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Stable integral equations for the iterative solution of electromagnetic scattering problems

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Abstract

We present an integral equation approach for which the discretization leads to systems of linear equations which can be easily solved iteratively. This concept of intrinsically well-conditioned integral equations is illustrated by two equations applied to the scattering of a plane wave by a perfectly conducting obstacle. The numerical performance obtained in both cases is encouraging and opens new perspectives on stable integral equation methods. ***To cite this article: D.P. Levadoux, C. R. Physique 7 (2006).***

Résumé

Des équations intégrales stables pour la résolution itérative de problèmes de diffraction d'ondes en électromagnétisme. On décrit une approche intégrale dont la finalité est de permettre la construction d'équations qui, après discrétisation, conduisent à des systèmes linéaires facilement résolus par les solveurs itératifs. Le concept d'équation intégrale intrinsèquement bien conditionnée est dégagé, illustré par deux équations résolvant le problème de la diffraction d'une onde par un corps parfaitement conducteur. Les résultats numériques sont probants et ouvrent des perspectives nouvelles concernant la stabilisation des équations intégrales. ***Pour citer cet article : D.P. Levadoux, C. R. Physique 7 (2006).***

Keywords: Integral equations; Electromagnetic scattering; Preconditioner; CFIE; GCFIE; GCSIE

Mots-clés : Équations intégrales ; Diffraction d'ondes en électromagnétisme ; Préconditionneur ; CFIE ; GCFIE ; GCSIE

1. Introduction

This article reports on some recent advances in research on integral equation methods for the iterative solution of high-frequency wave scattering problems. A first direction of investigations, building on the generalisation of the combined source equations using an admittance operator, has been initiated in 1998 at the Electromagnetism and Radar Department of ONERA. This research has been pursued since 2002 in a collaborative effort with the Laboratory of Mathematics of the University Paris XI at Orsay. The results of Section 5 have been obtained in this context through a PhD thesis [1]. A second direction of investigation has been undertaken, seeking to stabilise the combined field integral equation by means of a dedicated parametrix. The latter approach, without having attained the same level of

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maturity as the first one, nevertheless presents a complementary angle of view on the same problem. Therefore, it seems interesting to present these two approaches jointly.

We shall consider the solution of scattering problems through integral equation methods in the time-harmonic case. The purpose of our developments is to improve on the convergence properties of iterative solution techniques. Indeed, we aim at minimising the number of iterations needed to construct acceptable approximations to given problems. It is a well-known fact that integral equations lack stability in the sense that their discretization leads to ill-conditioned systems of linear equations and hence are difficult to solve iteratively. For that reason, it is common to resort to pre-conditioners; that means, multiplying the system with some matrix which supposedly gives a good approximation to the inverse such that the conditioning of the resulting system be better. However, the construction of such pre-conditioners can be very costly and the purely algebraic nature of the construction does not benefit from the properties which could be implied from the operators the matrices are derived from. The gain to be expected from such methods from numerical algebra are not amenable for theoretical work and the efficiency of the methods is, accordingly, quite doubtful.

For these reasons, we propose to work on the stabilisation of integral equation formulations, not in an 'a posteriori' way (after discretization of a given integral equation) but in an 'a priori' way and, indeed, before the integral equation formulation of the scattering problem itself. We have to recognise that there is not just one way to transform a scattering problem into an integral equation and we have in fact many possibilities to do so. The two formulations presented in the Sections 4 and 5, show in which way physical and mathematical information concerning the problem at hand can be used to establish integral equations which are intrinsically well-conditioned, and hence leading to fast iterative solution without pre-conditioners.

The history of research on the applicability of stable integral equations is quite recent and coincides with the emergence of modern computing machines. Starting from about 1960, when the numerical solution of integral equations became technically possible, the question of the stability of these equations has been clearly stated. At that time, it became apparent that a number of integral equations initially preferred for their ease of implementation and immediate relation to the physical problem, appeared to be actually ill-posed and leading to numerical inaccuracies. In the time harmonic case, for example, one noticed that the solvability of an integral equation can depend on the frequency. For certain frequencies, called irregular or internal resonance frequencies, the null-space of the operator is no longer trivial and hence uniqueness of the solution is lost. By adapting ideas which had been developed before by the Russian school, principally by Kupradze [2], Brackhage and Werner [3], and also Panich [4] contribute decisive solutions to these internal resonance problems. These authors observe that the linear combination of equations which do not show the same set of internal resonance frequencies, leads, when choosing the coefficients appropriately, to a new equation without irregular frequencies. This technique applied to electromagnetics has shown to be particularly fruitful. For example, Mitzner proposed a Combined Field Integral Equation (CFIE) [5] and Mautz and Harrington a Combined Source Integral Equation (CSIE) [6]. From these 'twin' equations, the CFIE has become the most popular and nowadays belongs to the standard options for large scale industrial electromagnetic computations.

Having, finally then, some stable equations as well as iterative solution methods well adapted to them (the GMRES [7] is very important in this context), research in the 1980s has turned to the important problem of the computational cost of the matrix-vector product evaluation, which can become an obstacle with very large problems. This product has to be carried out as much as the number of iterations (say n_{iter}) needed to get a satisfactory solution. Now, it is specific for integral equations that the system matrices are full. The numerical cost of a solution is therefore of the order of $\mathcal{O}(n_{\text{iter}}N^2)$ operations, for a rank N system matrix. For that reason, ideas have been put forward to make these matrices more sparse or to compress them, often by trying to benefit from the specifics of high frequency scattering, which is one of the most frequent situations where large matrices arise. Without claiming exhaustiveness, in a domain where contributions are numerous, we want to quote the methods proposed by Canning [8] and DeLaBourdonnaye [9], which have been exemplary for that period. The most significant breakthrough has been achieved by Rokhlin [10], with the Fast Multipole Method (FMM) which allowed for matrix vector products at a cost of $\mathcal{O}(N \log N)$ operations. This has initiated a revolution, making problems with millions of unknowns within the reach of integral equation methods.

Ever since the emergence of the FMM, we can note a renewed interest for the stability problems. The success of the FMM had been so impressive, that the hope to do even better seemed vain. On the other hand, by the very power of the new methods, the large problems being currently handled showed more clearly than before the problems of them being ill-conditioned. This explains, the recent appearing of new methods to stabilise the integral equations. In

addition to the unique-existence problem, which directed the search for numerically stable equations in the sixties, we find nowadays more detailed investigation into the spectral properties of the underlying operators. The first authors paving the way to these considerations are probably McLean and Tran [11] and Steinbach [12]. The emerging idea is that the construction of efficient pre-conditioners consists of discretizing the parametrix of the initial operators defining the equation one tries to solve. In fact, systems pre-conditioned in this way can be interpreted as discretizations of equations which one might qualify as intrinsically well-conditioned integral equations. This is the point of view we shall adopt in this paper.

The research we present here is based on the conviction that we can still improve on the stability of well-defined integral equations. We aim at the solution of very large systems with iterative algorithms. In this context, any method pretending to be efficient, has to consider fast matrix vector products. The methods we propose here are therefore compatible with the fast multipole method for the eventual matrix vector products.

2. Competitive integral equations, combined sources and combined fields

In this section, we recall the model problem to which our further developments will apply as well as the various integral equation formulations which can be chosen. This will be the occasion to single out two different approaches: combined source equations and combined field equations.

We consider the scattering by an obstacle with boundary Γ, equipped with an orientation given by a normal field $\mathbf{n}$, pointing out from the obstacle. We define the space W^+, in which we find the solution of the external boundary value problem, as the set of electric fields radiating to infinity and having a tangential trace on Γ. The boundary value problem is

$$\text{Find}\,\mathbf{E} \in W^+ \text{ such that } \mathbf{n} \times \mathbf{E} = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \tag{1}$$

This defines the scattering by a perfectly conducting obstacle in an incident wave given by $\mathbf{E}^{\text{inc}}$.

A natural way to solve such a problem by means of integral equations consists of a parameterisation of the space of admissible waves, W^+, by a functional which establishes a link between current distributions on Γ and electric fields in W^+,

$$V : \mathcal{D}'(\Gamma) \to W^+ \tag{2}$$

The integral equations associated to this potential V is then

$$\mathbf{n} \times V(\mathbf{u}) = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \tag{3}$$

Note that in this way, the unknown $\mathbf{u}$ does not necessarily have a physical interpretation, or, in mathematical terms, is not necessarily part of the Cauchy data of the field $\mathbf{E}$ solving the original boundary value problem (1).

The most easily accessible potentials are

$$L = \frac{1}{\mathrm{i}k} \nabla \times \nabla \times G, \qquad K = \nabla \times G$$

where G denotes the vector potential (depending on the wave number k) which associates to any field, $\mathbf{u}$, of tangent vectors on Γ, a vector field defined in all $x \in \mathbb{R}^3 \backslash \Gamma$ by:

$$G\mathbf{u}(x) = \frac{-1}{4\pi} \int_{\Gamma} \frac{\mathrm{e}^{\mathrm{i}k\|x-y\|}}{\|x-y\|} \mathbf{u}(y)\, \mathrm{d}y \tag{4}$$

where $\| \ \|$ denotes the euclidean norm on $\mathbb{R}^3$.

However, equations constructed along the lines of (3) with such potentials can be ill-conditioned (irregular frequency problem). A method to circumvent this problem is to combine the potentials. An illustration of this is the equation of Mautz–Harrington [6], where the underlying potential is $V\mathbf{u} = L\mathbf{u} - K(\alpha \mathbf{n} \times \mathbf{u})$. In the same spirit, we might take

$$V\mathbf{u} = L(\alpha \mathbf{n} \times \mathbf{u}) - K\mathbf{u}$$

This leads to the combined source equation:

$$\mathbf{n} \times L(\alpha \mathbf{n} \times \mathbf{u}) - \mathbf{n} \times K\mathbf{u} = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{(CSIE)} \tag{5}$$

which is a well-defined equation for all frequencies if $\Re e(\alpha) > 0$. It is this equation which we will generalise in Section 5.

Although natural from the theoretical point of view, the approach of combined sources has been superseded, in applications, by another construction. This second family of integral equations, consists of integral equations formulations where the unknowns are the Cauchy data of the boundary value problems. There derivation relies on the Stratton–Chu integral representations, which express the electric fields, **E**, and the magnetic fields, **H**, of a field in W^+ in terms of their tangential boundary traces.

$$\mathbf{E} = L(\mathbf{n} \times \mathbf{H}) - K(\mathbf{n} \times \mathbf{E}), \qquad \mathbf{H} = -K(\mathbf{n} \times \mathbf{H}) - L(\mathbf{n} \times \mathbf{E}) \tag{6}$$

Note that $\mathbf{E}^{\text{inc}}$ is not in W^+ (because it does not satisfy the radiation condition) and instead of the Stratton–Chu formulas we have the following equalities,

$$0 = L\big(\mathbf{n} \times \mathbf{H}^{\text{inc}}\big) - K\big(\mathbf{n} \times \mathbf{E}^{\text{inc}}\big), \qquad 0 = -K\big(\mathbf{n} \times \mathbf{H}^{\text{inc}}\big) - L\big(\mathbf{n} \times \mathbf{E}^{\text{inc}}\big) \tag{7}$$

To sum up, using (6) and (7) and the perfect conductor boundary conditions, we get the electric field integral equation and the magnetic field integral equation

$$L(\mathbf{J}) = -\mathbf{E}^{\text{inc}} \quad \text{(EFIE)}, \qquad \mathbf{n} \times K(\mathbf{J}) + \mathbf{J} = \mathbf{n} \times \mathbf{H}^{\text{inc}} \quad \text{(MFIE)} \tag{8}$$

where the unknown **J** equals $\mathbf{n} \times (\mathbf{H} + \mathbf{H}^{\text{inc}})$. In order to dispose of the problem of irregular frequencies, from which suffer both of these equations, Mitzner [5] suggested to combine the two to obtain an equation which is well-defined at all frequencies

$$\alpha L(\mathbf{J}) + \mathbf{n} \times K(\mathbf{J}) + \mathbf{J} = -\alpha \mathbf{E}^{\text{inc}} + \mathbf{n} \times \mathbf{H}^{\text{inc}} \quad \text{(CFIE)} \tag{9}$$

Although this equation has very good stability properties, we present, in Section 4, an improvement of this equation which makes it even more stable.

From the two competing equations, combined source and combined field, it is clear that the latter has been the more popular one. It seems that this success is primarily due to the fact that the unknowns are the physical current distributions on the obstacle. Bendali, in the introduction to his fundamental work on the EFIE [13], conjectures that "*the most used methods and the most interesting ones are those where the unknowns have a clear physical interpretation.*" However, the desire to deal with unknowns with a physical interpretation imposes a constraint which severely restricts the possibilities to construct stable formulations. The equations derived from general potentials are much richer and provide tools for the composition of formulations with better properties than their 'physical' counterparts. This is the conjecture, which we shall corroborate in Section 5, where we present an integral equation which remains very stable at high frequencies.

Putting aside this controversy, which formally separates the two equations, we want to emphasise that, essentially both are the fruit of the same stabilisation strategy which we shall now comment on more detail.

3. Comments on the stabilisation strategy

Let us define some notations for the rest of this paper. The Hilbert space of finite energy current on Γ will be denotes by H (i.e., $\mathbf{u} \in H$ if $\mathbf{u} \cdot \mathbf{n} = 0$ and $\int_\Gamma \|\mathbf{u}(x)\|^2 \, dx < +\infty$). The approximation space, X_h, which we shall use is the H_{div} conforming finite element space (Raviart and Thomas [14]), which can be defined on a triangulation of Γ with a mesh-size indicated by the parameter $h > 0$. The degree of freedom associated to the ith edge of the triangulation is denoted by $\mathbf{e}_i$.

The strategy which we want to develop, and which we wish to apply in the Sections 4 and 5, consists of constructing intrinsically well-conditioned equations by modifying classical equations (for example Mitzner's CFIE and the CSIE of Mautz and Harrington), such that the operator of the new integral equation appears as a compact perturbation of positive operators in the space H of finite energy currents on Γ. This needs some comments.

Suppose then that we want to solve the stabilised equation in H,

$$L\mathbf{x} = \mathbf{b} \tag{10}$$

We denote by $[L]_h$ the Galerkin matrix of L in the basis $(\mathbf{e}_i)$ of the approximation space X_h. If we write M_h for the mass-matrix pertaining to that basis, the fact that L is a compact perturbation of a positive operator, the spectral

condition number κ_h[1] of $M_h^{-1}[L]_h$ converges to the one of L. We note that, hence, the condition number κ_h becomes independent of the mesh-size and favourable to an iterative solution. The preferred discretization scheme for an iterative solution therefore seems to be

$$M_h^{-1}[L]_h \mathrm{X}_h = M_h^{-1} \mathrm{B}_h \tag{11}$$

where B_h is the vector $\mathbb{C}^{N(h)}$ listing the coupling between $\mathbf{b}$ and each of the basis elements $(\mathbf{e}_i)$.

One of the major problems is that, in practice, L involves a product of non-local operators of first order, which excludes a direct computation of the Galerkin matrix $[L]_h$. We, therefore, have to replace this matrix by a linear operator L_h, giving the best possible approximation to $[L]_h$. We propose the new scheme

$$M_h^{-1} L_h \mathrm{X}_h = M_h^{-1} \mathrm{B}_h \tag{12}$$

We shall see that the risk of inaccuracies in the results, due to the fact that this new scheme is not derived from a Galerkin type variational approach as in (11), is easily circumvented in practice. However, it is highly probable that, in contradistinction to $M_h^{-1}[L]_h$, the condition number of $M_h^{-1} L_h$ no longer converges to the one of L. Nevertheless, in the case where the iteration algorithm allows for an functional counterpart, i.e., can be given in a Hilbert space without limits on the dimension,[2] it is possible to give a certain consistency to the discretization scheme (12).

For example, the solution of Eq. (10), conjugate gradient algorithm (CG) constructs a sequence $(\mathbf{x}^i)$ of approximations of $\mathbf{x}$ by the recurrence relation $\mathbf{x}^i = \mathbf{x}^{i-1} + \alpha^i \mathbf{p}^i$. The sequences (α^i) and $(\mathbf{p}^i)$ are also derived from a recurrence relation involving the sequence of residues $(\mathbf{r}^i)$ according to

$$\mathbf{p}^i = f\big(\mathbf{r}^{i-1}, \mathbf{r}^{i-2}, \mathbf{p}^{i-1}\big), \qquad \alpha^i = g\big(\mathbf{r}^{i-1}, \mathbf{p}^i, L\mathbf{p}^i\big), \qquad \mathbf{r}^i = h\big(\mathbf{r}^{i-1}, \alpha^i, L\mathbf{p}^i\big)$$

where the functions f, g, and h are continuous and independent of L, $\mathbf{b}$ and the iteration index i. The functions f and g are also in the approximation space X_h if $\mathbf{r}^{i-1}$, $\mathbf{r}^{i-2}$ and $\mathbf{p}^{i-1}$ are.

We next introduce a Hilbert space X subspace of H, which in turn contains X_h. The L^2 projection of X on X_h will be denoted by Q^h (i.e., $\langle \mathbf{u}, \mathbf{u}_h \rangle = \langle Q^h \mathbf{u}, \mathbf{u}_h \rangle$, $\forall \mathbf{u} \in X$, $\forall \mathbf{u}_h \in X_h$). If we equip X_h with the norm of X, $Q^h : X \to X_h \subset X$ is still continuous on X. Recall that when the operator norms $(|Q^h|_X)_{h>0}$ are bounded, one calls Q^h stable on X. We shall denote by i_h, the mapping which associates to each numerical vector $\mathrm{U}_h \in C^{N(h)}$ the current $\mathbf{u}_h = \sum \mathrm{U}_h^j \mathbf{e}_j \in X_h$. With this isomorphism, we can translate in a natural way Eq. (12) into an equation on X_h more amenable for mathematical analysis

$$\mathcal{L}_h \mathbf{x}_h = i_h M_h^{-1} \mathrm{B}_h \quad \text{with } \mathcal{L}_h = i_h M_h^{-1} L_h i_h^{-1} \tag{13}$$

The numerical analysis of the stabilisation technique which we propose relies on two fundamental properties:

(P1) The projection Q^h, in L^2 sense, of X on X_h is stable;
(P2) If a sequence $(\mathbf{u}_h)$ of elements of X_h converges to $\mathbf{u}$ in X, then $(\mathcal{L}_h \mathbf{u}_h)$ converges to $L\mathbf{u}$ in X.

In practice, (P2) often arises as a consequence of (P1), which itself can be seen as a consequence of the regularity of the mesh underlying X_h (cf., for instance, [15,16] or [17]).

We shall now study the Conjugate Gradient method applied to Eq. (13), which generates, at the ith iteration, the quantities $\mathbf{x}_h^i$, α_h^i, $\mathbf{p}_h^i$ and $\mathbf{r}_h^i$. If the right hand side of the functional equation (10) is in X, then, due to (P1), $i_h M_h^{-1} \mathrm{B}_h$ tends to $\mathbf{b}$ in X. Through recurrence on i and by using (P2), we can show that $\mathbf{x}_h^i$, α_h^i, $\mathbf{p}_h^i$ and $\mathbf{r}_h^i$ converge to $\mathbf{x}^i$, α^i, $\mathbf{p}^i$ and $\mathbf{r}^i$, respectively, being the quantities produced by a CG method to the functional equation (10). In other words, the CG method applied to the functional equation (10) or to its discretized version (13) is asymptotically (in h) equivalent.

From the previous discussion, it appears that the iterative algorithm which seems the most appropriate for the solution of the system (12) results by translation, through i_h, of the CG method applied to Eq. (13). We get,

$$\begin{aligned} \mathrm{P}_h^i &= i_h^{-1} f\big(i_h \mathrm{R}_h^{i-1}, i_h \mathrm{R}_h^{i-2}, i_h \mathrm{P}_h^{i-1}\big), \qquad \alpha^i = g\big(i_h \mathrm{R}_h^{i-1}, i_h \mathrm{P}_h^i, i_h M_h^{-1} L_h \mathrm{P}_h^i\big) \\ \mathrm{R}_h^i &= i_h^{-1} h\big(i_h \mathrm{R}_h^{i-1}, \alpha^i, i_h M_h^{-1} L_h \mathrm{P}_h^i\big) \end{aligned} \tag{14}$$

[1] Ratio of the absolute values of the largest and smallest eigenvalues.
[2] As is the case with the most performing, non-stationary, methods of the Krylov type.

which is an unconditionally (on h) stable algorithm, because the convergence is identical (by giving $\mathbb{C}^{N(h)}$ the induced norm through i_h) to the CG applied to (13), itself identical (asymptotically in h) to the convergence for the functional equation (10).

In fact, the new algorithm (14) differs little from the CG, as almost all the operation contained in f, g and h commute with i_h (i.e., $i_h^{-1}\,\mathrm{Op}(i_h \mathrm{U}_h) = \mathrm{Op}(\mathrm{U}_h)$) like for example the operator evaluation on a vector. Only the scalar product has to be rewritten. In our applications, we have not taken this modification into account. We think that, nevertheless, it may be important to do so. This issue is currently being investigated.

4. Stabilisation of the CFIE equation by means of a parametrix

4.1. The PCFIE equation

The technique we are going to apply here, is undoubtedly the most directly leading to realising the objectives announced in the previous section. For a classical equation, $A\mathbf{x} = \mathbf{b}$, we propose a preconditioned equation $BA\mathbf{x} = B\mathbf{b}$ where B is a pre-conditioner of A in a weak sense.[3]

The are two favourable situation for finding such parametrices rapidly. For example, if the equation is elliptic[4] with principal symbol $\sigma(\xi)$, it suffices to solve the 'quantisation problem' consisting of finding an explicit definition of an operator which has the principal symbol $\sigma(\xi)^{-1}$. This is particularly well-adapted to acoustics, for example, where integral equation formulations are mostly elliptic. On the contrary, numerous integral equation formulations of electromagnetics are not elliptic in this sense, like, for example the EFIE. However, the latter equation has a particularity which makes it nevertheless a suitable candidate for this type of methods. A well-known consequence of the Calderón projection relations makes that the operator of this equation is its own parametrix. This result may be surprising, as the operator is of order 1, but is in a way related to its non-ellipticity. One can find examples of apparently well performing pre-conditioners for the EFIE in [19] and [20].

In this paper, we are interested in the CFIE of Mitzner (9), a very popular equation solved by many industrial numerical electromagnetics codes. An essential reason for its success is that the CFIE is an equation without internal resonance problems and, in principle, well-adapted to high-frequency computations (cf. the PhD theses on the FMM [21,22] and [23]). On the other hand, experience with very large systems (with more than a million degrees of freedom like an aircraft radar cross-section computation as several GHz) shows that convergence can be very slow even when applying pre-conditioners from numerical linear algebra.

This motivates our research on the stabilisation of this equation by means of a pseudo-differential technique indicated above. For the case of the CFIE, the Calderón projection relations do not directly lead to a parametrix. So we have to pursue a more profound analysis, which has already been the subject of a previous publication [24], from which we use the results here without reproducing the proofs. This analysis clearly shows the role played by the Helmholtz decomposition which expresses any current distributions on Γ into a solenoidal one and an irrotational one. One of the projectors underlying this decomposition is the L^2 projector on the solenoidal currents, denoted by Π_{loop}. We can show that there exists an operator B, explicitly defined and numerically constructible, of which the composition with the operator A of the CFIE, gives a compact perturbation of the identity.

$$B = \frac{\mathrm{i}4k}{\alpha} G_0 + 2\Pi_{\mathrm{loop}}$$

where G_0 is the tangential trace of the vector potential (4) for $k = 0$. This is another case where the non-ellipticity of the integral operators of electromagnetics comes into play: B is an order 0 operator which nevertheless regularises the order 1 operator A. We conjecture that the Pre-conditioned Combined Field Integral Equation (PCFIE)

$$BA\mathbf{J} = B\left(-\alpha \mathbf{E}^{\mathrm{inc}} + \mathbf{n} \times \mathbf{H}^{\mathrm{inc}}\right) \quad \text{(PCFIE)} \tag{15}$$

is intrinsically well-conditioned. An essential point is that we can also show that the operator B has no internal resonances, which makes the PCFIE (15) a well-conditioned integral equation for all frequencies.

[3] That means one only requires that the order of $BA - \mathrm{Id}$ be strictly negative.

[4] Adopting the interpretation this term has in the theory of pseudo-differential operators [18].

Note, finally, that this work has allowed us to establish a new result in the form of a discrete Inf–Sup inequality for A uniform in h. In this way it has been shown that the numerical solution of the CFIE converges to the exact solution, in the energy norm for the currents and in the $H^{-1/2}$ norm for the charges [24]. To the best of the author's knowledge, this is the only numerical convergence result for the CFIE thusfar.

4.2. Discretization scheme

With the notations and the ideas of Section 3, the discretization of the PCFIE reads,

$$M_h^{-1}[BA]_h \mathrm{J}_h = M_h^{-1}\left[B\mathbf{J}^{\mathrm{inc}}\right]_h \quad \text{with } \mathbf{J}^{\mathrm{inc}} = -\alpha\mathbf{E}^{\mathrm{inc}} + \mathbf{n}\times\mathbf{H}^{\mathrm{inc}}$$

However, the numerical computation of the Galerkin matrix $[BA]_h$ cannot be done in the classical way because the operator kernel of BA is not explicitly known. Also we do not have $[BA]_h = [B]_h[A]_h$ because the discretization is not done on an orthogonal basis. However, considering that $Q^h A$ is a good approximation of A on X_h, we propose to approximate $[BA]_h$ by $[BQ^h A]_h$, which is equal to $[B]_h M_h^{-1}[A]_h$. But $[B]_h$ cannot be computed directly either, because X_h does not have an exact Helmholtz decomposition. We therefore decide to approximate the behaviour of B on X_h by $\widetilde{B} = \mathrm{i}4kG_0/\alpha + 2\Pi_{\mathrm{loop}}^h$ where Π_{loop}^h is the L^2 projector on the solenoidal currents of X_h. The discrete version of the PCFIE (15) then becomes

$$P_h[A]_h \mathrm{J}_h = P_h\left[\mathbf{J}^{\mathrm{inc}}\right]_h \quad \text{with } P_h = M_h^{-1}\left[\widetilde{B}\right]_h M_h^{-1} \tag{16}$$

It is remarkable that the constraint to approximate the matrix $[BA]_h$ does not hinder the precision of the numerical solution. Indeed, the choice of the discretization which we made can be interpreted as the action of a pre-conditioner (the operator P_h) dedicated to the CFIE, which we apply to the variational Galerkin scheme which one conventionally uses for discretizing this equation. By the way, this is the point of view of our previous paper [24]. In this way, the solutions of the CFIE and the PCFIE are the same.

From the spectral point of view, things are less evident. One can show, indeed, that for sufficiently small h, the discrete equation (16) is not singular. But it appears difficult to show, in contrast to what happens with $\kappa(M_h^{-1}[BA]_h)$, that $\kappa(P_h[A]_h)$ converges to $\kappa(BA)$. Nevertheless, it has been shown in [24] that there exists a subspace X of H, on which, under the condition that property (P1) holds (i.e., a sufficiently fine mesh), we have property (P2) of Section 3. As a consequence, we can predict that, asymptotically in h, the convergence of a Krylov type iterative algorithm will be the same for the functional equation (15) and for the discretized version (16).

In an iterative solution, the most expensive operation is the matrix vector product. In the case of the PCFIE, the extra cost with respect to the CFIE is in the product of a vector with an operator which is the sum of a convolution operator (with $1/r$ kernel) and a projector (projecting on the solenoidal currents in X_h). The numerical cost of such an operation is a priori competitive, because the convolution part can be reduced using a fast multi-pole method and the projection can be done in $\mathcal{O}(N(h))$ operations. Of course, the inversion of the 'mass matrix' M_h, which appears in (16) is also done iteratively.

4.3. First numerical results

The numerical results which we present here are rather preliminary and only serve to illustrate the ideas. The electrostatic potential G_0 having not yet been discretized with an FMM technique, we could only deal with quite small problems. We have tested the PCFIE both on a sphere, with unit radius carrying 3072 degrees of freedom (DoF), and on a spherical cavity, as described in [19] carrying 2880 DoF (cf. Fig. 3). These objects have been illuminated by plane waves at frequencies of 75 and 150 MHz. The iterative solution algorithm we applied is the GCR [25].

We compare the performances of the PCFIE and the CFIE, both with a coupling coefficient of $\alpha = 1$. The curves on the Figs. 1 and 2 give the values of the residues (in l^2-norm) for each of the equations. We can observe the desired effects of the pre-conditioning. For all cases, the convergence of the PCFIE is the fastest. Above all, though, the convergence remains linear even when the residue attains very small values. Such a property is not innocuous (cf. [19] for a counter example). It confirms, that in spite of the approximations we had to introduce into the approximation of the Galerkin matrix, the regularisation procedure is well preserved. In fact, we can bet that the spectral contents of the residues (or the descent vectors) is more and more rich as the convergence progresses. We can easily imagine that the corrections are of spatially high frequency nature. At the same time, the quality of the product of such excitations and

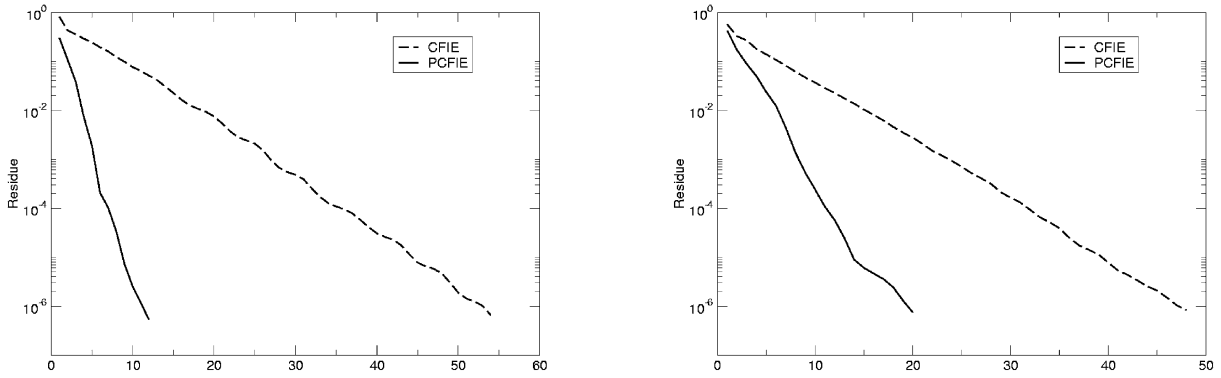

Fig. 1. Convergence curves for the PCFIE on a sphere at 75 MHz (left) and 150 MHz (right).

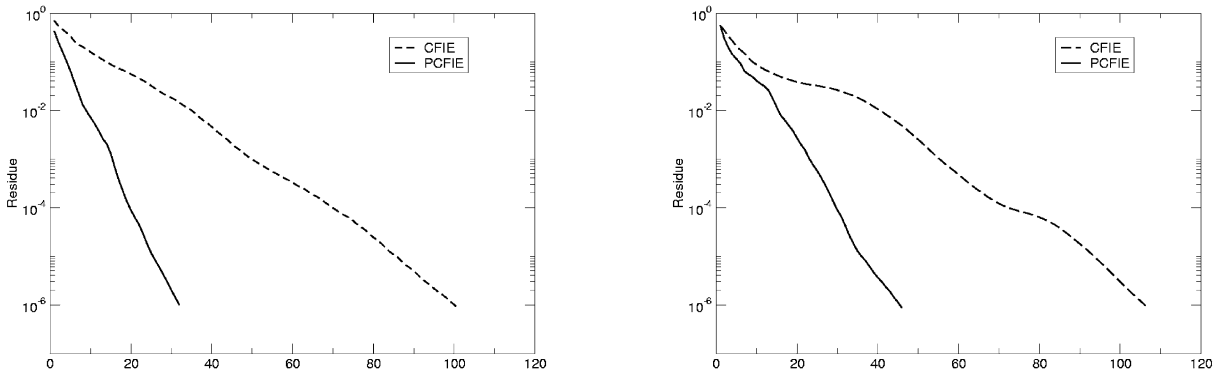

Fig. 2. Convergence curves for the PCFIE on a spherical cavity at 75 MHz (left) and 150 MHz (right).

the operator underlying the linear system depends essentially on the accurateness with which the principal symbol of the operator has been accounted for in the Galerkin matrix. Looking at the curves we obtain, this seems to be correct.

5. Generalisation of a CSIE equation using an admittance operator

5.1. The GCSIE formulation

The integral equation formalism presented in this section, is based on previous ideas (see [26]) initially given in the context of acoustics. This formulation, by the way, is not restricted to wave propagation alone. One finds in [27], for example, an axiomatic approach showing that the method can be translated to any physical situation where the modelling consists of boundary value problems for elliptic partial differential equations for which a numerically constructible Green kernel exists. In addition, more general impedance boundary conditions than those used in the present paper (i.e., those for perfect conductors) are possible.

One particularity of our formulation is that it relies on methods which inherit from both of the classical approaches (cf. Section 2). By means of a reformulation of the boundary value problem (1), relying on a particular choice of a potential, we aim at the construction of a well defined integral equation. But this construction involves the classical field representation formulas (6), and hence defines a link to the field representation based methods. In the end, though, the unknown of the resulting equation does not have a physical interpretation and we have to classify the method of the general potential type.

At the heart of the new formalism is the observation that the Stratton–Chu formulas (6) can be rewritten to involve only one of the traces $\mathbf{n} \times \mathbf{E}$ and $\mathbf{n} \times \mathbf{H}$. The well-posedness of the exterior boundary value problem (1), which has exactly one solution, allows for the definition of a relation between the components of the electromagnetic field on Γ in terms of an operator. For example, the trace of the magnetic field for any field in W^+ can be expressed

as the image of the electric field trace under an operator which establishes a bijection between the two field traces. We shall write $\mathbf{n} \times \mathbf{H} = Y^+(\mathbf{n} \times \mathbf{E})$. Mathematicians will speak of Y^+ as the Dirichlet-to-Neumann operator (in the scalar case) or of the Steklov–Poincaré operator, whereas physicists will speak of the admittance operator. With this operator, we can write for each $\mathbf{E} \in W^+$: $\mathbf{E} = LY^+(\mathbf{n} \times \mathbf{E}) - K(\mathbf{n} \times \mathbf{E})$. As a consequence, on Γ, we have $\mathbf{n} \times \mathbf{E} = \mathbf{n} \times L(Y^+\mathbf{n} \times \mathbf{E}) - \mathbf{n} \times K(\mathbf{n} \times \mathbf{E})$. This means that if one chooses to solve (1) by means of a source integral equation, the best choice ever is to choose the potential $V\mathbf{u} = LY^+\mathbf{u} - K\mathbf{u}$, because from the previous relation one has, $\mathbf{n} \times V = \mathrm{Id}$ and the solution of the corresponding equation (3), is nothing but the right-hand side.

Of course, this reasoning is entirely formal as, except from some canonical geometries, we do not have explicit knowledge of Y^+. However, if we can construct a numerical approximation, $\widetilde{Y}^+$, to this operator we suggest to adopt the potential $V\mathbf{u} = L\widetilde{Y}^+\mathbf{u} - K\mathbf{u}$. In this way, we obtain an equation which generalises the CSIE (5)

$$\mathbf{n} \times L\widetilde{Y}^+\mathbf{u} - \mathbf{n} \times K\mathbf{u} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \quad \text{(GCSIE)} \tag{17}$$

Indeed, if $\widetilde{Y}^+ = \alpha\mathbf{n}\times$, we get Mautz–Harrington's CSIE back. Because the operator of this equation reduces to the identity when $\widetilde{Y}^+ = Y^+$, we conjecture that (17) will be the better conditioned the better $\widetilde{Y}^+$ approximates Y^+.

5.2. A well-posed equation at high frequencies

The GSIE formalism is very general indeed. It is dependent of the choice of an approximation to the exterior admittance operator of the scattering obstacle. Various ways for the construction of such approximations can be pursued, the pertinence of which depends on the application context. For example, one does not want the same type of approximations at high frequencies as one wants at low frequencies. In this paper, we shall only be concerned with high frequency applications, and the approximation we choose will be very close to the one which has previously been proposed for acoustics and analysed in detail in [28].

Let us note that in the case of acoustics, another approach, based on a micro-local approximation of the Dirichlet-to-Neumann operator has been proposed in [29,30]. The results are encouraging, but the technique has not yet been applied to electromagnetics.

It is well known that in the high-frequency regime, scattering phenomena tend to localize (e.g., asymptotic theories prove that the energy at high frequency localize near the rays given by geometrical optics). We therefore conjecture that the admittance operator itself tends to localize at high frequencies, and $-2\mathbf{n} \times L$ being the admittance of an infinite metallic plate, we propose to set[5]

$$\widetilde{Y}^+ = -2\sum_p \chi_p \mathbf{n} \times L\chi_p \tag{18}$$

where (U_p, χ_p) is a quadratic partition of unity on Γ. This means that $(U_p)_p$ is a family of open sets of Γ such that $\bigcup_p U_p = \Gamma$, and $(\chi_p)_p$ is a family of smooth functions such that the support of χ_p is included in U_p for all p and $\sum_p \chi_p^2 = 1$.

Concerning the model (18), one should always keep in mind that each patch U_p has to be chosen sufficiently small in order to be locally comparable with the tangent plane. However, this high-frequency approximation of the admittance does not pretend (nor does it need to) cover globally non-convex objects, where multiple-reflections destroy localization. It is precisely the role of Eq. (17) to take into account more complex (and non local) phenomena such as for instance diffraction, creeping waves, or multiple-reflections.

It is possible to prove that under a natural condition on (χ_p), Eq. (17) equipped with (18) is well-posed on H for sufficiently high frequencies. The strategy is classical, and consists in using Fredholm's theory. We first demonstrate that the GCSIE's operator is a compact perturbation of a coercive order 0 operator, and then that is one-to-one. A complete proof is given in [1], but a sketch of the proof is the following.

If the one-to-one property is not very difficult to show, using the classical tool of Rellich's lemma, the compact perturbation property needs an accurate symbolic analysis of the equation. Indeed, the principal symbol of the operator,

[5] This elegant technique of localization has been suggested to the author during his PhD [28] by B.L. Michielsen.

curiously enough, is not the identity as is the case in the acoustics counter part of the equation [28]. More precisely, in electromagnetism, the symbol is

$$\sigma(\xi) = \mathrm{Id} + \frac{1}{2k\|\xi\|^4}\sum_p \big(\nabla\chi_p \cdot \mathbf{n}\times\xi\big)^2 \mathbf{n}\times\xi\otimes\xi\mathbf{n}$$

where $\xi \in T^*\Gamma$ is identified to a tangent vector of the Riemannian manifold Γ. So, one can not conclude we have to face a compact perturbation of identity. Nevertheless, $\sigma(\xi)$ is also the principal symbol of the operator

$$\mathrm{Id} - \frac{1}{2k}\Pi_{\mathrm{loop}}\sum_p \nabla\chi_p\nabla\chi_p\cdot\ \Pi_{\mathrm{loop}}$$

where Π_{loop} is one of the projectors of the Helmholtz decomposition already encountered in Section 4.1. On the other hand one can establish that this operator becomes coercive on H if

$$\max_{x\in\Gamma}\left(\sum_p \|\nabla\chi_p(x)\|^2\right) < 2k^2 \tag{19}$$

Consequently, under this condition, the underlying operator of the GCSIE is a compact perturbation of an index 0 operator, which we needed to prove.

In conclusion, we can say that the GCSIE formalism (17), with the admittance model given in (18), is a well-posed equation in the high-frequency regime, because in this context criterion (19) is always verified. Moreover, the operator of the equation appears as a compact perturbation of a coercive one, which is an indication of favourable spectral properties.

5.3. Discretization scheme

Following the same discretization strategy as used for the PCFIE, applying the general principles stated in Section 3, a first discretization scheme for the GCSIE is

$$M_h^{-1}\big([\mathbf{n}\times L\widetilde{Y}^+]_h \mathrm{J}_h - [\mathbf{n}\times K]_h \mathrm{J}_h\big) = -M_h^{-1}\big[\mathbf{n}\times\mathbf{E}^{\mathrm{inc}}\big]_h \tag{20}$$

The construction of the matrix $[\mathbf{n}\times K]_h$ is not problematic, since it is almost the same as the one built for the MFIE (8). The main difficulty for the discretization of the GCSIE remains the construction of the matrix $[\mathbf{n}\times L\widetilde{Y}^+]_h$. It is not possible to envisage a direct discretization of the operator resulting from the composition of the pseudo-differential operators $\mathbf{n}\times L$ and $\widetilde{Y}^+$. Therefore, we propose to compute this matrix as a product $[\mathbf{n}\times L]_h\widetilde{Y}_h^+$, where $\widetilde{Y}_h^+$ is a linear operator to construct. This construction must be such that the endomorphism $\mathcal{L}_h = i_h M_h^{-1}[\mathbf{n}\times L]_h\widetilde{Y}_h^+ i_h^{-1}$ satisfies the consistency property (P2) of Section 3. In other words, we have to search an endomorphism $\widetilde{\mathcal{Y}}_h^+$ on X_h, being a good approximation of $\widetilde{Y}^+$ in the sense that (P2) holds for $\mathcal{L}_h = i_h M_h^{-1}[\mathbf{n}\times L]_h i_h^{-1}\widetilde{\mathcal{Y}}_h^+$. Afterwards, the linear operator we want is

$$\widetilde{Y}_h^+ = i_h^{-1}\widetilde{\mathcal{Y}}_h^+ i_h \tag{21}$$

As the range of $\widetilde{Y}^+$ is not included in the approximation space X_h, it is natural to seek for an approximation of $\widetilde{Y}^+$ as $\widetilde{\mathcal{Y}}_h^+ = P^h\widetilde{Y}^+$ where P^h is a projector on X_h. Since $\widetilde{Y}^+$ can be split into the sum of two operators $-\mathrm{i}k\mathbf{n}\times\sum_p \chi_p G\chi_p$ and $(\mathrm{i}k)^{-1}\mathbf{n}\times\sum_p \chi_p\nabla G\nabla\cdot\chi_p$ of respective orders -1 and $+1$, it is probably more advisable to realize projections adapted to each of these terms. We define the projection $\Pi^h_{\mathrm{edge}}\mathbf{u} = \sum_i f_i(\mathbf{u})\mathbf{e}_i$ where $f_i(\mathbf{u})$ denotes the flux of $\mathbf{u}$ through the edge i (we suppose that the DoF are normalized by their flux). We also denote by Π^h_{node} the projection on the scalar P^1 finite element space S_h defined by $\Pi^h_{\mathrm{node}}u = \sum_i u(x_i)e_i$ with e_i the DoF attached to the node x_i. Noting that $\mathbf{n}\times\nabla(S_h)\subset X_h$, we propose to approximate $\widetilde{Y}^+$ on X_h by

$$\widetilde{\mathcal{Y}}_h^+ = \frac{1}{ik}\sum_p \chi_p^h\mathbf{n}\times\nabla\Pi^h_{\mathrm{node}}G\nabla\cdot\chi_p^h - \mathrm{i}k\sum_p \chi_p^h\Pi^h_{\mathrm{edge}}\mathbf{n}\times G\chi_p^h \tag{22}$$

The advantage of this approximation is that it takes into account, in an exact way, the surface curl $\mathbf{n}\times\nabla$, which is essential for the regularization obtained in the composition $\mathbf{n}\times L\widetilde{Y}^+$ the order of which is 0, whereas each of the factors is of order 1.

Numerical construction of the matrix $\widetilde{Y}_h^+$ from the formulas (21) and (22) does not pose any problems. However, it is very important to note that such a matrix is sparse because, because of the localizations, only the coefficients (i, j), corresponding to DoF $(\mathbf{e}_i, \mathbf{e}_j)$ with edges located on the same patch U_p, do not vanish.

Finally, the new numerical scheme replacing the initial one (20) is

$$M_h^{-1}\big([\mathbf{n}\times L]_h \widetilde{Y}_h^+ \jmath_h - [\mathbf{n}\times K]_h \jmath_h\big) = -M_h^{-1}\big[\mathbf{n}\times \mathbf{E}^{\text{inc}}\big]_h \tag{23}$$

At a first glance, this scheme may appear expensive, since we have to perform first the product with the sparse matrix $\widetilde{Y}_h^+$, and then two independent products involving $[\mathbf{n}\times L]_h$ and $[\mathbf{n}\times K]_h$. However, a judicious discretization of these two operators with FMM techniques, enables us to group the so-called 'transfer' and 'reconstruction' phases for L and K. Therefore, the additional cost of this scheme compared with a classical equation is the cost of two sparse products, one involving $\widetilde{Y}_h^+$ and the other a near FMM matrix, which is very competitive.

However, in contradistinction to the PCFIE case, the numerical analysis of the discretization scheme of the GCSIE is at this time not achieved. It remains to find a well-adapted functional space X allowing to exhibit the consistency property (P2) of the Section 3. It would prove the scheme to be unconditionally stable, regarding h, e.g., the convergence would be independent of the mesh size. We continue to work on this problem.

Concerning the practical implementation, let us note that the truncation functions χ_p are generated as the tensor product in three dimensions of a quadratic partition of unity over the real axis. A quadratic partition of the unity on Γ is induced by the partition on $\mathbb{R}^3$ by taking the restriction to the surface.

From a programming point of view, our scheme is relatively easy to implement, at least when starting from an existing code having the FMM algorithms. Meshes are not modified and the FMM machinery does not want any special tuning. This explains why the new equation has been implemented with success in ONERA's development environment PAME [31], where it appears as a new functionality of the industrial code *Elsem3D* [32].

5.4. Numerical results

In order to give an impression of the numerical behaviour of the GCSIE in situations usually known as 'hard' for integral equations, we have selected the two scattering objects shown in Fig. 3. The hollow sphere is the same as the one tested in [19]. It is a sphere of diameter 2.33 m, with a circular hole of radius approximately 80 cm. The other object is the Channel cavity described in [33] modelling, at a small scale, an air-intake of an aircraft (its length is about 1.36 m). We also tested scattering by a sphere with has a radius equal to 1 m.

In all following results, the mesh sizes, which necessarily depend on the frequency, comply with the criterion of some 7 edges per wavelength.

The iterative solver used implements a GCR-method [25] and, without explicit statement of the contrary, the stopping criterion on the relative residue is fixed to 10^{-4} in l^2 norm.

5.4.1. Phenomenological aspects

The accuracy of the GCSIE equation appears to be approximately the same as of the CFIE. For instance, Fig. 4 gives bi-static radar cross section (RCS) diagrams calculated with these two equations and shows the excellent accuracy of the GCSIE for RCS applications. But this is not surprising, because GCSIE is a source kind equation whose

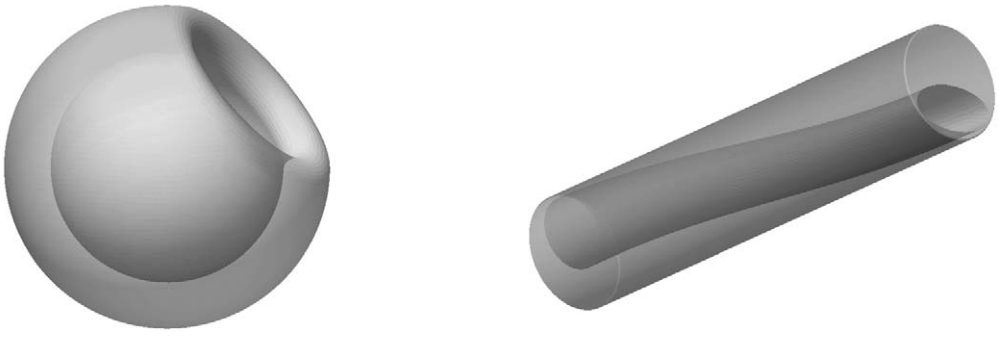

Fig. 3. Transparent view of the spherical cavity and the Channel air-intake used for tests.

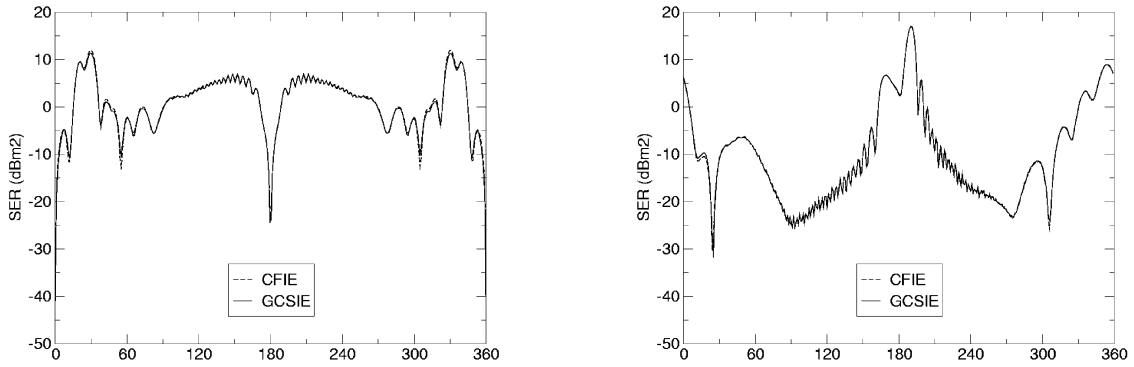

Fig. 4. Bi-static RCS of the hollow sphere at 2.8 GHz (left), and Channel cavity at 5 GHz (right).

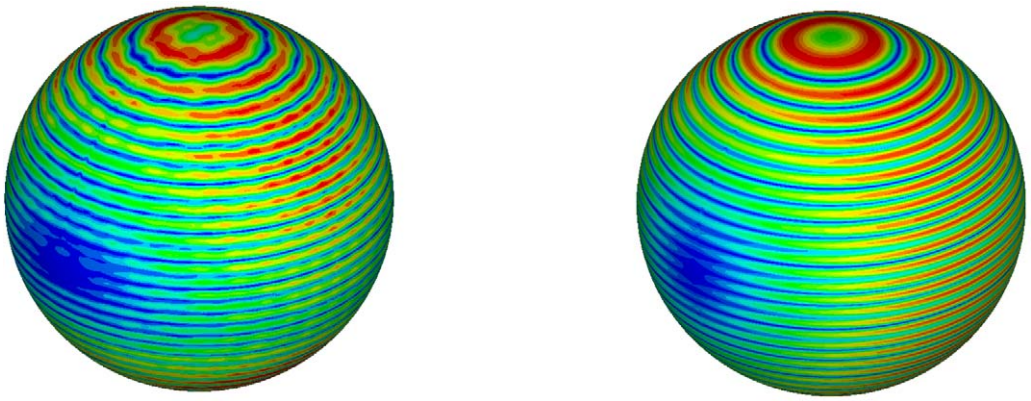

Fig. 5. Real part of the solution of the GCSIE (left) and the right hand side (right).

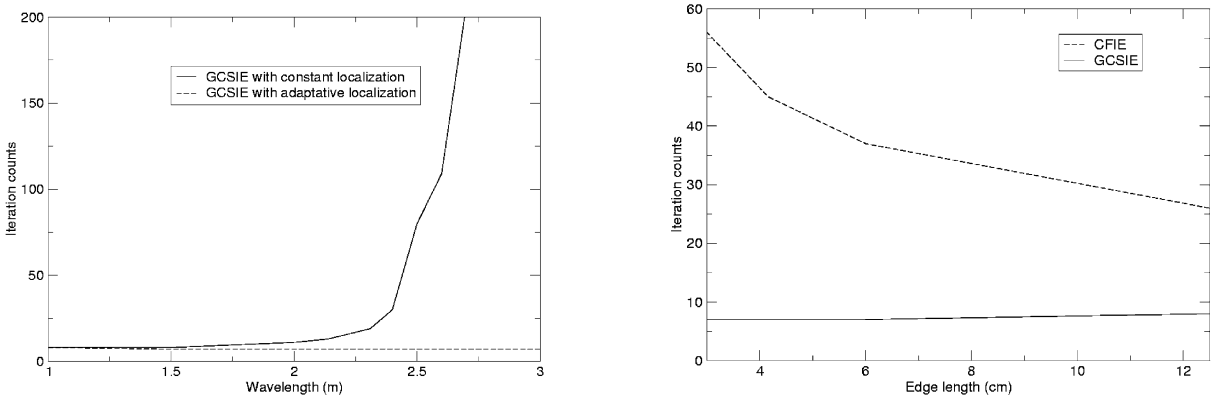

Fig. 6. Iteration counts for the sphere, as a function of the wavelength (left), and as a function of the mesh size at 0.6 GHz (right).

potentials L and K are rigorously approximated with a Galerkin method. Approximations which perturb the numerical scheme (23) only concern the construction of the $\widetilde{Y}^{+}$ operator and cannot affect the precision of the final result.

On the other hand, the accuracy of the model approximating the admittance and the quality of the discretization do indeed govern the convergence speed of the iterative method. Apart from the fact that the observed convergence speeds are very satisfying (cf. Fig. 7), the performance of the GCSIE implementation can also be judged by comparing the solution of the equation with the right-hand side (see Fig. 5). Indeed, these quantities are the same in the ideal case where $\widetilde{Y}^{+} = Y^{+}$. On a sphere, they appear to be very close, not only in the center of the illuminated region, but also

where the incident plane wave impacts tangentially to the sphere. This confirms that the model (18) is a pertinent one, providing a good approximation to the true admittance operator at high frequencies, even at grazing incidence.

In Section 5.2, we have seen that the theoretical analysis endorses the high-frequency assumption hidden behind the construction of the model (18). For a constant quadratic partition of unity, the condition (19) implies that, for a sufficiently high frequency, the equation GCSIE is well-posed, and that, on the contrary, the GCSIE may have a bad behaviour at low frequency. On a sphere (with a radius of 1 m), when the frequency decreases, we observe an explosion of the number of iterations required to attain a relative residue of 10^{-4} (Fig. 6, left). But, if we adapt the cut-off functions χ_p to the frequency in such a way that the condition (19) is verified (by taking larger supports), we can stabilize the iterative convergence.

Numerical experiments also reveal that the GCSIE is stable as to the mesh size. For the sphere at 0.6 GHz, Fig. 6 (right) shows that the number of iterations remains stable when the mesh becomes smaller and smaller.

5.4.2. Performance

Numerical experiments concerning the convergence speeds reveal a global advantage of the GCSIE over the conventional integral equations. Obviously so, over the EFIE, but also, and less trivially so, over the CFIE which, among the classical equations, is known to offer the best convergence results. In order to reach a certain residue, one spends, according to the geometry, two to ten times more iterations with the CFIE than with the GCSIE. Fig. 7 is very revealing of the convergence dynamics whether the scattering object is globally convex or whether it encloses a cavity. It is important to emphasize that, as observed in Section 4.3, the decrease of the residue does not show stagnation, even for very small values. Showing again that the regularization process working at the 'continuous' level of the equation has probably found a satisfactory translation at the discretization level. We should point out too that the gain in iteration count still holds firm facing a CFIE preconditioned with an approximated inverse. Such algebraic pre-conditioners are

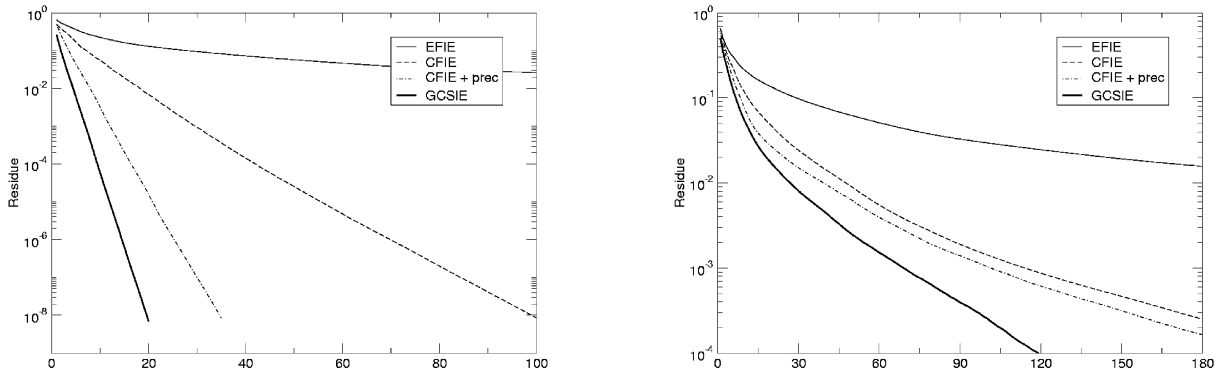

Fig. 7. Convergence curves for the sphere at 2 GHz meshed with 101568 DoF (left) and for the hollow sphere at 1.8 GHz meshed with 127509 DoF (right).

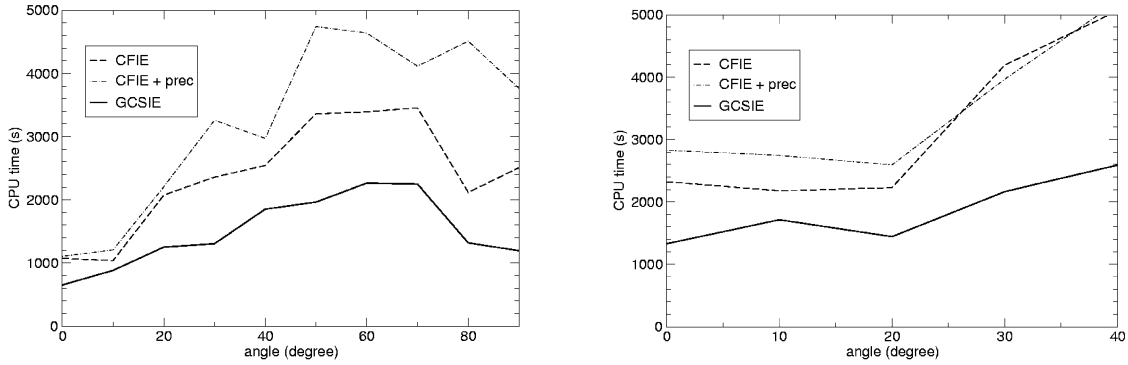

Fig. 8. Resolution time for the hollow sphere at 2.8 GHz with 264186 DoF (left) and the Channel cavity at 7 GHz with 309711 DoF (right).

being used at ONERA in a production context. It consists of building, iteratively, the inverse of a sparse version of the initial linear system to solve, in preserving only the biggest coefficients (around 60 per line) and cancelling the others. However, if this pre-conditioner works on convex objects, it unfortunately fails to apply to cavities. Contrary to this pre-conditioner, efficient only for convex obstacles, but whose behavior is uncertain for cavities, the GCSIE seems to be efficient in any configuration.

With regard to resolution time, we notice almost the same benefits as those observed for the convergence speed. Results presented in Fig. 8 correspond to computations run on 8 nodes of a parallel machine, where each node is equipped with 4 processors (1.25 GHz) and have a working memory of 4 Gb. The residue criterion was fixed to 10^{-4} for both the hollow sphere and the Channel cavity. The resolution times depend on the incidence angle of the plane wave excitation. Nevertheless, we can conclude that the resolution time for the GCSIE is about twice smaller than the one obtained with the most efficient classical equation.

6. Conclusion

We hope that we have shown that returning to the more or less forgotten integral equation methods of the past, which consisted of starting from general potentials instead of field representations, associated with an appropriate stabilisation strategy (as explained in Section 3), opens new perspectives in the field of pre-conditioning of integral equations. Indeed, it appears to us, that the two proposed equations illustrating this issue, make questionable two well-established ideas and opinions:

– Firstly, that integral equations based on the Cauchy data are always the most appropriate ones;
– Secondly, that only Galerkin methods are attractive as to the warrants they imply on the theoretical level.

We have seen how fruitful it can be to counter the first one and to exploit the richer set of general potentials in order to obtain intrinsically well-conditioned integral equations.

The second one is probably supported by the fact that the very popular EFIE and CFIE are first order and hence very difficult to discretize on low order finite elements other than by ‘energy’ methods. For example, the field corresponding to a 0-th order Raviart–Thomas element is singular on the edges and this complicates point collocation discretizations. However, the equations we have presented here, while being well-defined at all frequencies, are of order 0. Their iterative solution needs only consistency between the evaluation of the operator on the current distributions and the matrix vector product of the discretized version. Certainly, the discretization uses the Galerkin matrices, but these are only intermediate, as in the end, the ‘mass matrix’ serves to make the discretized system consistent with the underlying functional equation. Supposing a higher order discretization, we can therefore conjecture that the same level of stability will be obtained by simply using point collocation or Nyström like methods.

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