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C. R. Physique 7 (2006) 406–413
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Statistical mechanics of non-extensive systems/Mécanique statistique des systèmes non-extensifs

Gravity, dimension, equilibrium, and thermodynamics

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Available online 5 April 2006

Abstract

It is actually possible to interpret gravitation as a property of space in a purely classical way. We note that an extended self-gravitating system equilibrium depends directly on the number of dimensions of the space in which it evolves. Given these precisions, we review the principal thermodynamical knowledge in the context of classical gravity with arbitrary dimension of space. Stability analyses for bounded 3D systems, namely the Antonov instability paradigm, are then associated to some amazing properties of globular clusters and galaxies. ***To cite this article: J. Perez, C. R. Physique 7 (2006).***

Résumé

Gravité, dimension, équilibre et thermodynamique. Il est possible d'interpréter la force de gravitation comme une propriété de l'espace dans un cadre purement classique. Nous remarquons que l'équilibre des systèmes auto-gravitants étendus dépend directement de la dimension de l'espace. Etant donné ces précisions, nous passons en revue la thermodynamique connue, dans le contexte de la gravité classique, en dimension quelconque. Nous présentons l'analyse de stabilité pour des systèmes liés à trois dimensions, en particulier le paradigme de l'instabilité d'Antonov, et commentons la relation avec certaines propriétés fascinantes des amas globulaires et des galaxies. ***Pour citer cet article : J. Perez, C. R. Physique 7 (2006).***

Keywords: Classical gravitation; Thermodynamics; Theory

Mots-clés : Gravitation classique ; Thermodynamique ; Théorie

1. The nature of classical gravitation

1.1. Gravitational field

In a classical way, gravitation is a force $F = -\operatorname{grad} U$ acting on a test mass m, deriving from a scalar potential energy U which at any time t depends only on the position $\mathbf{r} \in \mathbb{R}^3$. This potential is generated by a mass density field $\rho(\mathbf{r}, t)$ such that the Poisson equation holds

$$U(\mathbf{r}, t) = m\psi(\mathbf{r}, t) \quad \text{and} \quad \Delta_3\psi(\mathbf{r}, t) = 4\pi G\rho(\mathbf{r}, t) \tag{1}$$

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1631-0705/$ – see front matter
doi:10.1016/j.crhy.2006.01.011

Inverting the Laplacian Δ_3, one writes the gravitational potential as

$$\psi(\mathbf{r},t) = -G\int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}' \tag{2}$$

Using a convolution product, this last relation can be written:

$$\psi = 4\pi G g_3 * \rho \tag{3}$$

where g_3 is a solution of the equation

$$\Delta_3 g_3 = \delta \tag{4}$$

which is different from harmonic polynomials. In other words g_3 is the Laplacian Green's function in $\mathbb{R}^3$

$$g_3(\mathbf{r}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}|} \tag{5}$$

Using information theory, the interpretation of Eq. (3) is clear: the Green's function is interpretable as an impulsional response of the corresponding operator; the gravitational potential $\psi(\mathbf{r},t)$ is then the response from space when it submits to the presence of a mass density distribution $\rho(\mathbf{r},t)$. Constant factors are also clearly interpretable: the Newton–Cavendish constant G fixes units and 4π is the value of the surface of unit radius sphere in $\mathbb{R}^3$; it is then attached to the radial nature of the gravitational interaction.

Let us note that the traditional link usually made between gravity and space in general relativity is already contained, using this formulation, in classical field gravity. Two main problems remain: the instantaneous character of the interaction, and the fundamental restriction to isotropic spaces (Laplacians act equally in all directions). Two major successes of the Einstein theory of gravitation were to solve these problems.

In a classical context, one can directly generalize Eq. (3) to obtain the classical definition of the gravitational potential in a n-dimensional space; we have:

$$\psi = G S_{n-1} g_n * \rho \tag{6}$$

where

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{with } \forall z \in \mathbb{R}_*^+\ \Gamma(z) := \int_0^{+\infty} \mathrm{e}^{-s} s^{z-1}\,\mathrm{d}s$$

represents the value of the surface of unit radius sphere in $\mathbb{R}^n$, and

$$\forall \mathbf{r} \in \mathbb{R}_*^n \quad g_n(\mathbf{r}) = \begin{cases} |\mathbf{r}| & \text{if } n=1 \\ \frac{1}{2\pi}\ln|\mathbf{r}| & \text{if } n=2 \\ -\frac{1}{(n-2)S_{n-1}}\frac{1}{|\mathbf{r}|^{n-2}} & \text{if } n>2 \end{cases}$$

is Green's function of the Laplacian operator in $\mathbb{R}^n$.

Representation of the gravitational interaction in $\mathbb{R}^n$ via relation (6) is not original, but it is presented here in a rational way.

1.2. Equilibrium

Let us describe the global dynamical properties of a system of N particles of same[1] mass m represented by their positions and impulsions merged in the vector $\Gamma_{i=1,\dots,N} := (\mathbf{r}_i(t), \mathbf{p}_i(t) = m\dot{\mathbf{r}}_i)$. Three tensors are of interest: the kinetic $\mathcal{K}$, potential $\mathcal{U}$ and inertial $\mathcal{I}$ tensor, whose components are, respectively:

$$\forall i,j = 1,\dots,N \quad \mathcal{K}_{ij} := \frac{\mathbf{p}_i\mathbf{p}_j}{2m}, \quad \mathcal{U}_{ij} := mGS_{n-1}\sum_j g_n(\mathbf{r}_i), \quad \mathcal{I}_{ij} := m\mathbf{r}_i\mathbf{r}_j$$

[1] This assumption is not essential but it simplifies notably the notations, mainly in the continuous case where $N \to \infty$.

In the continuous case where $N \to \infty$, the variable is no longer Γ_i but its probability density $f = f(\Gamma_1, \ldots, \Gamma_N, t)$. The dynamical tensors usually become:

$$\mathcal{K}_{ij} := \int \frac{p_i p_j}{2m} f \, \mathrm{d}\Gamma^N, \quad \mathcal{U}_{ij} := -m \int r_i \frac{\partial \psi}{\partial r_j} f \, \mathrm{d}\Gamma^N$$

$$\forall i, j = 1, \ldots, N \quad \mathcal{I}_{ij} := m \int r_i r_j f \, \mathrm{d}\Gamma^N \quad \text{where } \mathrm{d}\Gamma^N = \prod_{i=1}^{N} \mathrm{d}\Gamma_i$$

In the general conservative case f obeys the Liouville equation, which reduces under generic symmetry assumptions to the Collisionless Boltzmann Equation for gravitating systems (see [5] for example).

It is not too long to prove[2] the fundamental virial theorem:

Theorem 1. *If $\mathcal{U}$ is homogeneous of degree α, i.e., $\forall \lambda \in \mathbb{R}$, $\mathcal{U}(\lambda \mathbf{r}_1, \ldots, \lambda \mathbf{r}_N) = \lambda^\alpha \mathcal{U}(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ then*

$$2\,\mathrm{Tr}(\mathcal{K}) - \alpha\,\mathrm{Tr}(\mathcal{U}) = \frac{1}{2} \frac{\mathrm{d}^2\,\mathrm{Tr}(\mathcal{I})}{\mathrm{d}t^2}$$

It is quite natural to define an equilibrium state by

$$\frac{\mathrm{d}^2\,\mathrm{Tr}(\mathcal{I})}{\mathrm{d}t^2} = 0$$

in some mean sense. Hence, for a self-gravitating system in a n-dimensional space:

- if $n = 2$, $\mathcal{U}$ is not homogeneous: the virial theorem does not apply in this form;
- if $n \neq 2$, g_n, then $\mathcal{U}$ is an homogeneous function of degree $n - 2$ and equilibrium is characterized by the relation

$$2\,\mathrm{Tr}(\mathcal{K}) + (n-2)\,\mathrm{Tr}(\mathcal{U}) = 0$$

For extended self-gravitating systems like globular clusters or galaxies, a 3-dimensional space in the virial theorem seems compatible with observations.

2. Thermodynamics

If all particles have the same probability law, are independent and do not interact by pairs but only globally through their whole mean gravitating field, one can reduce the dimensionality of the phase space; the system is statistically equivalent to a test particle of mass m at position $\mathbf{r} \in \mathbb{R}^n$, with impulsion $\mathbf{p} \in \mathbb{R}^n$, described at any time t by a distribution function $f(\mathbf{r}, \mathbf{p}, t)$ and evolving in a mean field $\psi(\mathbf{r}, t)$. These two functions are solutions of the Vlasov–Poisson system:

$$\begin{cases} \dfrac{\partial f}{\partial t} - m \dfrac{\partial f}{\partial \mathbf{p}} . \dfrac{\partial \psi}{\partial \mathbf{r}} + \dfrac{\mathbf{p}}{m} . \dfrac{\partial f}{\partial \mathbf{r}} = 0 \\ \psi = G\, S_{n-1}\, g_n * \left[m \displaystyle\int f \, \mathrm{d}\mathbf{p} \right] \end{cases}$$

2.1. Definitions

Several quantities are, in general, used in thermodynamics:

- Phase space variable Γ

$$\Gamma = (\mathbf{r}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n$$

[2] From Newton fundamental principle in the discrete case or from the Liouville equation for the continuous case, ...

– Space (ν) or mass (ρ) density

$$\nu(\mathbf{r},t) = \int f(\Gamma,t)\,\mathrm{d}\mathbf{p} =: \rho/m$$

– 'Number of particles' N or system's mass M

$$N[f] := \int f(\Gamma,t)\,\mathrm{d}\Gamma = \int \nu(\mathbf{r},t)\,\mathrm{d}\mathbf{r} := M/m$$

– Energy

$$E[f] := K[f] + U[f] \quad \text{with} \quad \left| \begin{array}{l} K[f] := \dfrac{1}{2m}\displaystyle\int \mathbf{p}^2 f(\Gamma,t)\,\mathrm{d}\Gamma \\ U[f] := \dfrac{m^2}{2}\displaystyle\int \nu(\mathbf{r},t)\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r} \end{array} \right.$$

– Angular momentum[3]

$$J[f] = \int r p_\phi f(\Gamma,t)\,\mathrm{d}\Gamma$$

– Boltzmann entropy

$$S[f] := -\int f(\Gamma,t)\ln\bigl[f(\Gamma,t)\bigr]\,\mathrm{d}\Gamma$$

For precise considerations let us define some sets:

(i) Unbounded systems

$$\mathcal{G}_n(N,E) = \left\{ f \text{ s.t. } \forall\Gamma \left| \begin{array}{l} K < \infty,\ U < \infty \\ E < \infty,\ N < \infty \end{array} \right. \right\}$$

$$\mathcal{G}_n(N,E,J) = \mathcal{G}(N,E) \cap \bigl\{ f \text{ s.t. } \forall\Gamma\ J < \infty \bigr\}$$

(ii) Bounded systems $D \subset R^n$
We denote by $\mathrm{Supp}(f)$ the support of the distribution function f, i.e., the complementary of the largest open set of $\Gamma \in (\mathbb{R}^n \times \mathbb{R}^n)$ such that $f(\Gamma) = 0$. We then call

$$\mathcal{G}_n(D,N,E) = \bigl\{ f \in \mathcal{G}_n(N,E);\ \mathrm{Supp}(f) \subset D \bigr\}$$

$$\mathcal{G}_n(D,N,E,J) = \bigl\{ f \in \mathcal{G}_n(N,E,J);\ \mathrm{Supp}(f) \subset D \bigr\}$$

2.2. Thermodynamical equilibrium problem

The classical thermodynamical equilibrium problem concerns the existence of an entropy extremalizer. Considering a set $\mathcal{G}_n$ of acceptable distribution functions, it can be posed as:

$$\exists?\ f^+ \in \mathcal{G}_n \quad \text{s.t. } \forall f \in \mathcal{G}_n,\ S[f] \leqslant S[f^+]$$

For the sets considered in the previous section, this problem corresponds to a classical Euler–Lagrange one, whose solutions are the well-known isothermal spheres:

– For $f \in \mathcal{G}_n(E,N)$

$$f^+ = \exp\left\{ -\alpha - \beta\left(\frac{\mathbf{p}^2}{2m} + m\psi^+ \right) \right\}$$

Lagrange multipliers α and β correspond respectively to the conservation of N and E.

[3] The quantity p_ϕ indicates the tangential components of the impulsion $\mathbf{p}$, the quantity r stands for the Euclidian norm of the position $\mathbf{r}$ of the test particle.

– For $f \in \mathcal{G}_n(E, N, J)$

$$f^+ = \exp\left\{-\alpha - \beta\left(\frac{p_r^2}{2m} + \frac{(p_\phi - mr\omega)^2}{2m} + m\psi^+ - \frac{m^2\omega^2 r^2}{2}\right)\right\}$$

The additional constraint of J conservation corresponds to the introduction of the ω multiplier.

2.3. 2D thermodynamics: $\mathcal{G}_2$

We reproduce here the classical results obtained in [1] and [2]. The main one is twofold: we first produce a bound for entropy, we then study the existence of a distribution function which allow the reaching of that bound. Concerning the upper bound for the entropy, we prove that[4]

Theorem 2. $\forall f \in \mathcal{G}_2(E, N)$

$$S[f] \leqslant S^+(N, E) := \sup_{\mathcal{G}_2(N,E)} S[f] \leqslant \frac{2E}{N} + N \ln\left(\mathrm{e}\pi^2\right)$$

In 2D, the entropy of unbounded self-gravitating systems for which $E = cst$, and $N = cst$ is bounded from below.

Theorem 3. *In the notations of Theorem* 2

$$S^+(N, E, J) = S^+(N, E)$$

Adding the angular momentum constraint does not change the least upper bound on the entropy. The existence of a distribution function f^+ corresponding to this entropy maximizer, i.e., $S[f^+] = S^+$ is closely related to the set under consideration:

– In $\mathcal{G}_2(E, N)$: f^+ exists and is unique,

$$f^+ = \frac{\mathrm{e}^{2(E-N^2)/N^2}}{\pi^2} \frac{\mathrm{e}^{-p^2/N}}{(\mathrm{e}^{2(E-N^2)/N^2} + r^2)^2}$$

it generates the potential

$$\psi^+ = N \ln\left(\mathrm{e}^{2(E-N^2)/N^2} + r^2\right)$$

in the units of papers [1] and [2].
– There is no $f^+ \in \mathcal{G}_2(E, N, J)$ for which $S[f^+] = S^+$.
– In $\mathcal{G}_2(D, E, N)$ and $\mathcal{G}_2(D, E, N, J)$: f^+ exists and is unique.

2.4. 3D thermodynamics: $\mathcal{G}_3$

Introducing a new dimension changes drastically the situation concerning the classical equilibrium problem of gravitational thermodynamics. As a matter of fact, it is well known for a long time (see, for example, [5, p. 268]) that

– Entropy has no global maxima on $\mathcal{G}_3(E, N)$, $\mathcal{G}_3(E, N, J)$, $\mathcal{G}_3(D, E, N)$ and $\mathcal{G}_3(D, E, N, J)$.
– Entropy has no local maximum on $\mathcal{G}_3(E, N)$ and $\mathcal{G}_3(E, N, J)$.

The existence of local maximum for entropy in $\mathcal{G}_3(D, E, N)$ corresponds to an extensive literature initiated by the works of Antonov (see [5] for a review) in the early 1960s. It represents a beautiful problem of thermodynamics.

[4] In the units used in the original papers.

2.4.1. Entropy extremalizer in $\mathcal{G}_3(D,E,N)$

We reproduce below the main results obtained by Padmanabhan (see [3]) which clarify all previous works. If we denote by R the radius of the largest bowl contained in the spatial part of D, it is proven that any entropy extremalizer in $\mathcal{G}_3(D,E,N)$ must be of the form:

$$f^+ = \left(\frac{2\pi}{\beta}\right)^{-3/2} \nu_o \mathrm{e}^{-\beta E} \quad \text{with } m\nu_o = \rho(0)\mathrm{e}^{\beta\psi(0)} \tag{7}$$

The associated potential verifies the Poisson equation:

$$\Delta_3\psi = 4\pi G m\nu_o \mathrm{e}^{-\beta\psi} \tag{8}$$

with the limit condition $\psi(R) = -GM/R$. Introducing dimensionless variables

$$L_o = \sqrt{4\pi G\rho(0)\beta}, \quad M_o = 4\pi\rho(0)L_o^3, \quad \psi_o = \beta^{-1} = \frac{GM_o}{L_o}$$
$$x = r/L_o, \quad n = \rho/\rho(0), \quad \mu = M(r)/M_o, \quad y = \beta\big(\psi - \psi(0)\big)$$

the Poisson equation becomes

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{e}^{-y} \quad \text{with } y(0) = y'(0) = 0$$

Milne's functions, $v = \mu/x$ and $u = nx^3/\mu$, transform the Poisson equation into

$$\frac{u}{v}\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1-u}{u+v-3} \quad \text{with} \quad \left|\begin{array}{l} v = 0 \quad \text{when } u = 3 \\ \quad\text{and} \\ \left.\dfrac{\mathrm{d}v}{\mathrm{d}u}\right|_{(u,v)=(3,0)} = -5/3 \end{array}\right.$$

Hence, isothermal extremal spheres lie on a curve in the u–v plane. This curve is plotted on Fig. 1.

2.4.2. Antonov instability

In a meaningful remark, Patmanabhan notes that the dimensionless quantity

$$\lambda := \frac{RE}{GM^2} = \frac{1}{v}\left(u - \frac{3}{2}\right)$$

also lies on the same u–v plane. He then asks the fundamental question: Can E, R and M be accommodated by a suitable choice of $\rho(0)$ and β? As one can see on Fig. 2, the answer is clearly no. There exists a critical value $\lambda_c \simeq -0.335$, associated with the possibility to put a given isothermal extremal sphere in a given box!

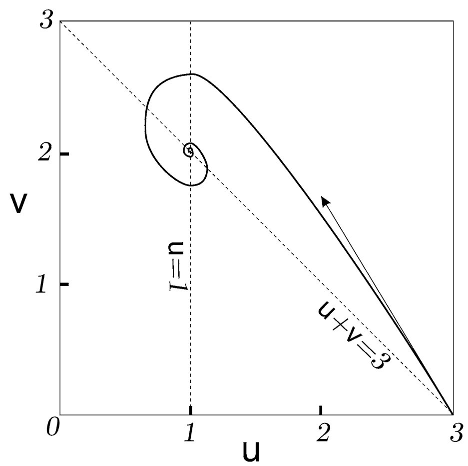

Fig. 1. Isothermal sphere in the Milne plane.

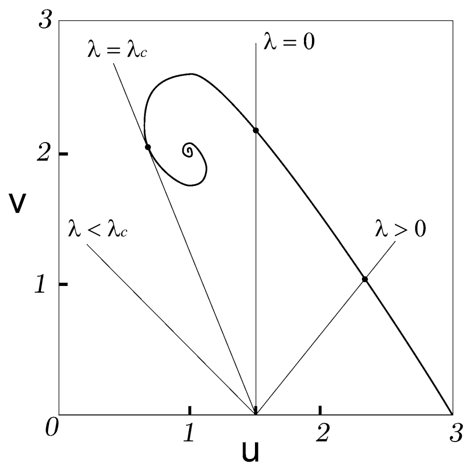

Fig. 2. Existence of an extremal isothermal sphere.

– If $\lambda < \lambda_c$ the isothermal sphere cannot exist, and the entropy extremum cannot exist;
– If $\lambda > \lambda_c$ the isothermal sphere exists, and the entropy extremum exists!

More exciting is the fact that the nature of the extremum depends on $\kappa = \rho(0)/\rho(R)$ the density concentration ratio of the isothermal sphere:

– If $\kappa > 709$, the extremum of the entropy is an unstable saddle point;
– If $\kappa < 709$, the extremum of the entropy is local maximum.

This last point is generally associated to the so-called gravothermal catastrophe; we prefer to call it the Antonov instability. As we will see in the next section, such an instability is certainly at the origin of some important characteristics of extended self gravitating systems.

3. The Antonov instability in astrophysics

3.1. Globular clusters in galaxies

Since the early 1980s, observations have shown that galactic globular clusters split in two categories which differ by some properties of their radial density profiles. On the one hand, there is a large family of about 120 clusters with a large constant density core which extends to almost the half mass radius (R_{50}) of the whole system. This large core is surrounded by a power law density decreasing halo. On the other hand, a small family of about 20 core collapsed clusters with a very high central density, which decreases monotonically outward with mainly two power law indexes. These two types of globular clusters are very well represented by two of their components, namely NGC 6388 for the core halo cluster and Trz2 for core collapsed cluster; see Fig. 1 of [6].

Such a behaviour of the radial density could be explained in a very simple way by the Antonov instability. As a matter of fact, if globular cluster formation results from the collapse of a small, hence homogeneous, region of some galaxy, the natural result is roughly an isothermal sphere (see [4]) with generally a contrast density κ less than the critical value. The evolution of the cluster in the galaxy produces a slow evaporation of the cluster (passing through the galactic plane in spiral galaxies, for example). Due to the negative specific heat of such gravitating systems, this evaporation makes the contrast density grow. When κ reaches the critical value, the Antonov instability triggers and transforms the core halo density profile into a collapsed core one. On Fig. 3 we represent the radial density profile after the collapse of an initially homogeneous system and of an initially inhomogeneous one (see [4]).

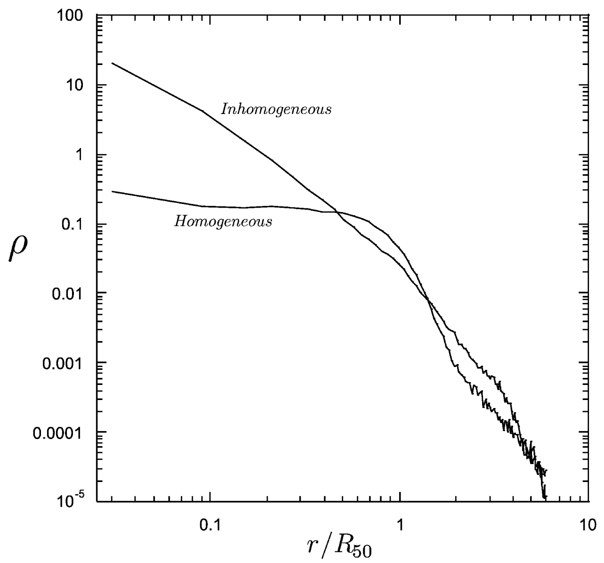

Fig. 3. Radial density profile obtained from the gravitational collapse of an homogeneous set of mass and an inhomogeneous one.

3.2. A paradigm for Super Massive Black Hole (SMBH) formation

From the accumulation of observational data, it becomes necessary to put a Super Massive (from 10^6 to perhaps 10^9 $M_\odot$) Black Hole in the dynamical center of galaxies. Except for the fact that such gravitational monsters are as old as their host, little is known about their formation process. In the context of a hierarchical galaxy formation scenario, the Antonov instability could produce a good paradigm. As a matter of fact, if galaxies are the net result of successive collapse and merger of gravitational structures, it can be modeled generically by a general collapse of inhomogeneous media. As shown by [4], in such a case, on reaching the center, small structures first collapse to form a quasi-isothermal sphere surrounded by the rest of the not yet collapsed large structures. Evaporating the smallest, the collapse of the largest could trigger the Antonov instability. A progenitor of a SMBH could then be formed. This process is not yet confirmed, but seems to correspond to all observed properties.

4. Conclusion

Classical gravity is an amazing topic. Although, 2D gravitating systems are well described by thermodynamics, their equilibrium is not well defined. On the other hand, provided that $n \geqslant 3$, we possess a powerful tool to describe the gravitational equilibrium for systems in $\mathbb{R}^n$, but the corresponding thermodynamics is not too efficient. However, in the restricted case of bounded systems in $\mathbb{R}^3$, the message from gravitational thermodynamics and in particular the Antonov instability, could be fundamental to explain some features of self gravitating systems.

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