alternatively, for $D = \xi = 0$ (Hamiltonian systems), the evolution of the system for $N \rightarrow +\infty$ is governed by the Vlasov equation [8]:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (10)$$

The Vlasov equation conserves mass and energy but also the entropy (and more generally an infinite class of functionals called the Casimirs). Furthermore, it admits an infinite number of stationary states (not necessarily Boltzmannian) that can be selected by the dynamics as a result of an incomplete collisionless violent relaxation on the coarse-grained scale [11,12]. The statistical equilibrium state in the microcanonical ensemble is selected by finite N effects (granularities) accounting for correlations between particles due to close encounters. The collisional relaxation of stellar systems is usually described by the mean-field Landau–Poisson system [8]. In $d = 3$ the multicomponent Landau equation can be written

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \frac{\partial}{\partial v^\mu} \sum_{b=1}^X \int K^{\mu\nu} \left(m_b f'_b \frac{\partial f_a}{\partial v^\nu} - m_a f_a \frac{\partial f'_b}{\partial v'^\nu} \right) \, d\mathbf{v}' \quad (11)$$

$$K^{\mu\nu} = 2\pi G^2 \frac{1}{u} \ln \Lambda \left(\delta^{\mu\nu} - \frac{u^\mu u^\nu}{u^2} \right) \quad (12)$$

where $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ is the relative velocity of the particles involved in an encounter and $\ln \Lambda = \int_0^{+\infty} dk/k$ is the gravitational Coulomb factor. We have set $f'_a = f_a(\mathbf{r}, \mathbf{v}', t)$ assuming that the encounters can be treated as local. The Landau–Poisson system monotonically increases the entropy while conserving the mass M_a of each species and the total energy E . The linearly dynamically stable stationary state is the mean-field Maxwell–Boltzmann distribution (9)

that is a local maximum of Boltzmann entropy at fixed mass and energy (microcanonical description). Therefore, the statistical equilibrium states of Hamiltonian and Brownian systems have the same distribution profiles but their stability may differ in case of ensemble inequivalence. This occurs in particular when the caloric curve presents turning points as discussed in Section 4.

3. Virial theorem in d dimensions

3.1. The general expression

Let us establish the Virial theorem associated with the stochastic equations (1). For convenience, we shall assume that the friction coefficient ξ is the same for all the particles but this assumption can be relaxed easily. The moment of inertia tensor is defined by

$$I_{ij} = \sum_{\alpha} m_{\alpha} x_i^{\alpha} x_j^{\alpha} \quad (13)$$

We introduce the kinetic energy tensor

$$K_{ij} = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{x}_i^{\alpha} \dot{x}_j^{\alpha} \quad (14)$$

and the potential energy tensor

$$W_{ij} = G \sum_{\alpha \neq \beta} m_{\alpha} m_{\beta} \frac{x_i^{\alpha} (x_j^{\beta} - x_j^{\alpha})}{|\mathbf{r}_{\beta} - \mathbf{r}_{\alpha}|^d} = -\frac{1}{2} G \sum_{\alpha \neq \beta} m_{\alpha} m_{\beta} \frac{(x_i^{\alpha} - x_i^{\beta})(x_j^{\alpha} - x_j^{\beta})}{|\mathbf{r}_{\beta} - \mathbf{r}_{\alpha}|^d} \quad (15)$$

where the second equality results from simple algebraic manipulations obtained by exchanging the dummy variables α and β . Taking the second derivative of Eq. (13), using the equation of motion (1), and averaging over the noise and on statistical realizations, we get

$$\frac{1}{2} \langle \ddot{I}_{ij} \rangle + \frac{1}{2} \xi \langle \dot{I}_{ij} \rangle = 2 \langle K_{ij} \rangle + \langle W_{ij} \rangle - \frac{1}{2} \oint (P_{ik} x_j + P_{jk} x_i) dS_k \quad (16)$$

For the sake of generality, we have accounted for the presence of a box enclosing the system and included the Virial of the pressure force $F_i = P_{ik} \Delta S_k$ on the boundary of the box: $(\ddot{I}_{ij})_{\text{box}} = \sum_{\text{box}} \sum_{\alpha} (F_i^{\alpha} x_j^{\alpha} + F_j^{\alpha} x_i^{\alpha}) = \sum_{\text{box}} \sum_{\alpha} (P_{ik}^{\alpha} x_j^{\alpha} + P_{jk}^{\alpha} x_i^{\alpha}) \Delta S_k$. The scalar Virial theorem is obtained by contracting the indices yielding

$$\frac{1}{2} \langle \ddot{I} \rangle + \frac{1}{2} \xi \langle \dot{I} \rangle = 2 \langle K \rangle + \langle W_{ii} \rangle - \oint P_{ik} x_i dS_k \quad (17)$$

where

$$I = \sum_{\alpha} m_{\alpha} r_{\alpha}^2, \quad K = \frac{1}{2} \sum_{\alpha} m_{\alpha} v_{\alpha}^2 \quad (18)$$

are the moment of inertia and the kinetic energy. On the other hand,

$$W_{ii} = -\frac{1}{2} G \sum_{\alpha \neq \beta} \frac{m_{\alpha} m_{\beta}}{|\mathbf{r}_{\beta} - \mathbf{r}_{\alpha}|^{d-2}} \quad (19)$$

3.2. The trace of the potential energy tensor

For $d \neq 2$, we find that

$$W_{ii} = (d - 2) W \quad (20)$$

where W is the potential energy

$$W = -\frac{G}{2(d-2)} \sum_{\alpha \neq \beta} \frac{m_\alpha m_\beta}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^{d-2}} \tag{21}$$

In that case, the scalar Virial theorem reads

$$\frac{1}{2} \langle \ddot{I} \rangle + \frac{1}{2} \xi \langle \dot{I} \rangle = 2 \langle K \rangle + (d-2) \langle W \rangle - \oint P_{ik} x_i dS_k \tag{22}$$

For $d = 2$, we have the simple result

$$W_{ii} = -\frac{1}{2} G \sum_{\alpha \neq \beta} m_\alpha m_\beta \tag{23}$$

For equal mass particles,

$$W_{ii} = -\frac{1}{2} G N(N-1) m^2 \tag{24}$$

which reduces to

$$W_{ii} \simeq -\frac{1}{2} G M^2 \tag{25}$$

for $N \gg 1$. We also note that $\sum_{\alpha \neq \beta} m_\alpha m_\beta = M^2 - \sum_\alpha m_\alpha^2$. For typical mass distributions, the first term is of order $\sim N^2 m^2$ and the second of order $N m^2$ (where m is a typical mass). Therefore, in the thermodynamic limit $N \rightarrow +\infty$, we get Eq. (25) even for a multicomponent system [10].

3.3. The equilibrium state

At equilibrium, the Virial theorem (17) reduces to

$$2 \langle K \rangle + \langle W_{ii} \rangle = \oint P_{ik} x_i dS_k. \tag{26}$$

For an unbounded domain ($P = 0$), we get $2 \langle K \rangle + \langle W_{ii} \rangle = 0$ (more precisely, $2 \langle K \rangle = -(d-2) \langle W \rangle$ for $d \neq 2$ and $\langle K \rangle = G M^2 / 4$ for $d = 2$ in the approximation (25)). If the system is at statistical equilibrium, then $\langle K \rangle = \frac{d}{2} N k_B T$ and $P_{ij} = p \delta_{ij}$ with $p = \sum_s \rho_s k_B T / m_s$ (where s labels the species of particles) [10]. If p_b is constant on the boundary of the box, the pressure term becomes

$$\oint p_b \mathbf{r} \cdot d\mathbf{S} = p_b \oint \mathbf{r} \cdot d\mathbf{S} = p_b \int \nabla \cdot \mathbf{r} dV = d p_b V \tag{27}$$

More generally, we define $P \equiv \frac{1}{dV} \oint p \mathbf{r} \cdot d\mathbf{S}$. In that case, the equilibrium Virial theorem becomes

$$d N k_B T + \langle W_{ii} \rangle = d P V \tag{28}$$

For an ideal gas without self-gravity ($W = 0$), we recover the perfect gas law

$$P V = N k_B T \tag{29}$$

Alternatively, for a self-gravitating gas in two dimensions, using Eq. (23), we get the exact equation of state

$$P V = N k_B (T - T_c) \tag{30}$$

with the exact critical temperature

$$k_B T_c = \frac{G \sum_{\alpha \neq \beta} m_\alpha m_\beta}{4N} \tag{31}$$

For equal mass particles, we get

$$k_B T_c = \frac{G m^2}{4} (N - 1) \tag{32}$$

and in the $N \rightarrow +\infty$ limit

$$k_B T_c = \frac{GM^2}{4N} \quad (33)$$

This relation is also valid for a multicomponent system in the thermodynamic limit (see remark after Eq. (25)) [10]. The equation of state (30) has been obtained by Lynden-Bell and Katz [13], Padmanabhan [1] and Chavanis and Sire [14] using different methods.

3.4. The critical dimension $d = 4$ for Hamiltonian systems

For Hamiltonian systems ($D = \xi = 0$), the total energy $E = K + W$ is conserved. Thus, in an unbounded domain ($P = 0$), the Virial theorem (17) can be written for $d \neq 2$:

$$\frac{1}{2}\ddot{I} = 2K + (d-2)W = (4-d)K + (d-2)E = 2E + (d-4)W \quad (34)$$

This is the extension of the usual Virial theorem in d dimensions (this equation is exact without averages). We note that the dimension $d = 4$ is *critical* as was also noticed in [7] using different arguments. In that case, $\dot{I} = 4E$ which yields after integration $I = 2Et^2 + C_1t + C_2$. For $E > 0$, $I \rightarrow +\infty$ indicating that the system evaporates (there can be equilibrium states if the system is confined within a box). For $E < 0$, I goes to zero in a finite time, indicating that the system forms a Dirac peak in a finite time. More generally, for $d \geq 4$, since $(d-4)W \leq 0$, we have $I \leq 2Et^2 + C_1t + C_2$ so that the system forms a Dirac peak in a finite time if $E < 0$ (this result remains valid for box-confined systems). Therefore, self-gravitating systems with $E < 0$ are not stable in a space of dimension $d \geq 4$. The study in [7] indicates that this observation remains true even if quantum effects (Pauli exclusion principle for fermions) are taken into account. This is a striking result because quantum mechanics stabilizes matter against gravitational collapse in $d < 4$ [5]. For $2 < d \leq 4$, since $(d-4)W \geq 0$ we conclude, according to Eq. (34), that the system evaporates if $E > 0$ (there can be equilibrium states if the system is confined within a box). An equilibrium is possible, but not compulsory, if $E < 0$ (the case of statistical equilibrium is discussed in Section 4). Finally, for $d < 2$, since $W > 0$, the energy is necessarily positive ($E > 0$). Since $(d-4)W < 0$, an equilibrium state is possible. In $d = 2$, the Virial theorem becomes with the approximation (25):

$$\frac{1}{2}\ddot{I} = 2K - \frac{GM^2}{2} \quad (35)$$

Since $K \geq 0$, an equilibrium state is possible.

3.5. The critical dimension $d = 2$ for Brownian systems

We consider a self-gravitating Brownian gas in the strong friction limit $\xi \rightarrow +\infty$. To leading order in $1/\xi$, the N -body distribution is given by

$$P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N, t) = e^{-\beta \sum_{\alpha} m_{\alpha} v_{\alpha}^2 / 2} \Phi_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t) + O(\xi^{-1}) \quad (36)$$

as can be deduced from Eq. (2) by canceling the term in brackets. From this expression, we find that $P_{ij} = p\delta_{ij}$ with $p = \sum_s \rho_s k_B T / m_s$ and $\langle K_{ij} \rangle = \frac{1}{2} N k_B T \delta_{ij}$ even for the out-of-equilibrium problem. From Eq. (16), we obtain the overdamped Virial theorem for a self-gravitating Brownian gas

$$\frac{1}{2} \xi \langle \dot{I}_{ij} \rangle = \langle W_{ij} \rangle + N k_B T \delta_{ij} - \frac{1}{2} \oint p (\delta_{ik} x_j + \delta_{jk} x_i) dS_k \quad (37)$$

We can obtain this result in a different manner. In the strong friction limit $\xi \rightarrow +\infty$, the inertial term in Eq. (1) can be neglected so that the stochastic equations of motion reduce to

$$\dot{x}_i^{\alpha} = \mu_{\alpha} m_{\alpha} \sum_{\beta \neq \alpha} \frac{G m_{\beta} (x_i^{\beta} - x_i^{\alpha})}{|\mathbf{r}_{\beta} - \mathbf{r}_{\alpha}|^d} + \sqrt{2D_{\alpha}^{\alpha}} R_i^{\alpha}(t) \quad (38)$$

where $D'_\alpha = k_B T \mu_\alpha$ is the diffusion coefficient in position space and $\mu_\alpha = (\xi m_\alpha)^{-1}$ is the mobility. Taking the derivative of the tensor of inertia (13) and using Eq. (38), we get

$$\dot{I}_{ij} = \frac{2}{\xi} W_{ij} + \sum_{\alpha} m_{\alpha} \sqrt{2D'_{\alpha}} [x_i^{\alpha} R_j^{\alpha}(t) + x_j^{\alpha} R_i^{\alpha}(t)] \tag{39}$$

Averaging over the noise using $\langle x_i^{\alpha} R_j^{\alpha} \rangle = \sqrt{2D'_{\alpha}} \delta_{ij}$ and on statistical realizations, we recover Eq. (37). The scalar Virial theorem obtained by contracting the indices reads

$$\frac{1}{2} \xi \langle \dot{I} \rangle = d N k_B T + \langle W_{ii} \rangle - d P V \tag{40}$$

In particular, in $d = 2$, using Eq. (23) we find that

$$\frac{1}{2} \xi \langle \dot{I} \rangle = 2 N k_B (T - T_c) - 2 P V \tag{41}$$

where T_c is defined in Eq. (31). For $T > T_c$, an equilibrium state is possible in a box, while the system evaporates ($I \rightarrow +\infty$) in an unbounded domain ($P = 0$). For $T < T_c$, the system creates a Dirac peak in a finite time ($I = 0$ at $t = t_{\text{end}}$). These dynamical evolutions are studied precisely in [6,15] by solving the Smoluchowski–Poisson system.

3.6. The generalized Einstein relation including self-gravity in $d = 2$

In an unbounded domain ($P = 0$), the Virial relation (41) reduces to

$$\frac{1}{2} \xi \langle \dot{I} \rangle = 2 N k_B (T - T_c) \tag{42}$$

Defining the spatial dispersion of the particles by

$$\langle r^2 \rangle = \frac{\langle \sum m_{\alpha} r_{\alpha}^2 \rangle}{\sum m_{\alpha}} = \frac{\langle I \rangle}{M} \tag{43}$$

we can rewrite the foregoing relation in the form

$$\frac{d \langle r^2 \rangle}{dt} = \frac{4 N k_B T}{\xi M} (1 - T_c / T) \tag{44}$$

Integrating this relation, we obtain

$$\langle r^2 \rangle(t) = \frac{4 N k_B T}{\xi M} (1 - T_c / T) t + \langle r^2 \rangle_0 \tag{45}$$

This relation suggests to introducing an effective diffusion coefficient

$$D_{\text{eff}}(T) = \frac{N k_B T}{\xi M} (1 - T_c / T) \tag{46}$$

so that

$$\langle r^2 \rangle(t) = 4 D_{\text{eff}}(T) t + \langle r^2 \rangle_0 \tag{47}$$

For $T \gg T_c$ when gravitational effects become negligible with respect to thermal motion, the diffusion coefficient is given by the celebrated Einstein formula $D = N k_B T / \xi M$. However, Eq. (46) shows that the diffusion is less and less effective as temperature decreases and gravitational effects come into play. *Therefore, relation (46) provides a generalization of the Einstein relation to the case of self-gravitating Brownian particles in $d = 2$.* For $T > T_c$, we have a diffusive motion (evaporation) with an effective diffusion coefficient depending linearly on the distance $(T - T_c)$ to the critical temperature. For $T = T_c$, the effective diffusion constant vanishes $D(T_c) = 0$ so that the moment of inertia is conserved. Finally, for $T < T_c$ the effective diffusion coefficient is negative, implying finite time collapse. In particular, $\langle r^2 \rangle = 0$ for $t_{\text{end}} = \langle r^2 \rangle_0 / 4 |D(T)|$ (where $\langle r^2 \rangle_0$ is calculated from the center of mass). This corresponds to the formation of a Dirac peak $\rho(\mathbf{r}) = M \delta(\mathbf{r})$ at $\mathbf{r} = \mathbf{0}$ containing the whole mass [15].

4. Thermodynamics of self-gravitating classical and quantum particles

In this section we present a gallery of caloric curves for self-gravitating classical and quantum particles (fermions) at statistical equilibrium in different dimensions of space. From the series of equilibria, we can directly infer the thermodynamical stability limits of the system in microcanonical and canonical ensembles by using the turning point argument of Poincaré. We also directly see the critical parameters (energy and temperature) below which the system collapses. The microcanonical ensemble (MCE) is appropriate to isolated Hamiltonian systems (in which case the control parameter is the energy E) and the stable states are maxima of entropy S at fixed mass M and energy E . The canonical ensemble (CE) is appropriate to dissipative systems like self-gravitating Brownian particles or self-gravitating gaseous systems in contact with a thermal bath of non-gravitational origin (in which case the control parameter is the temperature T) and the stable states are minima of free energy $F = E - TS$ at fixed mass M . The series of equilibria $\beta(E)$ (representing critical points of the thermodynamical potentials) are the *same* for the two systems but their stability (related to the sign of the second order variations of the thermodynamical potentials) can be different. A turning point of energy is associated with a loss of microcanonical stability (for Hamiltonian systems) and a turning point of temperature is associated with a loss of canonical stability (for Brownian systems). Stability can be regained at the next turning point if the curve turns in the reversed direction (as is the case for fermions). We refer to Katz [16], Padmanabhan [1] and Chavanis [17,18,5] for more precise statements of these results and illustrations. In the following, the caloric curves are given for box-confined systems. This is necessary to have existence of statistical equilibrium states in $d > 2$ (otherwise the system evaporates). We also briefly discuss the case of unbounded systems in $d = 1$ and $d = 2$.

4.1. Classical particles

In $d = 1$, there exists statistical equilibrium states for any accessible value of energy $E \geq 0$ in MCE and any value of temperature T in CE (see Fig. 1(a)). Since the series of equilibria is monotonic, they are stable (global maxima of S in MCE and global minima of F in CE) and the ensembles are equivalent. There also exists stable statistical equilibrium states in an unbounded domain. The density profile is known analytically [19,6,15]. According to the Virial theorem, we have $2K = W$ and $K = \frac{1}{2}Nk_B T$ so that the caloric curve is simply $E = \frac{3}{2}Nk_B T$ [15].

In $d = 2$, there are statistical equilibrium states for any value of energy E in MCE and for $T > T_c = GMm/4k_B$ in CE (see Fig. 1(a)). Their density profile is known analytically (see, e.g., [6]). Since the curve $\beta(E)$ is monotonic, they are stable (global maxima of S in MCE and global minima of F in CE) and the ensembles are equivalent. For $T \leq T_c$ the self-gravitating Brownian gas (canonical description) undergoes an isothermal collapse and ultimately forms a Dirac peak [6]. In an unbounded domain, there are stable statistical equilibrium states in the microcanonical ensemble (Hamiltonian systems) for any value of the energy [20]. According to the Virial theorem, we have $K = GM^2/4$ and $K = Nk_B T$ so that these equilibrium states have the same temperature $T_c = GMm/4k_B$, independently on their energy E . By contrast, in the canonical ensemble (Brownian systems), there is no equilibrium state in an unbounded domain as discussed in [15] and in Section 3.6.

In $d = 3$, the series of equilibria forms a spiral (see Fig. 1(b)). Since the series of equilibria presents turning points, the ensembles are not equivalent. There exists stable statistical equilibrium states in MCE for $E > E_c = -0.335GM^2/R$ (Antonov energy); they have a density contrast $R \leq 709$. There exists stable statistical equilibrium states in CE for $T > T_c = GMm/2.52k_B R$ (Emden temperature); they have a density contrast $R \leq 32.1$. They corresponds to long-lived *metastable* states (local maxima of S in MCE and local minima of F in CE) whose lifetime $\sim e^N$ increases exponentially with the number of particles [21]. The states with density contrast $32.1 \leq R \leq 709$ are stable in MCE and unstable in CE; they have negative specific heats. For $E < E_c$ a self-gravitating Hamiltonian system (e.g., a globular cluster) undergoes a gravothermal catastrophe [22]. There is first a self-similar collapse leading to a finite time singularity (the central density becomes infinite in a finite time) and ultimately the system forms a binary star surrounded by a hot halo [23]. This structure has an infinite entropy at fixed energy (see Appendix A of [6]). For $T < T_c$ a self-gravitating Brownian system undergoes an isothermal collapse [9]. There is first a self-similar collapse leading to a finite time singularity [6] and ultimately the system forms a Dirac peak in the post-collapse regime [24]. This structure has an infinite free energy (see Appendix B of [6]). There is no equilibrium state (no maximum of entropy of minimum of free energy) in an unbounded domain: the system evaporates or collapses [6,15].

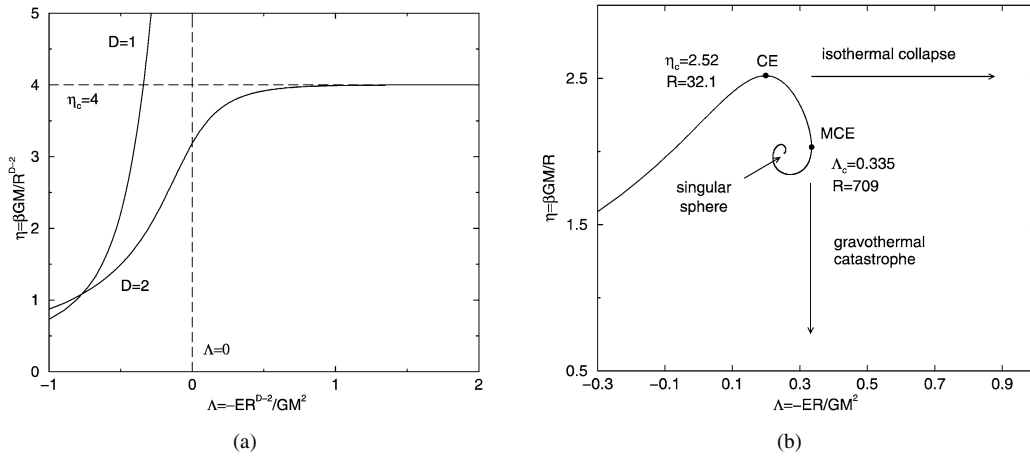


Fig. 1. Series of equilibria for classical isothermal spheres in $d = 1$, $d = 2$ and $d = 3$ dimensions.

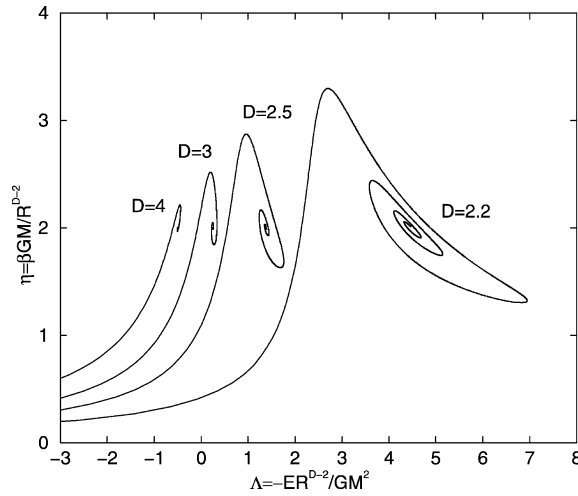


Fig. 2. Series of equilibria for classical isothermal spheres with $2 < d < 10$.

In $d \geq 3$, the phenomenology is the same as in $d = 3$ (see Fig. 2) but we note, for curiosity, that the spiral disappears for $d \geq 10$ [6]. On the other hand for $d \geq 4$ and $E < E_c$, the system forms a Dirac peak in the MCE instead of a ‘binary star + hot halo’ structure (see Appendix A of [6]).

4.2. Self-gravitating fermions

We now present the series of equilibria for the self-gravitating Fermi gas (described by the Fermi–Dirac entropy) in different dimensions of space and for different values of the degeneracy parameter μ . The degeneracy parameter can be viewed as a normalized Planck constant $\mu \sim \hbar^{-d}$, as an effective inverse small-scale cut-off $\mu \sim 1/\varepsilon$ or as the system size $\mu \sim R^{d(4-d)/2}$ (the classical case is recovered for $\hbar \rightarrow 0$, $\varepsilon \rightarrow 0$, $R \rightarrow +\infty$ or $\mu \rightarrow +\infty$). We refer to [5,7,25] for details.

In $d = 1$, there exists statistical equilibrium states for any accessible value of energy $E \geq E_g$ (where E_g is the ground state) in MCE and any value of temperature T in CE (see Fig. 3(a)). They are stable (global maxima of S in MCE and global minima of F in CE).

In $d = 2$, there exists statistical equilibrium states for any accessible value of energy $E \geq E_g$ in MCE and any value of temperature T in CE (see Fig. 3(b)). They are stable (global maxima of S in MCE and global minima of F in CE). For $\mu \rightarrow +\infty$, $E_g \rightarrow -\infty$ and we recover the classical caloric curve displaying the critical temperature T_c .

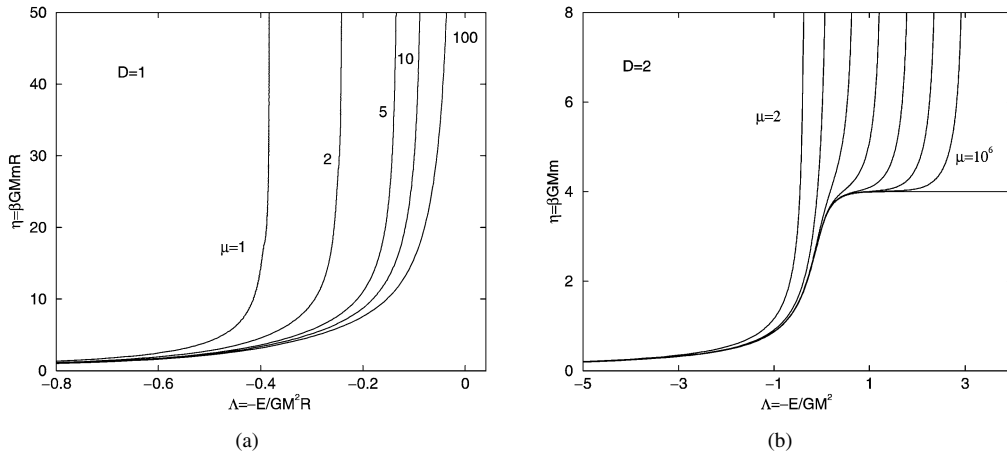


Fig. 3. Series of equilibria for an isothermal gas of self-gravitating fermions in $d = 1$ and $d = 2$ dimensions for different values of the degeneracy parameter (various system sizes).

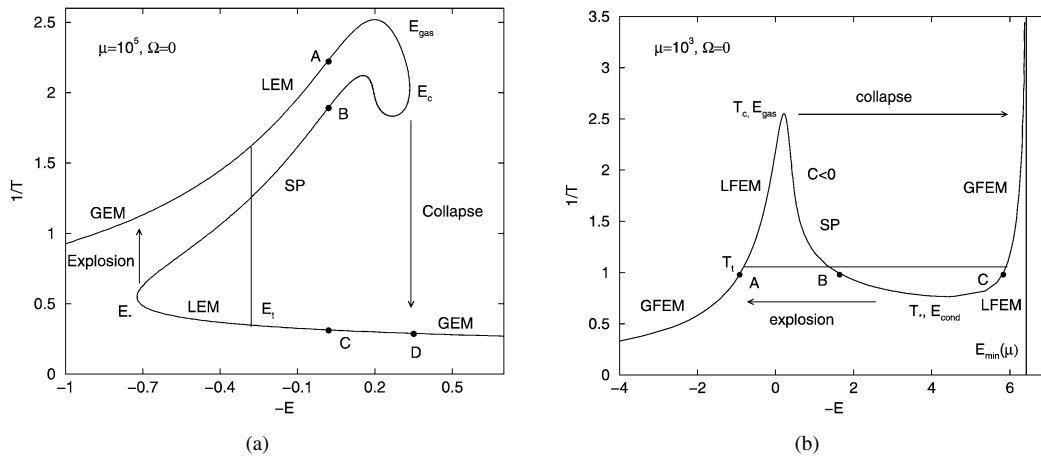


Fig. 4. Series of equilibria for an isothermal gas of self-gravitating fermions in $d = 3$ dimensions for large (left, Z-shape) and small (right, N-shape) values of the degeneracy parameter.

Below T_c , the classical Brownian gas collapses and creates a Dirac peak (‘black hole’) [6]. When quantum mechanics is accounted for, the ‘black hole’ is replaced by a ‘fermion ball’ (or white dwarf star) consisting of a dense degenerate nucleus surrounded by a dilute atmosphere [7].

In $d = 3$, the nature of phase transitions depends on the value of the degeneracy parameter [5]. For high values of μ (low small-scale cut-off), the series of equilibria has a Z-shape structure resembling a dinosaur’s neck (see Fig. 4(a)). Starting from a gaseous configuration (upper branch) and decreasing the energy, the system first passes from a stable gaseous phase (global maximum of S) for $E > E_t$ to a metastable gaseous phase (local maximum of S) for $E < E_t$. The microcanonical first order phase transition at E_t is avoided due to the long lifetime ($\sim e^N$) of the metastable gaseous states. At the critical energy E_c , the gaseous metastable phase disappears and the system collapses (gravothermal catastrophe). However, the collapse is stopped by quantum mechanics (Pauli exclusion principle) when the core of the system becomes degenerate. Therefore, the system ends up in the condensed phase (lower branch). If we now increase the energy the system first passes from a stable condensed phase (global maximum of S) for $E < E_t$ to a metastable condensed phase (local maximum of S) for $E > E_t$. Again, the microcanonical first order phase transition at E_t is avoided due to the long lifetime of the metastable condensed states. At the critical energy E_* , the condensed metastable phase disappears and the system explodes and returns to the gaseous phase. We can thus describe an hysteretic cycle in the MCE. For low values of μ (large small-scale cut-off), the series of equilibria $\beta(E)$ has a N-shape structure (see Fig. 4(b)). Since there is no turning point of energy, all the states are stable in MCE

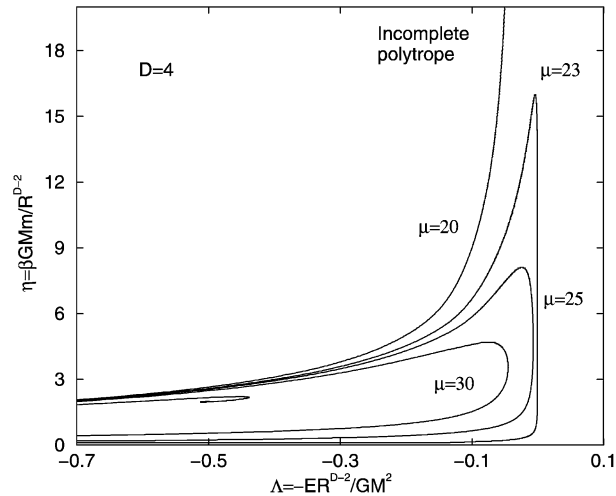


Fig. 5. Series of equilibria for an isothermal gas of self-gravitating fermions in $d = 4$ dimensions for different values of the degeneracy parameter.

(global maxima of S), for Hamiltonian systems, and there is no special phase transition for this value of degeneracy parameter, just a condensation (clustering) of the system as E decreases. Negative specific heats are possible in MCE. By contrast, since the curve $E(\beta)$ has a Z-shape structure, there is a phase transition in the CE, for Brownian systems. Starting from a gaseous configuration (left branch) and decreasing the temperature, the system first passes from a gaseous stable phase (global minimum of F) for $T > T_t$ to a gaseous metastable phase (local minimum of F) for $T < T_t$. The canonical first order phase transition at T_t is avoided due to the long lifetime ($\sim e^N$) of the metastable gaseous states. At the critical temperature T_c , the gaseous metastable phase disappears and the system collapses (isothermal collapse). However, the collapse is stopped by quantum mechanics (Pauli exclusion principle) when the core of the system becomes degenerate. Therefore, the system ends up in the condensed phase (right branch). We get a ‘fermion ball’ instead of a ‘Dirac peak’ for classical particles. If we now increase the temperature the system first passes from a stable condensed phase (global minimum of F) for $T < T_t$ to a metastable condensed phase (local minimum of F) for $T > T_t$. Again, the canonical first order phase transition at T_t is avoided due to the long lifetime of the metastable condensed states. At the critical temperature T_* , the condensed metastable phase disappears and the system explodes and returns to the gaseous phase. We can thus describe an hysteretic cycle in the CE [26]. We note that when quantum effects are taken into account, there exists an equilibrium state for any value of accessible energy $E > E_g$ in MCE and any value of temperature in CE [5].

In $d \geq 4$, the situation is different [7]. In that case, there is no equilibrium state for sufficiently small energies and temperatures (see Fig. 5). This is similar to the Antonov instability for classical particles in $d = 3$ but this now occurs for fermions. Therefore, quantum mechanics cannot stabilize matter against gravitational collapse in $d \geq 4$, contrary to what happens in $d = 3$. This is consistent with our previous observation that classical white dwarf stars (the $T = 0$ limit of the self-gravitating Fermi gas) would be unstable in a space of dimension $d \geq 4$ [14]. This is also similar to a result found by Ehrenfest [27] at the molecular level (in Bohr’s model). The case of relativistic white dwarf stars in d dimensions is treated in [28].

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