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Discontinuous Galerkin methods for Maxwell's equations in the time domain

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Abstract

In this article, we describe a new high-order Discontinuous Galerkin approach to Maxwell's equations in the time domain. This approach is based on hexahedral meshes and uses a mass-lumping technique. Thanks to the orthogonality of the basis functions and a judicious choice of the approximation spaces, it provides an efficient solver for these equations in terms of storage and CPU time. ***To cite this article: G. Cohen et al., C. R. Physique 7 (2006).***

Résumé

Une méthode Galerkin discontinu pour la résolution des équations de Maxwell en régime instationnaire. Dans cet article on décrit une nouvelle approche Galerkin Discontinu d'ordre spatial élevé pour résoudre les équations de Maxwell dans le domaine temporel. Cette approche est basée sur un maillage formé d'hexahèdres et utilise une technique de compression de matrice de masse naturelle par l'utilisation de formule de quadrature de Gauss pour évaluer les termes intégraux dans la méthode. De plus, à cause de l'orthogonalité des fonctions de bases et de l'espace d'approximation choisis, on réduit aussi considérablement les matrices de rigidité et de saut, ce qui entraîne au final une méthode de résolution des équations de Maxwell en temporel, précise et efficace en termes de coûts mémoire et temps CPU. ***Pour citer cet article : G. Cohen et al., C. R. Physique 7 (2006).***

Keywords: Maxwell's equation in time domain; Discontinuous Galerkin method; Centered scheme; Numerical scheme for Maxwell's equations

Mots-clés : Équations de Maxwell dans le domaine temporel ; Méthode Galerkin discontinu ; Schéma centré ; Schéma numériques pour les équations de Maxwell

1. Introduction

For many years, many scientists have proposed numerical and physical models to improve the accuracy of the solution of electromagnetic problems in various domains such as antenna design, Electromagnetic compatibility (EMC), wave propagation, detection, ... In this research, several methods have been studied and developed to solve Maxwell equations in time and frequency domains. In this article, we restrict our purpose to methods in the time domain. The

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first and most popular of these methods is based on a finite difference scheme (FDTD) proposed by Yee [1] in which the objects studied are approximated by a staircase mesh. However, the FDTD presents two major inconveniences. The first is the staircase representation of the objects which implies parasitic diffraction near the geometry. The second is due to its numerical dispersive character which induces a delay for the high frequency content of the signal. To improve the FDTD, different authors have proposed other solutions [2,3]. In this article, we first give an overview of some of these methods developed for EMC, with their advantages and inconveniences, and finally we describe a new high order spatial discontinuous Galerkin method which can be a good alternative for solving a large set of problems.

2. Overview of methods

In some particular Electromagnetic problems, ElectroMagnetic Compatibility (EMC) problems for example, the fact that knowing with accuracy the fields near the geometry is very important because cables are generally located in the close vicinity of the surfaces. To avoid the difficulty related to the staircase mesh in the FDTD method, several solutions have been investigated by different authors. In these solutions, the time domain integral method [4,5] seems to be a good alternative to FDTD. In this method, we only need to mesh the surface of the objects and not a volume around them as in the case of the FDTD. However, this important gain in meshing is limited by a complex numerical scheme where one needs to store integral terms in order to avoid having to re-compute them at each iteration in time, and to add many unknowns in the case of heterogeneous objects. Up to now, these points limit the interest of this method for treating large problems.

An other alternative to the FDTD is the Finite Volume approach (FVTD) [6] where the shape of the cells in the mesh can be very different (tetrahedra, hexahedra, prisms, ...). In this method the unknowns located inside each cell are evaluated by the sum of flux terms taken across the boundaries of the cell. As for the FDTD, this method requires one to mesh a computational domain around the object, bounded by absorbing conditions. Compared to the FDTD, the possibilities of refining the mesh locally in the geometry permit one to obtain a gain in storage and CPU time for a large set of problems (Fig. 1). However, the dissipative or dispersive errors coming from the numerical schemes used in the FVTD approaches limit, as for FDTD, the ability of this method to treat accurately problems in cavities or for long observation times. In addition, at each iteration in time, the number of operations is more important

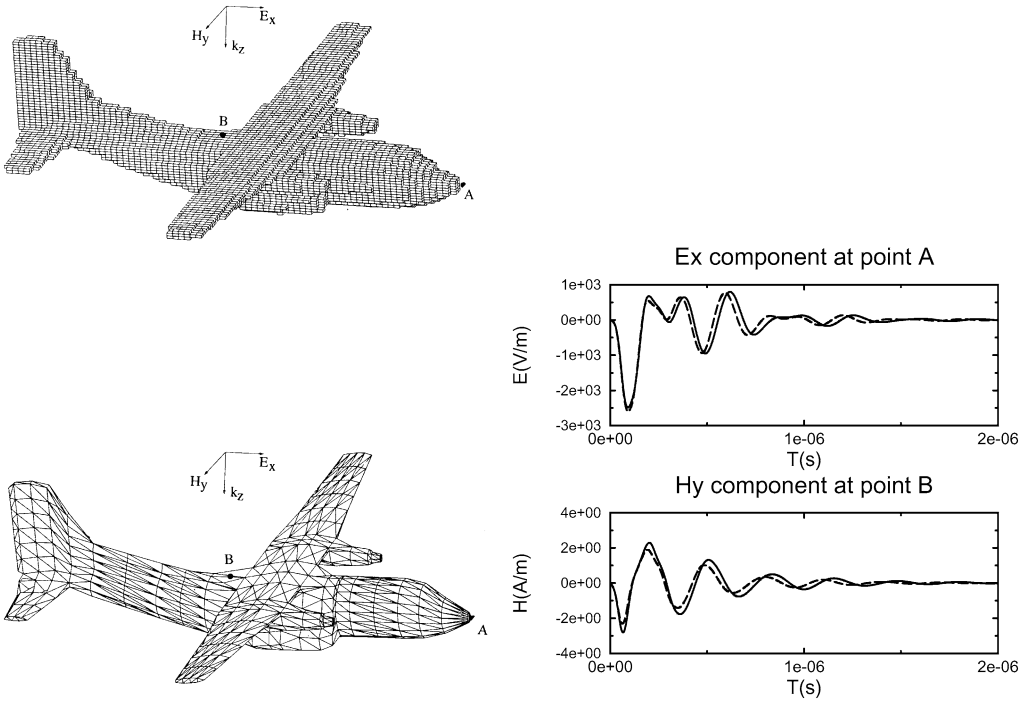

Fig. 1. Scattered fields evaluated near object: factor 1.5 in CPU time between FVTD and FDTD (the dotted line is the FDTD solution, the other the FVTD).

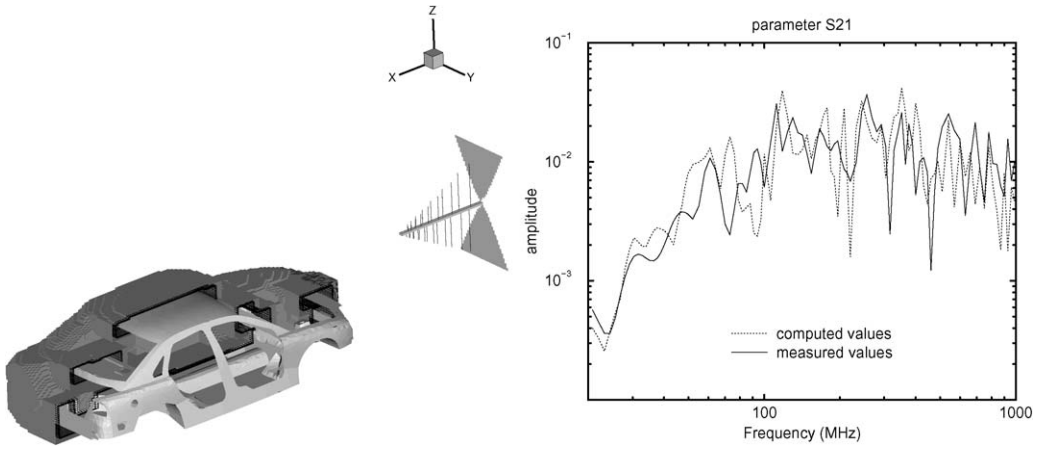

Fig. 2. Hybrid FVTD/FDTD model and comparison of simulation and measurement results.

than in FDTD, with a stability condition which imposes a smaller time step value. Consequently, for the same mesh, the CPU time is more important in this method than in FDTD. A possibility to avoid this problem is to consider hybrid FVTD/FDTD methods where the FVTD method is locally used around the objects (Fig. 2). This approach has been used with success in the GEMCAR European project [7]. However, the difficulty of hybrid methods remains guaranteeing mathematically a stability condition [8].

In all the previous methods, we have improved the geometry model, but the numerical schemes in space remained of order 1 or 2 only. The next step in these methods was, logically, to increase the order of the schemes in order to reduce the dispersive and dissipative errors. This approach provides more accurate methods which can be efficiently used in cavity problems and for long observation times. This has been done in the finite volume method by taking in each cell a solution based on a polynomial approximation of order larger than one. This type of approach is related to a family of methods known under the name of Discontinuous Galerkin methods (DGM) which are actually widely studied in the community [9–11]. Of course, a small memory storage and a small CPU time remain the main challenges of these methods. In the following, we present an original DGM which satisfies these two criteria.

3. A new discontinuous Galerkin formulation for Maxwell's equations

Solving Maxwell's equations in the time domain on a non regular mesh by using an explicit algorithm in time is a difficult challenge. Finite element techniques provide a n-diagonal mass matrix which must be inverted at each time-step. A natural alternative is the use of Finite Volume or Discontinuous Galerkin methods which provide block-diagonal mass matrices. A substantial literature on finite volume techniques was produced for these equations, but this approach is generally dissipative, difficult to extend to high-order approximations and requires an important storage. However, Hesthaven [9] recently developed a low-storage high-order Discontinuous Galerkin method on tetrahedra with a judicious choice of the location of the degrees of freedom. In this article, we present a non dissipative Discontinuous Galerkin method on a hexahedral mesh which uses a mass-lumping technique based on Gauss quadrature rule. In addition, the introduction of a judicious approximation space avoids the storage of the stiffness matrix. Moreover, this matrix is very sparse, which provides an algorithm in $O(r^4)$ instead of $O(r^6)$ for tetrahedra, r being the order of the method.

3.1. The formulation

To introduce our Galerkin Discontinuous method, we consider the electromagnetic problem, described in a domain Ω by the Maxwell's equations:

$$\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} = \nabla \times \mathbf{H} \tag{1}$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \tag{2}$$

$$\mathbf{E}(x,0)=\mathbf{E}_0(x),\quad \mathbf{H}(x,0)=\mathbf{H}_0(x)\quad \text{on } \Omega \tag{3}$$

where $(\mathbf{E},\mathbf{H})$ are the electric and magnetic fields and $\mathbf{n}$ the outward normal to the boundary of Ω. ε, σ and μ are the permittivity, conductivity and permeability matrices, respectively, which can be used to take into account anisotropic media. In this model, on the boundary $\partial\Omega$ of the domain, to simulate unbounded space, we use a PML formalism where a condition $\mathbf{n}\times\mathbf{E}=0$ ends the layers [12].

Let $\mathcal{T}_h$ be a hexahedral mesh of Ω such that $\Omega=\bigcup_{i=1}^{N_e} K_i$. In our approach, for each $K_i\in\mathcal{T}_h$, we rewrite (1)–(3) by adding terms which define jumps of electric and magnetic tangential components across the boundary of K_i. We obtain on K_i :

$$\int_{K_i}\varepsilon\frac{\partial\mathbf{E}}{\partial t}\cdot\boldsymbol{\psi}\,\mathrm{d}x+\int_{K_i}\sigma\mathbf{E}\cdot\boldsymbol{\psi}\,\mathrm{d}x=\int_{K_i}\nabla\times\mathbf{H}\cdot\boldsymbol{\psi}\,\mathrm{d}x+\alpha\int_{\partial K_i}[\![\mathbf{H}\times\mathbf{n}]\!]_{\partial K_i}^{K_i}\cdot\boldsymbol{\psi}\,\mathrm{d}s+\beta\int_{\partial K_i}[\![\mathbf{n}\times(\mathbf{E}\times\mathbf{n})]\!]_{\partial K_i}^{K_i}\cdot\boldsymbol{\psi}\,\mathrm{d}s \tag{4}$$

$$\int_{K_i}\mu\frac{\partial\mathbf{H}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}x=-\int_{K_i}\nabla\times\mathbf{E}\cdot\boldsymbol{\phi}\,\mathrm{d}x+\gamma\int_{\partial K_i}[\![\mathbf{E}\times\mathbf{n}]\!]_{\partial K_i}^{K_i}\cdot\boldsymbol{\phi}\,\mathrm{d}s+\delta\int_{\partial K_i}[\![\mathbf{n}\times(\mathbf{H}\times\mathbf{n})]\!]_{\partial K_i}^{K_i}\cdot\boldsymbol{\phi}\,\mathrm{d}s \tag{5}$$

where α, β, γ, δ are positive constants, $\boldsymbol{\phi}\in[H(\mathcal{T}_h)]^3$, $\boldsymbol{\psi}\in[H(\mathcal{T}_h)]^3$ and $[\![\mathbf{v}]\!]_{\partial K_i}^{K_i}$ is the jump of a vector-valued function $\mathbf{v}$ across the boundary of K_i. $H(\mathcal{T}_h)$ defines the functional space given by $\{v\in L^2(\Omega);\ \forall K\in\mathcal{T}_h,\ v|_K\in H^1(K)\}$.

Now we choose α, β, γ and δ so that the last formulation (4), (5) is equivalent to the Maxwell's equations (1)–(3) and to ensure a well posed problem, the energy $\varepsilon\int_\Omega E\cdot E\,\mathrm{d}x+\mu\int_\Omega H\cdot H\,\mathrm{d}x$ of this new formulation must be bounded in time.

A study of the energy of (4), (5) for $\sigma=0$ shows that this system is dissipative for $\beta\geqslant 0$, $\delta\geqslant 0$ and $1+\alpha-\gamma=0$. Moreover, if $\beta=\delta=0$, we get an energy conservation. So, in order to obtain a non-dissipative formulation and two equivalent problems, we set $\beta=\delta=0$ and $-\alpha=\gamma=1/2$ for any interior face and $\gamma=1$ and $\beta=\delta=\alpha=0$ on a metallic face.

Now, let $\hat{K}=[0,1]^3$ be the unit cube and a hexaedric cell $K_i\in\mathcal{T}_h$; $\mathbf{F}_i$ is a trilinear conform mapping such that $\mathbf{F}_i(\widehat{K})=K_i$. DF_i defines its Jacobian matrix and $J_i=\det(DF_i)$. The approximate problem is then defined in the following approximation of $[H(\mathcal{T}_h)]^3$:

$$\mathbf{V}_h^r=\left\{\mathbf{v}\in\left[L^2(\Omega)\right]^3\text{ such that }|Ji|DF_i^{-1}\mathbf{v}|_{K_i}\circ F_i\in\left[Q_r\left(\widehat{K}\right)\right]^3\right\} \tag{6}$$

where $Q_r(\widehat{K})$ is the set of polynomials of order less or equal to r in each variable. In the following, we shall call a Q_r approximation for the Discontinuous Galerkin method, an approximation based on $\mathbf{V}_h^r$.

3.1.1. Basis functions

To define the basis functions of $\mathbf{V}_h^r$, first, we define local basis functions on $\widehat{K}$. Let $\hat{\vec{\xi}}_{i,j,k}=(\hat{\xi}_i,\hat{\xi}_j,\hat{\xi}_k)$, $0\leqslant i\leqslant r$, $0\leqslant j\leqslant r$, $0\leqslant k\leqslant r$, be a set of points of $\widehat{K}$, where $\hat{\xi}_\ell$ represents a Gauss quadrature point on the interval $[0,1]$. We define the set of the $(r+1)^3$ Lagrange interpolation polynomials $\theta_{i,j,k}\in Q_r$ such that $\theta_{i,j,k}(\hat{\vec{\xi}}_{\ell,m,n})=\delta_{i\ell}\delta_{jm}\delta_{kn}$, where δ_{ij} is the Kronecker symbol. Then, we define the following set $\widehat{\mathcal{B}}$ of vector-valued basis functions on $\widehat{K}$: $\boldsymbol{\theta}_{i,j,k}^{(1)}=(\theta_{i,j,k},0,0)^T$, $\boldsymbol{\theta}_{i,j,k}^{(2)}=(0,\theta_{i,j,k},0)^T$, $\boldsymbol{\theta}_{i,j,k}^{(3)}=(0,0,\theta_{i,j,k})^T$. Fig. 3 shows the location of the degrees of freedom in $\widehat{K}$ for a Q_3 approximation.

Finally, for each hexahedron K_ℓ we define $3(r+1)^3$ functions $\boldsymbol{\phi}_{\ell,i,j,k}^{(n)}$ of the basis $\mathcal{B}_v$ of $\mathbf{V}_h^r$ such that $\boldsymbol{\phi}_{\ell,i,j,k}^{(n)}\circ\mathbf{F}_\ell=DF_\ell^{*-1}\boldsymbol{\theta}_{i,j,k}^{(n)}$.

3.1.2. Discrete formulation

In problem (4), (5), by using the above definitions of basis functions to approach the fields and by computing all the integral by the Gauss rule, we get the following discrete formulation of the problem, with a leapfrog scheme in time:

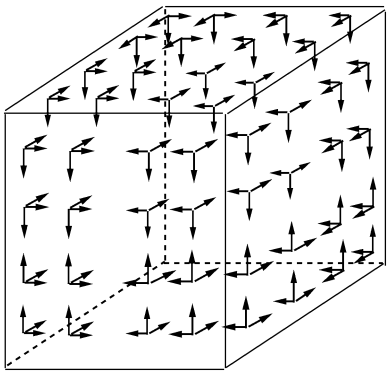

Fig. 3. Degrees of freedom for Q_3 approximation on the unit cube.

$$B_\varepsilon \frac{\mathbf{E^{n+1}} - \mathbf{E^n}}{\Delta t} + B_\sigma \frac{\mathbf{E^{n+1}} + \mathbf{E^n}}{2} + R_h \mathbf{H^{n+1/2}} + \alpha S_h \mathbf{H^{n+1/2}} + \mathbf{J^{n+1/2}} = 0 \tag{7}$$

$$B_\mu \frac{\mathbf{H^{n+1/2}} - \mathbf{H^{n-1/2}}}{\Delta t} + R_h \mathbf{E^n} + \gamma S_h^* \mathbf{E^n} = 0 \tag{8}$$

In this system:

- B_ε, B_σ, B_μ are 3×3 block-diagonal symmetric mass matrices. This property comes from the orthogonality of the basis functions and a adequate numbering of the degrees of freedom;
- the stiffness matrix R_h and the jump matrices S_h, S_h^* are very sparse and require almost no storage. Only the storage of the integral terms computed on the reference element $\widehat{K}$ and the value of the sign of J_i on each element K_i are necessary. This is due to the somehow strange definition of $\mathbf{V}_h^r$ which induces the following properties: For any $\boldsymbol{\phi} \in \mathbf{V}_h^r$, $\boldsymbol{\psi} \in \mathbf{V}_h^r$ such that $\boldsymbol{\phi} \circ \mathbf{F}_\ell = DF_\ell^{*-1} \hat{\boldsymbol{\phi}}$ and $\boldsymbol{\psi} \circ \mathbf{F}_\ell = DF_\ell^{*-1} \hat{\boldsymbol{\psi}}$, we have

$$\int_{K_i} \nabla \times \boldsymbol{\phi} \cdot \boldsymbol{\psi} \, \mathrm{d}x = \operatorname{sign} J_i \int_{\widehat{K}} \widehat{\nabla} \times \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\psi}} \, \mathrm{d}x$$

and

$$\int_{\partial K_i} (\boldsymbol{\phi} \times n_i) \cdot \boldsymbol{\psi} \, \mathrm{d}\sigma = \operatorname{sign} J_i \int_{\partial \widehat{K}} \left(\hat{\boldsymbol{\phi}} \times \hat{n}\right) \times \hat{\boldsymbol{\psi}} \, \mathrm{d}\hat{\sigma}$$

where $\widehat{\nabla}$ is the gradient operator on $\widehat{K}$. In the numerical scheme, the block-diagonal structure of B_ε, B_σ and B_μ leads to a quasi explicit scheme of resolution in time without the need to store big inverse mass matrices. In particular, this induces a fast method ($\mathrm{O}(r^4)$ number of operations per iteration compared to $\mathrm{O}(r^6)$ for classical approaches [9]). So, this approach allows us to have a high order spatial approximation with a low memory storage ($24(r+1)^3 + 1$ value to store by cell for a Q_r approximation).

The advantage of this method is strongly related to the use of a hexahedral mesh. Unfortunately, such meshes introduce a practical difficulty. In fact, hexahedral meshes are not easy to generate, contrary to tetrahedral meshes. A solution to this problem consists of constructing a hexahedral mesh by using a tetrahedral one in which each tetrahedron is split into four hexahedra, as is shown in Fig. 4. This strategy enables us to apply our DGM approach to very complex structures. In this approach to generate meshes, the size of the tetrahedra split is taken bigger than the necessary size of the same cells for a method based upon a tetrahedral mesh. Thus, the additional cost in terms of number of elements in the mesh is limited, and by using the particular properties in terms of storage of the scheme proposed, we can say that, despite this kind of mesh, the method stays more advantageous in memory storage than classical Discontinuous Galerkin methods based upon tetrahedric cells.

For the numerical scheme obtained, stability conditions based on plane waves analysis or energetic techniques have been established [10].

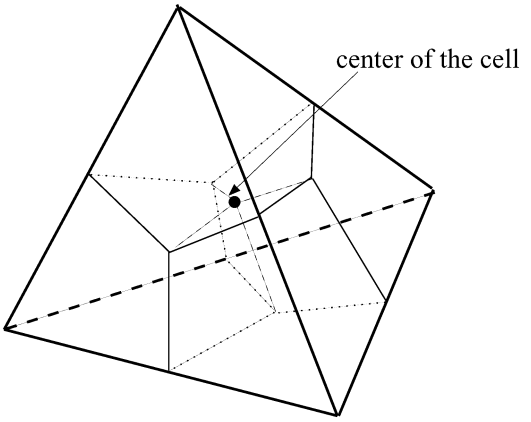

Fig. 4. Decomposition of a tetraedric cell into four hexaedric cells.

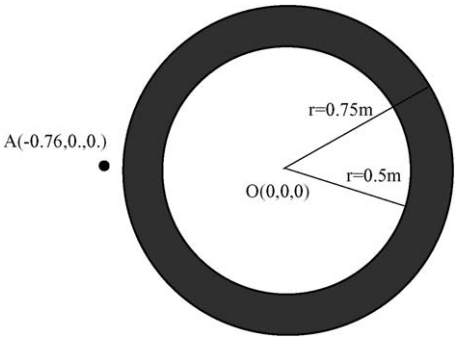

Fig. 5. The experimental configuration with the test-point A.

3.2. Numerical results

In this section, we present an example to show the behaviour of this method on problems where a scattered field is observed for a time corresponding to several wavelengths (100). The problem consists of evaluating at a test point A, located at 1 cm of the object, the field scattered by a perfectly conductive sphere (radius 50 cm) coated with a dielectric layer (thickness: 25 cm) and illuminated by a plane wave defined by $(k_x = 1, E_y = 377 f(t,x), H_z = f(t,x))$ with $f(t,x) = \exp(-(5e8(t - (x + 20/3e8))^2)$ (Fig. 5). In the dielectric medium, the wave velocity is three times smaller than in vacuum. The presence of the dielectric material implies an unsteady solution on long times. To show the interest of the DGM, we compare the solution obtained by DGM for Q_3 and Q_5 approximations with the same mesh and FDTD with 10 and 20 points per wavelengths in the mesh. In Fig. 6, the slight difference between Q_3 and Q_5 results show the good behaviour of the DGM solution in time. The high order spatial scheme avoids the dispersion error which clearly appears in the FDTD results. We also notice in this figure the good agreement of all the solutions at the beginning of the experiment, whereas, after a time corresponding to 120 wavelength, we can see the poor accuracy of the FDTD solution, even for a fine mesh. In terms of CPU time, the DGM Q_3 solutions takes 1400 mn versus 1100 mn for the FDTD with 20 points per wavelength. To obtain a similar accuracy in the FDTD method, these results clearly show that we must take a smaller spatial step in the FDTD mesh. Therefore, the DGM will take less CPU time than the FDTD. In terms of storage, the DGM requires about 10 times less memory than FDTD. Note that, in this example, the important storage of FDTD forced us to restrict our mesh to 20 cells per wavelength (these

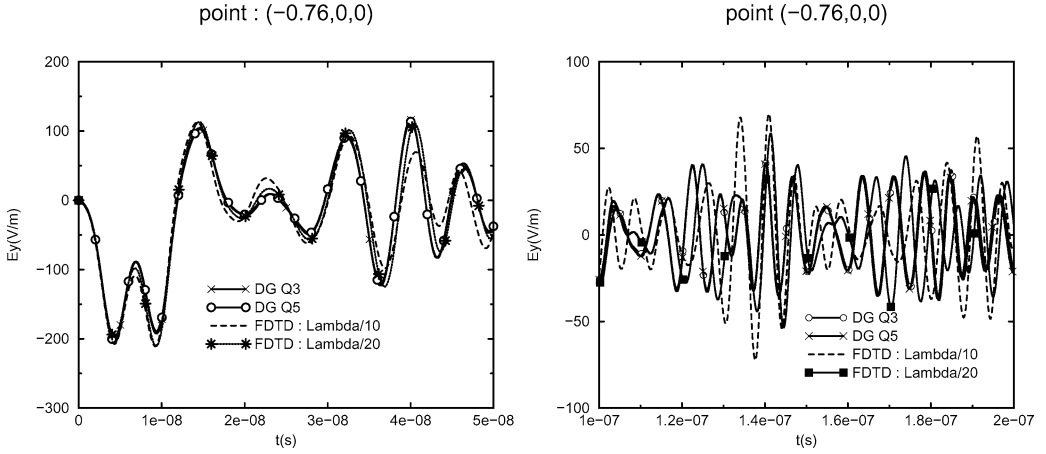

Fig. 6. Fields at the test-point A for different ranges in time.

results have been obtained on a Pentium 4 processor with 2 GBytes of memory). The CPU time of the DGM could be substantially reduced by the use of a local time-stepping.

4. Conclusion

In this article, after a brief overview on different methods to solve Maxwell equations in the time domain, we have presented an original high spatial order Discontinuous Galerkin method which ensures a very accurate solution with a low memory storage and a low CPU time. This method is based on a spatial non dissipative scheme and on an approximation upon mesh constituted of hexahedra. This particularity permits us to obtain easily a high degree of spatial approximation by Cartesian products. Such meshes can be obtained by splitting tetraedric cells of a non-structured classical mesh into 4 hexaedric cells. This possibility makes the method very attractive, even for complex geometries and in particular for cavity problems and for problems where the signals are unsteady during a long time. However, the size of some local cells in the split mesh imposes a very small time step for the method in order to ensure its stability. Actually, to correct this lack of performance, some local time step strategies are studied and introduced in the method. The gains obtained are very promising.

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