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Stochastic invertible mappings for Tsallis distributions

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Abstract

We devise mappings between Gaussian distributions and power-law distributions, nowadays also called Tsallis distributions. To a given Tsallis distributed vector X, one can associate a Gaussian distributed vector N in the fashion $N = aX$ where a is a random variable independent of X whose properties we are going to characterize here. We not only show that this mapping is invertible but also construct the adequate inversion operation. As an application of this stochastic mapping, we revisit the problem posed to Tsallis practitioners by the zeroth law of thermodynamics, that has bedeviled them for 15 years. ***To cite this article: C. Vignat, A. Plastino, C. R. Physique 7 (2006).***

Résumé

Mappings stochastiques inverses pour les distributions de Tsallis. Nous définissons des relations bijectives entre les distributions Gaussiennes et les distributions de Tsallis ; à un vecteur aléatoire X suivant une distribution de Tsallis, il est possible d'associer un vecteur aléatoire Gaussien N de la façon suivante : $N = aX$ où a est une variable aléatoire indépendante de X dont nous caractérisons les propriétés. Nous montrons que cette association est bijective et construisons explicitement l'association inverse. Nous appliquons ce résultat au problème du principe zéro de la thermodynamique tel qu'il se pose dans le cadre des statistiques de Tsallis. ***Pour citer cet article : C. Vignat, A. Plastino, C. R. Physique 7 (2006).***

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1. Introduction

Recently, Beck and Cohen (BC) [1] have advanced an interesting generalization, that they call 'superstatistics', of the exponential factor entering Boltzmann–Gibbs (BG) equilibrium probability distribution (PD) for the canonical ensemble, advanced originally by Gibbs [2–4], namely,

$$p_G(i) = \frac{\exp(-\beta E_i)}{Z_{\mathrm{BG}}} \tag{1}$$

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where E_i is the energy of the pertinent microstate, labelled by i, $\beta = 1/k_B T$ the inverse temperature, k_B the Boltzmann's constant, and Z_{BG} the partition function. Its associated exponential term $F_{\mathrm{BG}} = \exp(-\beta E)$ is called the BG factor. Assuming that the inverse temperature is a stochastic variable, BC effect a multiplicative convolution that leads to a 'generalized' statistical factor F_{GS}

$$F_{\mathrm{GS}} = \int_0^\infty \beta f(\beta)\,\mathrm{d}\beta \exp(-\beta E) \equiv f \circ F_{\mathrm{BG}} \tag{2}$$

where $f(\beta)$ satisfies

$$\int_{-\infty}^{\infty} \mathrm{d}\beta\, f(\beta) = 1 \tag{3}$$

The $\circ$-sign denotes multiplicative convolution between two PDs, $f_X \circ f_Y$ being the PD of the ratio of the two corresponding random variables X and Y. While β is the inverse temperature, the integration variable may also be any convenient intensive parameter. BC-superstatistics is a *statistics of statistics* that takes into account fluctuations of intensive parameters. BC also show that, if $f(\beta)$ is a χ distribution, the result is nonextensive thermostatistics (NEXT), currently a very active field, with applications to several branches [5–7]. NEXT's framework is characterized by power-law distributions (PLD), which are certainly common in physics (as for instance in critical phenomena [8]). PLD naturally emerge in maximizing Tsallis q-information measure (q is called the nonextensivity index)

$$H_q(f) = \frac{1}{1-q}\left(1 - \int_{-\infty}^{+\infty} f(x)^q\,\mathrm{d}x\right) \tag{4}$$

subject to appropriate constraints. For the canonical distribution, only one constraint is needed, apart from normalization: the mean energy E, i.e., the mean value of the pertinent Hamiltonian $\mathcal{H}(x)$:

$$E = \langle \mathcal{H}(x) \rangle \tag{5}$$

It is important to remark that, as $q \to 1$, Tsallis entropy reduces to Shannon entropy

$$H_1(f) = -\int_{-\infty}^{+\infty} f(x) \log f(x) \tag{6}$$

One deals with PLDs for many physical systems [9–13]. This is an incentive for furthering research into the nonextensive formalism along multiple paths. It is in such a spirit that we revisit here the BC-road [1] and generalize its scope by showing that, for a fixed temperature T (or for any other adequate intensive parameter), there exists a mapping between *power law PDs*, on the one hand, and Gaussian PDs on the other one. Using such a mapping one can transform a Tsallis PD into a Gaussian PD and vice versa via multiplicative convolution with a chi random variable. This mapping is able to nitidly reveal the intimate relation between the orthodox Gibbs–Boltzmann statistics [2] and NEXT. As a first application of our new mapping we confront Tsallis' troubles with thermodynamics' zero-th law, as was first revealed by Raggio and Guerberoff [14] (see also [15–19], a by no means exhaustive list, and references therein).

2. Details of the formalism

2.1. *Case $q > 1$ or $q < 0$*

Consider first a k-variate vector $Y = (Y_1, \ldots, Y_k)^t$ following a Tsallis distribution with parameter $q > 1$ (or $q < 0$). This distribution, defined as the maximum Tsallis entropy distribution with given covariance matrix K, reads

$$f_Y(Y) = \frac{\Gamma(\frac{q}{q-1} + \frac{k}{2})}{\Gamma(\frac{q}{q-1})|\pi\Sigma|^{1/2}} (1 - Y^t \Sigma^{-1} Y)_+^{(p-k)/2-1} \tag{7}$$

with $p = k + \frac{2q}{q-1}$, $\Sigma = pK$ and with the notation $x_+ = \max(0, x)$. As discussed in [20], if $p \in \mathbb{N}$ then f_Y can be obtained as the k-variate marginal of a uniform distribution on the sphere in $\mathbb{R}^p$. Additionally, a stochastic representation for Y writes

$$Y = \frac{\Sigma^{1/2} N}{\sqrt{N^t N + a^2}} \tag{8}$$

where $N = [N_1, \dots, N_k]^t$ is a k-variate unit covariance Gaussian vector and a is an independent chi random variable with $(p-k)$ degrees of freedom.

Theorem 1. *If b_p is a chi random variable with p degrees of freedom independent of vector Y distributed as in* (7), *then the product $Z = b_p Y$ is Gaussian with identity covariance.*

Proof. This result is based on the polar factorization property [22] of Tsallis distributions: if Y writes as (8) then Y and $\sqrt{N^t N + a^2}$ are independent random variables. Since $\sqrt{N^t N + a^2}$ is chi-distributed with p degrees of freedom, the result follows. However, an analytical proof is given in Appendix A.

Thus a Tsallis system with $q > 1$ can be 'Gausssianized' simply by multiplying each of its components by an independent, scalar chi random variable with $p = k + \frac{2q}{q-1}$ degrees of freedom.[1]

2.2. *Case $\frac{k}{k+2} < q < 1$*

Let us now consider a k-variate random vector $X = (X_1, \dots, X_k)^t$ following a Tsallis distribution with $\frac{k}{k+2} < q < 1$. Then

$$f_X(X) = \frac{\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{1-q} - \frac{k}{2})|\pi \Lambda|^{1/2}} (1 + X^t \Lambda^{-1} X)^{-(k+m)/2} \tag{9}$$

with $\Lambda = (m-2)K$. A stochastic representation for X can be written:

$$X = \frac{\Lambda^{1/2} N}{a} \tag{10}$$

where a is chi distributed with m degrees of freedom, independent of the unit covariance Gaussian vector N. The equivalent of Theorem 1 can be expressed as follows (see the proof in Appendix A):

Theorem 2. *If X is distributed according to* (9) *and if b_{m+k} is a chi random variable independent of X with* $(m+k)$ *degrees of freedom then random vector*

$$Z = \frac{b_{m+k}}{\sqrt{1 + X^t \Lambda^{-1} X}} X$$

is Gaussian with identity covariance. Moreover, random variable $c_m = \frac{b_{m+k}}{\sqrt{1+X^t \Lambda^{-1} X}}$ is chi distributed with m degrees of freedom.

Thus a Tsallis random variable X with $\frac{k}{k+2} < q < 1$ can be 'Gaussianized' as $Z = c_m X$ but, contrary to the case $q > 1$, random variable c_m is now dependent of X.

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[1] Chi distributions are restricted to integer degrees of freedom; if $p \notin \mathbb{N}$ then the χ distribution $f_a(a) = \frac{2^{1-m/2}}{\Gamma(m/2)} a^{m-1} \exp(-a^2/2)$ should be extended to the distribution of the square-root of a gamma random variable with shape parameter equal to $2m$. For the sake of simplicity, we will speak of χ distribution in this case too.

3. Application: the zeroth law of thermodynamic

We deal with two independent systems (in the sense that their mutual interaction is negligible) whose states are described by two Tsallis-random vectors (independent as well) Y_1 and Y_2 with, say, $q > 1$ and both $\Sigma_1 = \Sigma_2 = I_k$ where I_k is the $k \times k$ identity matrix. If we consider the system $Y = [Y_1^t, Y_2^t]^t$, we immediately realize that it is not Tsallis-distributed, since its distribution

$$f_Y(Y_1, Y_2) \propto (1 - Y_1^t Y_1)^{(p-k)/2-1}(1 - Y_2^t Y_2)^{(p-k)/2-1} \tag{11}$$

can not be expressed as a function of $Y_1^t Y_1 + Y_2^t Y_2$ (except in the Gaussian case $q = 1$) [14–18]. This fact shows that we face severe NEXT-difficulties in describing the zeroth law of thermodynamics. We propose circumventing the problem in the following fashion: if we 'pre-multiply' Y_1 and Y_2 by the same chi-distributed random variable a_p as defined in Theorem 1, then the system described by $Z = a_p[Y_1^t, Y_2^t]^t$ is the merging of two Gaussian systems. Moreover, the covariance of Z can be written

$$EZZ^t = Ea_p^2 EYY^t = I_{2k} \tag{12}$$

so that $a_p Y_1$ and $a_p Y_2$ are both uncorrelated and Gaussian, and thus independent. Finally, considering a new chi-distributed random variable b_{p-k} with $p - k$ degrees of freedom and chosen independent of Z, the Tsallis distributed vector

$$Y_{\text{Tsallis}} = \frac{Z}{\sqrt{Z^t Z + b_{p-k}^2}} = \frac{Y}{\sqrt{Y^t Y + b_{p-k}^2/a_p^2}} \tag{13}$$

—and NOT vector Y—turns out to be the 'true' representative of the composite system '1 + 2' that arises after the merging of the two nonextensive systems '1' and '2' represented by, respectively, Y_1 and Y_2. Note this crucial fact: such a vector, by virtue of the random variable a_p, is no longer characterized by a fixed temperature T (or, more generally, by a fixed value of an appropriate intensive system's parameter τ), but instead by a *superposition of temperatures* 'centered' at T (resp., a superposition of the intensive parameter centered at τ), exactly in the spirit of superstatistics. Of course, for extensive systems at equilibrium the temperature of '1' and '2' would be equal to the common temperature T of both '1' and 2'.

In other words, we encounter a 'stochastic' formulation of the zeroth law that would apply for nonextensive thermostatistics: *given two independent systems A, B of equal temperature T, the pertinent temperature of the associated composite system $A + B$ at equilibrium fluctuates around T.*

3.1. 'Normal modes' or diagonalizable Hamiltonians

We extend this approach to more general types of Hamiltonians:

$$H = \sum_{i=1}^{k} c_i |z_i|^{p_i}, \quad p_i \text{ is an integer power} \tag{14}$$

In this case, the $q > 1$ Tsallis distribution compatible with (14) can be written:

$$f(Y) = A_q \left(1 - \sum_{i=1}^{k} \lambda_i |Y_i|^{p_i}\right)^{1/(q-1)} \tag{15}$$

with $A_q = \frac{\Gamma(\frac{1}{q-1} + \frac{k}{p} + 1)}{\Gamma(\frac{1}{q-1} + 1)} \prod_{i=1}^{k} \frac{p_i \lambda_i^{1/p_i}}{2\Gamma(1/p_i)}$ and with $1/p = \sum_{i=1}^{k} 1/p_i$.

Then Theorem 1 can be applied provided the mixing variable a_n is modified as follows (see proof in Appendix A).

Theorem 3. *If A is a diagonal matrix whose element $A_{i,i} = a^{1/p_i}$ where a is* (i) *a chi random variable with $n - k + 2\sum_{i=1}^{k} \frac{1}{p_i}$ degrees of freedom and* (ii) *independent of component X_i distributed as in* (15)*, then the product $Z = AX$ is distributed as*

$$f_Z(Z) \propto \exp\left(-\sum_{i=1}^{k} c_i |z_i|^{p_i}\right) \tag{16}$$

with $c_i > 0$.

An identical line of reasoning as that above allows one then to generalize the stochastic zeroth law obtained there to all classical systems described by the above type of Hamiltonian.

4. Conclusion

We have shown in this paper that Tsallis and Boltzmann distributions can be related in a simple stochastic way, namely through multiplication by a random variable. This approach has allowed us to revisit the zeroth law of thermodynamics, showing that when two systems are put in contact, the resulting temperature should be considered as randomly fluctuating; this result is coherent with Beck and Cohen theory of superstatistics.

Appendix A. Proofs

A.1. Proof of Theorem 1

Using [21, 3.471.3], we have

$$\exp(-cx) = \frac{c^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \exp(-c/u) u^{-\alpha-1} (1-ux)_+^{\alpha-1}\,\mathrm{d}u \tag{A.1}$$

Replacing x by $\sum_{i=1}^k x_i^2$ and u by v^{-1} in (A.1) we obtain

$$\begin{aligned}\exp\left(-c\sum_{i=1}^k x_i^2\right) &= \frac{c^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \exp(-cv) v^{\alpha-1} \left(1 - v^{-1}\sum_{i=1}^k x_i^2\right)_+^{\alpha-1} \mathrm{d}v \\ &= \frac{c^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \exp(-cv) v^{\alpha-1+k/2} \left(\frac{1}{\sqrt{v}}\right)^k \left(1 - \sum_{i=1}^k \frac{x_i^2}{v}\right)_+^{\alpha-1} \mathrm{d}v\end{aligned}$$

Now, as a classical Gamma integral,

$$\int_0^{+\infty} \exp(-cv) v^{\alpha-1+k/2}\,\mathrm{d}v = \frac{\Gamma(\alpha+k/2)}{c^{\alpha+k/2}}$$

so that

$$f_v(v) = \frac{c^{\alpha+k/2}}{\Gamma(\alpha+k/2)} \exp(-cv) v^{\alpha-1+k/2}$$

is the distribution of a chi-squared random variable with $k+2\alpha$ degrees of freedom and variance $\sigma^2 = 1/2c$. Thus

$$\begin{aligned}\left(\frac{1}{2\pi\sigma^2}\right)^{k/2} \exp\left(-\frac{\sum_{i=1}^k x_i^2}{2\sigma^2}\right) &= \frac{\Gamma(\alpha+k/2)}{c^{k/2}(\sigma\sqrt{2\pi})^k\Gamma(\alpha)} \int_0^{+\infty} \sigma\chi_{2\alpha+k}(v) v^{-k/2} \left(1 - \sum_{i=1}^k \frac{x_i^2}{v}\right)_+^{\alpha-1} \mathrm{d}v \\ &= \int_0^{+\infty} \sigma\chi_{2\alpha+k}(v) S_{X\sqrt{v}}^{(\alpha)}(x_1,\ldots,x_k)\,\mathrm{d}v\end{aligned}$$

where $S_X^{(\alpha)}$ denotes a k-variate Tsallis distribution obtained as the marginal of the uniform distribution on the sphere S_p

$$S_X^{(\alpha)}(x) = \frac{\Gamma(\alpha+k/2)}{(2\pi\sigma^2 c)^{k/2}\Gamma(\alpha)} (1 - x^t x)^{\alpha-1}$$

where

$$\frac{p-k}{2}=\alpha \quad \leftrightarrow \quad p=2\alpha+k$$

A.2. Proof of Theorem 2

This result is based on the following duality result: if X is distributed according to (9) then $\frac{X}{\sqrt{1+X^t\Lambda^{-1}X}}=Y$ where Y is distributed as in (9) with $\Lambda=\Sigma$ and $p-k=m$. Thus

$$Y=\frac{X}{\sqrt{1+X^t\Lambda^{-1}X}}$$

with $p-k=m$ and by Theorem 1, $b_pY=\frac{b_{m+k}X}{\sqrt{1+X^t\Lambda^{-1}X}}$ is normal. Moreover, the distribution of random variable $c_m=b_p/\sqrt{1+Y^t\Lambda^{-1}Y}$ can be computed as

$$\begin{aligned}
f_c(c) &\propto \int (1+Y^tY)^{p/2}c^{p-1}\exp\left(-\frac{c^2(1+Y^tY)}{2}\right)(1+Y^tY)^{-(m+k)/2}\,\mathrm{d}Y \\
&= c^{p-1}\exp\left(-\frac{c^2}{2}\right)\int \exp\left(-\frac{c^2Y^tY}{2}\right)\mathrm{d}Y \\
&\propto c^{p-1}\exp\left(-\frac{c^2}{2}\right)c^{-k}=c^{m-1}\exp\left(-\frac{c^2}{2}\right)
\end{aligned}$$

so that c_m is a chi random variable with m degrees of freedom.

A.3. Proof of Theorem 3

Starting from equality (A.1), we obtain

$$\begin{aligned}
\exp\left(-c\sum_{i=1}^{k}|x_i|^{p_i}\right) &= \frac{c^\alpha}{\Gamma(\alpha)}\int_0^{+\infty}\exp(-cv)v^{\alpha-1}\left(1-\sum_{i=1}^{k}\left(\frac{|x_i|}{v^{1/p_i}}\right)^{p_i}\right)_+^{\alpha-1}\mathrm{d}v \\
&= \frac{c^\alpha}{\Gamma(\alpha)}\int_0^{+\infty}\exp(-cv)v^{\alpha-1+1/p}\left(\prod_{i=1}^{k}\frac{1}{v^{1/p_i}}\right)\left(1-\sum_{i=1}^{k}\left(\frac{|x_i|}{v^{\frac{1}{p_i}}}\right)^{p_i}\right)_+^{\alpha-1}\mathrm{d}v
\end{aligned}$$

with $\frac{1}{p}=\sum_{i=1}^{k}\frac{1}{p_i}$. Following the same path as in proof of Theorem 3, we deduce that vector with ith component $a^{p_i}X_i$, where X_i is distributed as

$$f_{X_i}(x)=(1-|x|^{p_i})_+^{\alpha-1}, \tag{A.2}$$

is distributed as vector Z with

$$f_Z(Z)\propto\exp\left(-\sum_{i=1}^{k}c_i|z_i|^{p_i}\right) \tag{A.3}$$

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