Observation of black holes and extreme gravitational events/Observation des trous noirs et des événements gravitationnels extrêmes

General relativistic dynamics of compact binary systems

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Abstract

The equations of motion of compact binary systems have been derived in the post-Newtonian (PN) approximation of general relativity. The current level of accuracy is 3.5PN order. The conservative part of the equations of motion (neglecting the radiation reaction damping terms) is deducible from a generalized Lagrangian in harmonic coordinates, or equivalently from an ordinary Hamiltonian in ADM coordinates. As an application, we investigate the problem of the dynamical stability of circular binary orbits against gravitational perturbations up to the 3PN order. We find that there is no innermost stable circular orbit or ISCO at the 3PN order for equal masses. ***To cite this article: L. Blanchet, C. R. Physique 8 (2007).***
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Résumé

Dynamique des systèmes binaires compacts en relativité générale. Les équations du mouvement d'un système binaire d'objets compacts ont étées calculées dans l'approximation post-newtonienne (PN) de la relativité générale. Le niveau d'approximation atteint l'ordre 3.5PN. La partie conservative des équations du mouvement (obtenue en négligeant les termes de freinage de rayonnement) se déduit d'un lagrangien généralisé en coordonnées harmoniques, et, de façon équivalente, d'un hamiltonien ordinaire en coordonnées ADM. Comme application nous étudions le problème de la stabilité dynamique, vis-à-vis des perturbations gravitationnelles, des orbites binaires circulaires à l'ordre 3PN. Nous trouvons qu'il n'y a pas de dernière orbite stable circulaire ou ISCO à l'ordre 3PN pour des masses égales. ***Pour citer cet article : L. Blanchet, C. R. Physique 8 (2007).***
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1. Introduction

The problem of the dynamics of two compact bodies is part of a program aimed at unravelling the information contained in the gravitational-wave signals emitted by inspiralling and coalescing compact binaries—systems of neutron stars or black holes driven into coalescence by emission of gravitational radiation. The treatment of the problem is post-Newtonian, i.e. based on formal expansions, when the speed of light c tends to infinity, of general relativity theory around Newton's theory. The early, classic works of Lorentz and Droste [1], Eddington and Clark [2], Einstein, Infeld and Hoffmann [3], Fock [4], Papapetrou [5] and others led to a good understanding of the equations of motion of N bodies at the first post-Newtonian approximation (1PN, corresponding to order $\sim 1/c^2$). In the 1970s, a series of

works [6–8] led to a nearly complete control of the problem of motion at the second post-Newtonian approximation (2PN $\sim 1/c^4$). Then, starting in the early 1980s, motivated by the observation of secular orbital effects in the Hulse–Taylor binary pulsar PSR1913+16 [9,10], several groups solved the two-body problem at the 2.5PN level (while completing on the way the derivation of the 2PN dynamics) [11–18]. The 2.5PN term constitutes the first contribution of gravitational *radiation reaction* in the equations of motion (such a term is the analogue of the Abraham–Lorentz reaction force in electromagnetism), and is directly responsible for the decrease of the binary pulsar orbital period by emission of gravitational radiation.

In the late 1990s, motivated by the aim of deriving high-accuracy templates for the data analysis of the international network of interferometric gravitational-wave detectors LIGO/VIRGO, two groups embarked on the derivation of the equations of motion at the third post-Newtonian (3PN) level. One group used the Arnowitt–Deser–Misner (ADM) Hamiltonian formalism of general relativity [19–22] and worked in a corresponding ADM-type coordinate system. Another group used a direct post-Newtonian iteration of the equations of motion in harmonic coordinates [23–28]. The end results of these two approaches have been proved to be physically equivalent [22,27]. However, both approaches, even after exploiting all symmetries and pushing their Hadamard-regularization-based methods to the maximum, left undetermined one dimensionless parameter at the 3PN order. The appearance of this unknown parameter was related with the choice of the regularization method used to cure the self-field divergencies of point particles. Both lines of works regularized the self-field divergencies by some version of the Hadamard regularization method. Finally, the completion of the equations of motion at the 3PN order was made possible thanks to the powerful *dimensional* self-field regularization, which could fix up uniquely the value of the ambiguity parameter [29,30]. This result is also in complete agreement with the recent finding of [31,32], who derived the 3PN equations of motion in harmonic gauge using a 'surface-integral' approach without the use of self-field regularization. Finally, the 3.5PN terms, which constitute a 1PN relative modification of the radiation reaction force (and are relatively easier to derive), have been added in Refs. [33–38].

In Section 2 of the present article we give the final result for the equations of motion of compact binaries at the 3.5PN order. The equations are presented in ready to use quasi-Newtonian form, in the reference frame associated with the center-of-mass position. In Section 3 we discuss the Lagrangian and Hamiltonian formulations of the conservative part of the equations, obtained by neglecting the radiation reaction terms occurring at the 2.5PN and 3.5PN orders. Finally, in Section 4, we investigate, following Ref. [28], the question of the stability of circular orbits, against linear gravitational perturbations, up to the 3PN order.

2. Equations of motion in the center-of-mass frame

In the present article we employ the so-called harmonic-coordinates approach [23,24,27,28] which derived the 3PN binary's equations of motion in harmonic coordinates. From these equations, obtained at first in a general frame, we translate the origin of the coordinates to the binary's center-of-mass by imposing that the binary's center-of-mass vector position is $G_i = 0$. The center-of-mass vector is nothing but the conserved integral of the motion that is associated, via the Noether theorem, with the *boost symmetry* of the Lagrangian from which the 3PN equations of motion are derived [27,28]. The condition $G_i = 0$ results in the 3PN-accurate relationship between the individual positions of the particles y_1^i and y_2^i in the center-of-mass frame, and the relative position $x^i = y_1^i - y_2^i$ and velocity $v^i = v_1^i - v_2^i = \mathrm{d}x^i/\mathrm{d}t$. The center-of-mass equations of motion are then obtained by replacing the individual positions and velocities by their center-of-mass expressions, given in terms of x^i and velocity v^i, applying as usual the order-reduction of all accelerations where necessary. Order reduction means that any acceleration (or derivative of acceleration) in a sub-dominant post-Newtonian term is to be replaced by its explicit expression given as a function of the positions and velocities as deduced from the lower-order equations of motion themselves.

We shall denote the orbital separation by $r = |\mathbf{x}|$, and pose $\mathbf{n} = \mathbf{x}/r$ and $\dot{r} = \mathbf{n} \cdot \mathbf{v}$. The mass parameters are the total mass $m = m_1 + m_2$, the mass difference $\delta m = m_1 - m_2$, the reduced mass $\mu = m_1 m_2/m$, and the very useful symmetric mass ratio $\nu = \mu/m$, which is such that $0 < \nu \leqslant 1/4$, with $\nu = 1/4$ in the case of equal masses, and $\nu \to 0$ in the 'test-mass' limit for one of the bodies. We write the relative acceleration in the center-of-mass frame in the form (we pose $G = 1$)

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{m}{r^2}\big[(1 + \mathcal{A})n^i + \mathcal{B}v^i\big] + \mathcal{O}\left(\frac{1}{c^8}\right) \tag{1}$$

where the first term represents the famous Newtonian approximation, and where the post-Newtonian remainder term $\mathcal{O}(c^{-8})$ indicates the level of accuracy of the expression, which is here 3.5PN order. We find [28,38] that the coefficients $\mathcal{A}$ and $\mathcal{B}$ are

$$
\begin{aligned}
\mathcal{A} =\ & \frac{1}{c^2}\left\{-\frac{3\dot{r}^2\nu}{2}+v^2+3\nu v^2-\frac{m}{r}(4+2\nu)\right\} \\
& +\frac{1}{c^4}\left\{\frac{15\dot{r}^4\nu}{8}-\frac{45\dot{r}^4\nu^2}{8}-\frac{9\dot{r}^2\nu v^2}{2}+6\dot{r}^2\nu^2 v^2+3\nu v^4-4\nu^2 v^4\right. \\
& \left.\qquad +\frac{m}{r}\left(-2\dot{r}^2-25\dot{r}^2\nu-2\dot{r}^2\nu^2-\frac{13\nu v^2}{2}+2\nu^2 v^2\right)+\frac{m^2}{r^2}\left(9+\frac{87\nu}{4}\right)\right\} \\
& +\frac{1}{c^5}\left\{-\frac{24\dot{r}\nu v^2}{5}\frac{m}{r}-\frac{136\dot{r}\nu}{15}\frac{m^2}{r^2}\right\} \\
& +\frac{1}{c^6}\left\{-\frac{35\dot{r}^6\nu}{16}+\frac{175\dot{r}^6\nu^2}{16}-\frac{175\dot{r}^6\nu^3}{16}+\frac{15\dot{r}^4\nu v^2}{2}-\frac{135\dot{r}^4\nu^2 v^2}{4}+\frac{255\dot{r}^4\nu^3 v^2}{8}\right. \\
& \qquad -\frac{15\dot{r}^2\nu v^4}{2}+\frac{237\dot{r}^2\nu^2 v^4}{8}-\frac{45\dot{r}^2\nu^3 v^4}{2}+\frac{11\nu v^6}{4}-\frac{49\nu^2 v^6}{4}+13\nu^3 v^6 \\
& \qquad +\frac{m}{r}\left(79\dot{r}^4\nu-\frac{69\dot{r}^4\nu^2}{2}-30\dot{r}^4\nu^3-121\dot{r}^2\nu v^2+16\dot{r}^2\nu^2 v^2\right. \\
& \qquad\qquad \left.+20\dot{r}^2\nu^3 v^2+\frac{75\nu v^4}{4}+8\nu^2 v^4-10\nu^3 v^4\right) \\
& \qquad +\frac{m^2}{r^2}\left(\dot{r}^2+\frac{32573\dot{r}^2\nu}{168}+\frac{11\dot{r}^2\nu^2}{8}-7\dot{r}^2\nu^3+\frac{615\dot{r}^2\nu\pi^2}{64}-\frac{26987\nu v^2}{840}\right. \\
& \qquad\qquad \left.+\nu^3 v^2-\frac{123\nu\pi^2 v^2}{64}-110\dot{r}^2\nu\ln\left(\frac{r}{r_0'}\right)+22\nu v^2\ln\left(\frac{r}{r_0'}\right)\right) \\
& \qquad \left.+\frac{m^3}{r^3}\left(-16-\frac{437\nu}{4}-\frac{71\nu^2}{2}+\frac{41\nu\pi^2}{16}\right)\right\} \\
& +\frac{1}{c^7}\left\{\frac{m}{r}\left(\frac{366}{35}\nu v^4+12\nu^2 v^4-114\nu^2 v\dot{r}^2-12\nu^2 v^2\dot{r}^2+112\nu\dot{r}^4\right)\right. \\
& \qquad \left.+\frac{m^2}{r^2}\left(\frac{692}{35}\nu v^2-\frac{724}{15}\nu^2 v^2+\frac{294}{5}\nu\dot{r}^2+\frac{376}{5}\nu^2\dot{r}^2\right)+\frac{m^3}{r^3}\left(\frac{3956}{35}\nu+\frac{184}{5}\nu^2\right)\right\} \\
\mathcal{B} =\ & \frac{1}{c^2}\{-4\dot{r}+2\dot{r}\nu\}+\frac{1}{c^4}\left\{\frac{9\dot{r}^3\nu}{2}+3\dot{r}^3\nu^2-\frac{15\dot{r}\nu v^2}{2}-2\dot{r}\nu^2 v^2+\frac{m}{r}\left(2\dot{r}+\frac{41\dot{r}\nu}{2}+4\dot{r}\nu^2\right)\right\} \\
& +\frac{1}{c^5}\left\{\frac{8\nu v^2}{5}\frac{m}{r}+\frac{24\nu}{5}\frac{m^2}{r^2}\right\} \\
& +\frac{1}{c^6}\left\{-\frac{45\dot{r}^5\nu}{8}+15\dot{r}^5\nu^2+\frac{15\dot{r}^5\nu^3}{4}+12\dot{r}^3\nu v^2-\frac{111\dot{r}^3\nu^2 v^2}{4}-12\dot{r}^3\nu^3 v^2-\frac{65\dot{r}\nu v^4}{8}\right. \\
& \qquad +19\dot{r}\nu^2 v^4+6\dot{r}\nu^3 v^4+\frac{m}{r}\left(\frac{329\dot{r}^3\nu}{6}+\frac{59\dot{r}^3\nu^2}{2}+18\dot{r}^3\nu^3-15\dot{r}\nu v^2-27\dot{r}\nu^2 v^2-10\dot{r}\nu^3 v^2\right) \\
& \qquad \left.+\frac{m^2}{r^2}\left(-4\dot{r}-\frac{18169\dot{r}\nu}{840}+25\dot{r}\nu^2+8\dot{r}\nu^3-\frac{123\dot{r}\nu\pi^2}{32}+44\dot{r}\nu\ln\left(\frac{r}{r_0'}\right)\right)\right\} \\
& +\frac{1}{c^7}\left\{\frac{m}{r}\left(-\frac{626}{35}\nu v^4-\frac{12}{5}\nu^2 v^4+\frac{678}{5}\nu v^2\dot{r}^2+\frac{12}{5}\nu^2 v^2\dot{r}^2-120\nu\dot{r}^4\right)\right. \\
& \qquad \left.+\frac{m^2}{r^2}\left(\frac{164}{21}\nu v^2+\frac{148}{5}\nu^2 v^2-\frac{82}{3}\nu\dot{r}^2-\frac{848}{15}\nu^2\dot{r}^2\right)+\frac{m^3}{r^3}\left(-\frac{1060}{21}\nu-\frac{104}{5}\nu^2\right)\right\}
\end{aligned}
\tag{2}
$$

The 3.5PN equations of motion play a crucial role when deriving the high-accuracy templates which will be used for analysing (hopefully, in the near future) the gravitational wave signals from compact binary inspiral in the data analysis of the LIGO and VIRGO detectors.

At the 3PN order we find some logarithmic terms, depending on some arbitrary constant r_0'. The presence of these logarithms reflects in fact the use of a specific harmonic coordinate system. It is indeed known that the logarithms at the 3PN order in Eqs. (2), together with the constant r_0' therein, can be removed by applying a gauge transformation. This shows that there is no physics associated with them, and that these logarithms and the constant r_0' will never appear in any physical result derived from these equations, because the physical results must be gauge invariant (r_0' is sometimes referred to as a 'gauge constant'). The gauge transformation at 3PN order whose effect is to remove the logarithms is given in [24]. Notice that after applying this gauge transformation we are still within the class of harmonic coordinates. The resulting modification of the equations of motion affects only the coefficients of the 3PN order in Eqs. (2); let us denote them by $\mathcal{A}_{3PN}$ and $\mathcal{B}_{3PN}$. The new values of these coefficients, say $\mathcal{A}'_{3PN}$ and $\mathcal{B}'_{3PN}$, obtained after removal of the logarithms by the latter harmonic gauge transformation, are then [39]

$$
\begin{aligned}
\mathcal{A}'_{3PN} = \frac{1}{c^6}\Bigg\{ &-\frac{35\dot{r}^6 v}{16} + \frac{175\dot{r}^6 v^2}{16} - \frac{175\dot{r}^6 v^3}{16} + \frac{15\dot{r}^4 v v^2}{2} - \frac{135\dot{r}^4 v^2 v^2}{4} + \frac{255\dot{r}^4 v^3 v^2}{8} \\
&-\frac{15\dot{r}^2 v v^4}{2} + \frac{237\dot{r}^2 v^2 v^4}{8} - \frac{45\dot{r}^2 v^3 v^4}{2} + \frac{11 v v^6}{4} - \frac{49 v^2 v^6}{4} + 13 v^3 v^6 \\
&+\frac{m}{r}\bigg(79\dot{r}^4 v - \frac{69\dot{r}^4 v^2}{2} - 30\dot{r}^4 v^3 - 121\dot{r}^2 v v^2 + 16\dot{r}^2 v^2 v^2 \\
&\qquad\quad + 20\dot{r}^2 v^3 v^2 + \frac{75 v v^4}{4} + 8 v^2 v^4 - 10 v^3 v^4 \bigg) \\
&+\frac{m^2}{r^2}\bigg(\dot{r}^2 + \frac{22717\dot{r}^2 v}{168} + \frac{11\dot{r}^2 v^2}{8} - 7\dot{r}^2 v^3 + \frac{615\dot{r}^2 v \pi^2}{64} - \frac{20827 v v^2}{840} + v^3 v^2 - \frac{123 v \pi^2 v^2}{64} \bigg) \\
&+\frac{m^3}{r^3}\bigg(-16 - \frac{1399 v}{12} - \frac{71 v^2}{2} + \frac{41 v \pi^2}{16} \bigg) \Bigg\} \\[2mm]
\mathcal{B}'_{3PN} = \frac{1}{c^6}\Bigg\{ &-\frac{45\dot{r}^5 v}{8} + 15\dot{r}^5 v^2 + \frac{15\dot{r}^5 v^3}{4} + 12\dot{r}^3 v v^2 - \frac{111\dot{r}^3 v^2 v^2}{4} - 12\dot{r}^3 v^3 v^2 - \frac{65\dot{r} v v^4}{8} \\
&+ 19\dot{r} v^2 v^4 + 6\dot{r} v^3 v^4 + \frac{m}{r}\bigg(\frac{329\dot{r}^3 v}{6} + \frac{59\dot{r}^3 v^2}{2} + 18\dot{r}^3 v^3 - 15\dot{r} v v^2 - 27\dot{r} v^2 v^2 - 10\dot{r} v^3 v^2 \bigg) \\
&+\frac{m^2}{r^2}\bigg(-4\dot{r} - \frac{5849\dot{r} v}{840} + 25\dot{r} v^2 + 8\dot{r} v^3 - \frac{123\dot{r} v \pi^2}{32} \bigg) \Bigg\}
\end{aligned}
\tag{3}
$$

These gauge-transformed coefficients are useful because they do not yield the usual complications associated with logarithms. However, they must be handled with care in applications such as [39], since one must ensure that all other quantities in the problem (energy, angular momentum, gravitational-wave fluxes etc.) are defined in the same specific harmonic gauge avoiding logarithms. In the following we shall no longer use the coordinate system leading to Eqs. (3). Notably the expression we shall derive below for the Lagrangian will be valid in the 'standard' harmonic coordinate system in which the equations of motion are given by (1) with (2).

3. Lagrangian and Hamiltonian formulations

The Lagrangian for the relative center-of-mass motion is obtained from the 3PN center-of-mass equations of motion (1), (2) in which one ignores the radiation-reaction terms at the 2.5PN and 3.5PN orders. We are indeed interested in the conservative part of the equations of motion, excluding the terms associated with gravitational radiation; only the conservative part is deducible from a Lagrangian. It is known that the Lagrangian in harmonic coordinates will necessarily be a generalized one (from the 2PN order), i.e. depending not only on the positions and velocities of the particles, but also on their *accelerations* [13,15]. It is also known [15,27] that one can always restrict ourselves to a Lagrangian that is linear in the accelerations.

The conservative part of the center-of-mass equations of motion (1), (2) then take the form (after systematic order-reduction of the accelerations) of the generalized Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial a^i}\right) = \mathcal{O}\left(\frac{1}{c^8}\right) \tag{4}$$

where $L[x^i, v^i, a^i]$ denotes the generalized center-of-mass Lagrangian—which is linear in the accelerations appearing at 2PN and 3PN orders. We recall that there is a large freedom for choosing a Lagrangian because we can always add to it the total time derivative of an arbitrary function. As a matter of convenience, we choose below a particular center-of-mass Lagrangian that is 'close' (in the sense that many coefficients are identical) to some 'fictitious' Lagrangian obtained from the general-frame one given in Ref. [27] by the mere Newtonian center-of-mass replacements $y_1^i \to \frac{m_2}{m}x^i$, $y_2^i \to -\frac{m_1}{m}x^i$. We point out that such a fictitious Lagrangian is *not* the correct Lagrangian for describing the center-of-mass relative motion. Indeed, the actual relations connecting the center-of-mass variables y_1^i and y_2^i to the relative position x^i and velocity v^i, involve many post-Newtonian corrections, so the actual center-of-mass Lagrangian must contain some extra terms in addition to those of the latter fictitious one. However, we find that these extra terms arise only from the 2PN order. Our result (when divided by the reduced mass $\mu = m\nu$) is then

$$
\begin{aligned}
\frac{L}{\mu} ={}& \frac{v^2}{2} + \frac{m}{r} + \frac{1}{c^2}\left\{\frac{v^4}{8} - \frac{3\nu v^4}{8} + \frac{m}{r}\left(\frac{\dot{r}^2\nu}{2} + \frac{3v^2}{2} + \frac{\nu v^2}{2}\right) - \frac{m^2}{2r^2}\right\} \\
&+ \frac{1}{c^4}\left\{\frac{v^6}{16} - \frac{7\nu v^6}{16} + \frac{13\nu^2 v^6}{16}\right. \\
&\qquad + \frac{m}{r}\left(\frac{3\dot{r}^4\nu^2}{8} - \frac{\dot{r}^2 a_n\nu r}{8} + \frac{\dot{r}^2\nu v^2}{4} - \frac{5\dot{r}^2\nu^2 v^2}{4} + \frac{7a_n\nu r v^2}{8} + \frac{7v^4}{8} - \frac{5\nu v^4}{4} - \frac{9\nu^2 v^4}{8} - \frac{7\dot{r}\nu r a_v}{4}\right) \\
&\qquad \left.+ \frac{m^2}{r^2}\left(\frac{\dot{r}^2}{2} + \frac{41\dot{r}^2\nu}{8} + \frac{3\dot{r}^2\nu^2}{2} + \frac{7v^2}{4} - \frac{27\nu v^2}{8} + \frac{\nu^2 v^2}{2}\right) + \frac{m^3}{r^3}\left(\frac{1}{2} + \frac{15\nu}{4}\right)\right\} \\
&+ \frac{1}{c^6}\left\{\frac{5v^8}{128} - \frac{59\nu v^8}{128} + \frac{119\nu^2 v^8}{64} - \frac{323\nu^3 v^8}{128}\right. \\
&\qquad + \frac{m}{r}\left(\frac{5\dot{r}^6\nu^3}{16} + \frac{\dot{r}^4 a_n\nu r}{16} - \frac{5\dot{r}^4 a_n\nu^2 r}{16} - \frac{3\dot{r}^4\nu v^2}{16} + \frac{7\dot{r}^4\nu^2 v^2}{4} - \frac{33\dot{r}^4\nu^3 v^2}{16} - \frac{3\dot{r}^2 a_n\nu r v^2}{16}\right. \\
&\qquad\qquad - \frac{\dot{r}^2 a_n\nu^2 r v^2}{16} + \frac{5\dot{r}^2\nu v^4}{8} - 3\dot{r}^2\nu^2 v^4 + \frac{75\dot{r}^2\nu^3 v^4}{16} + \frac{7a_n\nu r v^4}{8} - \frac{7a_n\nu^2 r v^4}{2} + \frac{11v^6}{16} \\
&\qquad\qquad \left.- \frac{55\nu v^6}{16} + \frac{5\nu^2 v^6}{2} + \frac{65\nu^3 v^6}{16} + \frac{5\dot{r}^3\nu r a_v}{12} - \frac{13\dot{r}^3\nu^2 r a_v}{8} - \frac{37\dot{r}\nu r v^2 a_v}{8} + \frac{35\dot{r}\nu^2 r v^2 a_v}{4}\right) \\
&\qquad + \frac{m^2}{r^2}\left(-\frac{109\dot{r}^4\nu}{144} - \frac{259\dot{r}^4\nu^2}{36} + 2\dot{r}^4\nu^3 - \frac{17\dot{r}^2 a_n\nu r}{6} + \frac{97\dot{r}^2 a_n\nu^2 r}{12} + \frac{\dot{r}^2 v^2}{4}\right. \\
&\qquad\qquad - \frac{41\dot{r}^2\nu v^2}{6} - \frac{2287\dot{r}^2\nu^2 v^2}{48} - \frac{27\dot{r}^2\nu^3 v^2}{4} + \frac{203a_n\nu r v^2}{12} + \frac{149a_n\nu^2 r v^2}{6} \\
&\qquad\qquad \left.+ \frac{45v^4}{16} + \frac{53\nu v^4}{24} + \frac{617\nu^2 v^4}{24} - \frac{9\nu^3 v^4}{4} - \frac{235\dot{r}\nu r a_v}{24} + \frac{235\dot{r}\nu^2 r a_v}{6}\right) \\
&\qquad + \frac{m^3}{r^3}\left(\frac{3\dot{r}^2}{2} - \frac{12041\dot{r}^2\nu}{420} + \frac{37\dot{r}^2\nu^2}{4} + \frac{7\dot{r}^2\nu^3}{2} - \frac{123\dot{r}^2\nu\pi^2}{64} + \frac{5v^2}{4} + \frac{387\nu v^2}{70}\right. \\
&\qquad\qquad \left.- \frac{7\nu^2 v^2}{4} + \frac{\nu^3 v^2}{2} + \frac{41\nu\pi^2 v^2}{64} + 22\dot{r}^2\nu\ln\left(\frac{r}{r_0'}\right) - \frac{22\nu v^2}{3}\ln\left(\frac{r}{r_0'}\right)\right) \\
&\qquad \left.+ \frac{m^4}{r^4}\left(-\frac{3}{8} - \frac{18469\nu}{840} + \frac{22\nu}{3}\ln\left(\frac{r}{r_0'}\right)\right)\right\} \tag{5}
\end{aligned}
$$

Witness the acceleration terms present at the 2PN and 3PN orders; our notation is $a_n = \mathbf{a} \cdot \mathbf{n}$ and $a_v = \mathbf{a} \cdot \mathbf{v}$ for the scalar products between $a^i = \mathrm{d}v^i/\mathrm{d}t$ and the direction n^i and velocity v^i. We recall here that it is, in general, forbidden to order-reduce the accelerations in a Lagrangian.

We next consider the problem of the Hamiltonian associated with (the conservative part of) the equations of motion (1), (2). This problem is not straightforward in harmonic coordinates because of the presence of accelerations at 2PN and 3PN orders in the Lagrangian (5). To proceed, the best is again to change coordinates, and transform the harmonic coordinate system into a new system which avoids the appearance of the accelerations terms, and also, as it will turn out, of the logarithms at the 3PN order. This new coordinate change will thus contain a piece which is identical to the one we used to remove the logarithms of the equations of motion in Section 2. However, this new coordinate system is not harmonic; it was introduced long ago by Arnowitt, Deser and Misner in their study of the Hamiltonian formulation of general relativity and is called the ADM coordinate system. In Ref. [27] the 'contact' transformation between the particles' variables in harmonic coordinates and those in ADM coordinates was determined. By contact transformation we mean the relation between the particles' trajectories $y_A^i(t)$ in one coordinate system, and the corresponding trajectories $Y_A^i(t)$ in another, say $\delta y_A^i(t) = Y_A^i(t) - y_A^i(t)$. Notice that in the contact transformation, the time t is to be viewed as a 'dummy' variable. The contact transformation is not a coordinate transformation between the spatial vectors in both coordinates, but takes also into account the fact that the time coordinate changes as well; i.e. $\delta y_A^i = \xi^i(y_A) - \xi^0(y_A)v_A^i/c$, where $\xi^\mu(y_A)$ denotes the four-dimensional change between the coordinates, when evaluated at the position $y_A = [t, y_A^i(t)]$.

There is a unique contact transformation such that the 3PN harmonic-coordinates Lagrangian (5) is changed into another Lagrangian whose Legendre transform coincides with the 3PN ADM-coordinates Hamiltonian derived in [19] (see [27,28] for details). In a first stage this yields the expression for the ADM-coordinates Lagrangian, in which we use names appropriate to the ADM variables $X^i = x^i + \delta x^i$, which means the separation distance R, the relative square velocity V^2, and the radial velocity $\dot{R} = \mathbf{N} \cdot \mathbf{V}$. This is an ordinary Lagrangian, depending only on the positions and velocities and without accelerations, $L^{\mathrm{ADM}}[X^i, V^i]$, and as we said which is free of logarithms at the 3PN order. Its explicit expression is

$$
\begin{aligned}
\frac{L^{\mathrm{ADM}}}{\mu} = {}& \frac{m}{R} + \frac{V^2}{2} + \frac{1}{c^2}\left\{ \frac{V^4}{8} - \frac{3\nu V^4}{8} + \frac{m}{R}\left(\frac{\nu \dot{R}^2}{2} + \frac{3V^2}{2} + \frac{\nu V^2}{2} \right) - \frac{m^2}{2R^2} \right\} \\
& + \frac{1}{c^4}\left\{ \frac{V^6}{16} - \frac{7\nu V^6}{16} + \frac{13\nu^2 V^6}{16} + \frac{m}{R}\left(\frac{3\nu^2 \dot{R}^4}{8} + \frac{\nu \dot{R}^2 V^2}{2} - \frac{5\nu^2 \dot{R}^2 V^2}{4} + \frac{7V^4}{8} - \frac{3\nu V^4}{2} - \frac{9\nu^2 V^4}{8} \right) \right. \\
& \left. + \frac{m^2}{R^2}\left(\frac{3\nu \dot{R}^2}{2} + \frac{3\nu^2 \dot{R}^2}{2} + 2V^2 - \nu V^2 + \frac{\nu^2 V^2}{2} \right) + \frac{m^3}{R^3}\left(\frac{1}{4} + \frac{3\nu}{4} \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{5V^8}{128} - \frac{59\nu V^8}{128} + \frac{119\nu^2 V^8}{64} - \frac{323\nu^3 V^8}{128} \right. \\
& + \frac{m}{R}\left(\frac{5\nu^3 \dot{R}^6}{16} + \frac{9\nu^2 \dot{R}^4 V^2}{16} - \frac{33\nu^3 \dot{R}^4 V^2}{16} + \frac{\nu \dot{R}^2 V^4}{2} - 3\nu^2 \dot{R}^2 V^4 \right. \\
& \left. + \frac{75\nu^3 \dot{R}^2 V^4}{16} + \frac{11V^6}{16} - \frac{7\nu V^6}{2} + \frac{59\nu^2 V^6}{16} + \frac{65\nu^3 V^6}{16} \right) \\
& + \frac{m^2}{R^2}\left(-\frac{5\nu \dot{R}^4}{12} + \frac{17\nu^2 \dot{R}^4}{12} + 2\nu^3 \dot{R}^4 + \frac{39\nu \dot{R}^2 V^2}{16} - \frac{29\nu^2 \dot{R}^2 V^2}{8} \right. \\
& \left. - \frac{27\nu^3 \dot{R}^2 V^2}{4} + \frac{47V^4}{16} - \frac{15\nu V^4}{4} - \frac{25\nu^2 V^4}{16} - \frac{9\nu^3 V^4}{4} \right) \\
& + \frac{m^3}{R^3}\left(\frac{77\nu \dot{R}^2}{16} + \frac{5\nu^2 \dot{R}^2}{4} + \frac{7\nu^3 \dot{R}^2}{2} + \frac{3\nu \dot{R}^2 \pi^2}{64} + \frac{13V^2}{8} - \frac{409\nu V^2}{48} \right. \\
& \left. \left. - \frac{5\nu^2 V^2}{8} + \frac{\nu^3 V^2}{2} - \frac{\nu \pi^2 V^2}{64} \right) + \frac{m^4}{R^4}\left(-\frac{1}{8} - \frac{109\nu}{12} + \frac{21\nu\pi^2}{32} \right) \right\}
\end{aligned} \tag{6}
$$

Next we apply the ordinary Legendre transform to obtain the corresponding Hamiltonian, $H^{\mathrm{ADM}}[X^i, P^i]$, which is a function of X^i and the conjugate momentum $P^i = \partial L^{\mathrm{ADM}}/\partial V^i$. We find

$$\frac{H^{\text{ADM}}}{\mu} = \frac{P^2}{2} - \frac{m}{R} + \frac{1}{c^2}\left\{-\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{m}{R}\left(-\frac{P_R{}^2\nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2}\right) + \frac{m^2}{2R^2}\right\}$$

$$+ \frac{1}{c^4}\left\{\frac{P^6}{16} - \frac{5\nu P^6}{16} + \frac{5\nu^2 P^6}{16} + \frac{m}{R}\left(-\frac{3P_R{}^4\nu^2}{8} - \frac{P_R{}^2 P^2\nu^2}{4} + \frac{5P^4}{8} - \frac{5\nu P^4}{2} - \frac{3\nu^2 P^4}{8}\right)\right.$$

$$\left. + \frac{m^2}{R^2}\left(\frac{3P_R{}^2\nu}{2} + \frac{5P^2}{2} + 4\nu P^2\right) + \frac{m^3}{R^3}\left(-\frac{1}{4} - \frac{3\nu}{4}\right)\right\}$$

$$+ \frac{1}{c^6}\left\{-\frac{5P^8}{128} + \frac{35\nu P^8}{128} - \frac{35\nu^2 P^8}{64} + \frac{35\nu^3 P^8}{128}\right.$$

$$+ \frac{m}{R}\left(-\frac{5P_R{}^6\nu^3}{16} + \frac{3P_R{}^4 P^2\nu^2}{16} - \frac{3P_R{}^4 P^2\nu^3}{16} + \frac{P_R{}^2 P^4\nu^2}{8}\right.$$

$$\left. - \frac{3P_R{}^2 P^4\nu^3}{16} - \frac{7P^6}{16} + \frac{21\nu P^6}{8} - \frac{53\nu^2 P^6}{16} - \frac{5\nu^3 P^6}{16}\right)$$

$$+ \frac{m^2}{R^2}\left(\frac{5P_R{}^4\nu}{12} + \frac{43P_R{}^4\nu^2}{12} + \frac{17P_R{}^2 P^2\nu}{16} + \frac{15P_R{}^2 P^2\nu^2}{8} - \frac{27P^4}{16} + \frac{17\nu P^4}{2} + \frac{109\nu^2 P^4}{16}\right)$$

$$+ \frac{m^3}{R^3}\left(-\frac{85P_R{}^2\nu}{16} - \frac{7P_R{}^2\nu^2}{4} - \frac{25P^2}{8} - \frac{335\nu P^2}{48} - \frac{23\nu^2 P^2}{8} - \frac{3P_R{}^2\nu\pi^2}{64} + \frac{\nu P^2\pi^2}{64}\right)$$

$$\left. + \frac{m^4}{R^4}\left(\frac{1}{8} + \frac{109\nu}{12} - \frac{21\nu\pi^2}{32}\right)\right\} \tag{7}$$

We denote $P^2 = \mathbf{P}^2$ and $P_R = \mathbf{N}\cdot\mathbf{P}$. The previous result is in perfect agreement with the center-of-mass Hamiltonian derived in Ref. [19].

4. Dynamical stability of circular orbits

As an application let us investigate the problem of the stability, against dynamical gravitational perturbations, of circular orbits at the 3PN order. We want in particular to discuss the existence (or non-existence) of an innermost stable circular orbit (ISCO) at various post-Newtonian orders, which would constitute the analogue for two black holes with comparable masses of the famous orbit $r_{\text{ISCO}} = 6M/c^2$ in the Schwarzschild metric. Notice that we are concerned here with the stability of the orbit with respect to *purely gravitational* perturbations appropriate to the motion of black holes; however it is known that for neutron stars instead of black holes the ISCO is determined by the hydrodynamical instability rather than by the effect of general relativity.

We propose to use two different methods for this problem, one based on a perturbation at the level of the equations of motion (1), (2) in harmonic coordinates, the other one consisting of perturbing the Hamiltonian equations in ADM coordinates for the Hamiltonian (7). We shall find a criterion for the stability of orbits and shall present it in the form of an invariant expression (which is the same in different coordinate systems). We shall check that our two methods agree on the result.

We deal first with the perturbation of the equations of motion, following the approach proposed in Section III.A of Ref. [40]. We introduce polar coordinates (r, φ) in the orbital plane and pose $u = \dot{r}$ and $\omega = \dot{\varphi}$. Then Eq. (1) yields the system of equations

$$\dot{r} = u \tag{8a}$$

$$\dot{u} = -\frac{m}{r^2}[1 + \mathcal{A} + \mathcal{B}u] + r\omega^2 \tag{8b}$$

$$\dot{\omega} = -\omega\left[\frac{m}{r^2}\mathcal{B} + \frac{2u}{r}\right] \tag{8c}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by Eqs. (2) as functions of r, u and ω (through $v^2 = u^2 + r^2\omega^2$). In the case of an orbit which is circular, apart from the adiabatic inspiral due to the 2.5PN and 3.5PN radiation-reaction effects, we have $\dot{r} = \dot{u} = \dot{\omega} = 0$ hence $u = 0$. Eq. (8b) gives thereby the angular velocity ω_0 of the circular orbit as

$$\omega_0^2 = \frac{m}{r_0^3}(1 + \mathcal{A}_0) \tag{9}$$

Solving iteratively this relation at the 3PN order using the equations of motion (1), (2) we obtain ω_0 as a function of the circular-orbit radius r_0 in harmonic coordinates (the result agrees with the one of Refs. [23,24]),

$$\omega_0^2 = \frac{m}{r_0^3}\left\{1 + \frac{m}{r_0 c^2}(-3 + \nu) + \frac{m^2}{r_0^2 c^4}\left(6 + \frac{41}{4}\nu + \nu^2\right)\right.$$
$$\left. + \frac{m^3}{r_0^3 c^6}\left(-10 + \left[-\frac{75707}{840} + \frac{41}{64}\pi^2 + 22\ln\left(\frac{r_0}{r_0'}\right)\right]\nu + \frac{19}{2}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\} \tag{10}$$

The circular-orbit radius r_0 should not be confused with the constant r_0' entering the logarithm at the 3PN order and which is issued from Eqs. (2).

Now we investigate the equations of linear perturbations around the circular orbit defined by the constants r_0, $u_0 = 0$, or, rather, if we were to include the radiation-reaction damping, $u_0 = \mathcal{O}(c^{-5})$, and ω_0. We pose

$$r = r_0 + \delta r \tag{11a}$$
$$u = \delta u \tag{11b}$$
$$\omega = \omega_0 + \delta\omega \tag{11c}$$

where δr, δu and $\delta\omega$ denote some perturbations of the circular orbit. Then a system of linear equations follows as

$$\dot{\delta r} = \delta u \tag{12a}$$
$$\dot{\delta u} = \alpha_0 \delta r + \beta_0 \delta\omega \tag{12b}$$
$$\dot{\delta\omega} = \gamma_0 \delta u \tag{12c}$$

where the coefficients, which solely depend on the unperturbed circular orbit, read [40]

$$\alpha_0 = 3\omega_0^2 - \frac{m}{r_0^2}\left(\frac{\partial\mathcal{A}}{\partial r}\right)_0 \tag{13a}$$

$$\beta_0 = 2r_0\omega_0 - \frac{m}{r_0^2}\left(\frac{\partial\mathcal{A}}{\partial\omega}\right)_0 \tag{13b}$$

$$\gamma_0 = -\omega_0\left[\frac{2}{r_0} + \frac{m}{r_0^2}\left(\frac{\partial\mathcal{B}}{\partial u}\right)_0\right] \tag{13c}$$

In obtaining Eqs. (13) we use the fact that $\mathcal{A}$ is a function of the square u^2 through $v^2 = u^2 + r^2\omega^2$, so that $\partial\mathcal{A}/\partial u$ is proportional to u and thus vanishes in the unperturbed configuration (because $u = \delta u$). On the other hand, since the radiation reaction is neglected, $\mathcal{B}$ also is proportional to u [see Eqs. (2)], so only $\partial\mathcal{B}/\partial u$ can contribute at the zeroth perturbative order. Now by examining the fate of perturbations that are proportional to some $e^{i\sigma t}$, we arrive at the condition for the frequency σ of the perturbation to be real, and hence for stable circular orbits to exist, as being [40]

$$\widehat{C}_0 = -\alpha_0 - \beta_0\gamma_0 > 0 \tag{14}$$

Substituting into this $\mathcal{A}$ and $\mathcal{B}$ at the 3PN order we then arrive at the orbital-stability criterion

$$\widehat{C}_0 = \frac{m}{r_0^3}\left\{1 + \frac{m}{r_0 c^2}(-9 + \nu) + \frac{m^2}{r_0^2 c^4}\left(30 + \frac{65}{4}\nu + \nu^2\right)\right.$$
$$\left. + \frac{m^3}{r_0^3 c^6}\left(-70 + \left[-\frac{29927}{840} - \frac{451}{64}\pi^2 + 22\ln\left(\frac{r_0}{r_0'}\right)\right]\nu + \frac{19}{2}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\} \tag{15}$$

where we recall that r_0 is the radius of the orbit in harmonic coordinates.

Our second method is to use the Hamiltonian equations based on the 3PN Hamiltonian in ADM coordinates given by Eq. (7). We introduce the polar coordinates (R, Ψ) in the orbital plane—we assume that the orbital plane is equatorial, given by $\Theta = \pi/2$ in the spherical coordinate system (R, Θ, Ψ)—and make the substitution

$$P^2 = P_R{}^2 + \frac{P_\Psi^2}{R^2} \tag{16}$$

into the Hamiltonian. This yields a 'reduced' Hamiltonian that is a function of R, P_R and P_Ψ, namely $\mathcal{H} = \mathcal{H}[R, P_R, P_\Psi]$, and describes the motion in polar coordinates in the orbital plane (henceforth we denote $\mathcal{H} = H^{\mathrm{ADM}}/\mu$). The Hamiltonian equations then read

$$\frac{dR}{dt} = \frac{\partial \mathcal{H}}{\partial P_R} \tag{17a}$$

$$\frac{d\Psi}{dt} = \frac{\partial \mathcal{H}}{\partial P_\Psi} \tag{17b}$$

$$\frac{dP_R}{dt} = -\frac{\partial \mathcal{H}}{\partial R} \tag{17c}$$

$$\frac{dP_\Psi}{dt} = 0 \tag{17d}$$

Evidently the constant P_Ψ is nothing but the conserved angular-momentum integral. For circular orbits we have $R = R_0$ (a constant) and $P_R = 0$, so

$$\frac{\partial \mathcal{H}}{\partial R}\big[R_0, 0, P_\Psi^0\big] = 0 \tag{18}$$

which gives the angular momentum P_Ψ^0 of the circular orbit as a function of R_0, and

$$\omega_0 = \left(\frac{d\Psi}{dt}\right)_0 = \frac{\partial \mathcal{H}}{\partial P_\Psi}\big[R_0, 0, P_\Psi^0\big] \tag{19}$$

which yields the angular frequency of the circular orbit ω_0—the same as in Eq. (10)—in terms of R_0,

$$\omega_0^2 = \frac{m}{R_0^3}\left\{1 + \frac{m}{R_0 c^2}(-3 + \nu) + \frac{m^2}{R_0^2 c^4}\left(\frac{21}{4} - \frac{5}{8}\nu + \nu^2\right)\right.$$
$$\left. + \frac{m^3}{R_0^3 c^6}\left(-7 + \left[-\frac{2015}{48} + \frac{167}{64}\pi^2\right]\nu - \frac{31}{8}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\} \tag{20}$$

The last equation, which is equivalent to $R = \mathrm{const} = R_0$, i.e.

$$\frac{\partial \mathcal{H}}{\partial P_R}\big[R_0, 0, P_\Psi^0\big] = 0 \tag{21}$$

is automatically verified because $\mathcal{H}$ is a quadratic function of P_R and hence $\partial \mathcal{H}/\partial P_R$ is zero for circular orbits.

We consider now a perturbation of the circular orbit defined by

$$P_R = \delta P_R \tag{22a}$$

$$P_\Psi = P_\Psi^0 + \delta P_\Psi \tag{22b}$$

$$R = R_0 + \delta R \tag{22c}$$

$$\omega = \omega_0 + \delta\omega \tag{22d}$$

It is easy to verify that the Hamiltonian equations (17), when worked out at the linearized order, read as

$$\delta\dot{P}_R = -\pi_0\delta R - \rho_0\delta P_\Psi \tag{23a}$$

$$\delta\dot{P}_\Psi = 0 \tag{23b}$$

$$\delta\dot{R} = \sigma_0\delta P_R \tag{23c}$$

$$\delta\omega = \rho_0\delta R + \tau_0\delta P_\Psi \tag{23d}$$

where the coefficients, which depend on the unperturbed orbit, are given by

$$\pi_0 = \frac{\partial^2 \mathcal{H}}{\partial R^2}[R_0, 0, P_\Psi^0] \tag{24a}$$

$$\rho_0 = \frac{\partial^2 \mathcal{H}}{\partial R \partial P_\Psi}[R_0, 0, P_\Psi^0] \tag{24b}$$

$$\sigma_0 = \frac{\partial^2 \mathcal{H}}{\partial P_R^2}[R_0, 0, P_\Psi^0] \tag{24c}$$

$$\tau_0 = \frac{\partial^2 \mathcal{H}}{\partial P_\Psi^2}[R_0, 0, P_\Psi^0] \tag{24d}$$

By looking to solutions proportional to some $e^{i\sigma t}$ one obtains some real frequencies, and therefore one finds stable circular orbits, if and only if

$$\widehat{C}_0 = \pi_0 \sigma_0 > 0 \tag{25}$$

Using the Hamiltonian (7) we readily obtain

$$\widehat{C}_0 = \frac{m}{R_0^3}\left\{1 + \frac{m}{R_0 c^2}(-9 + \nu) + \frac{m^2}{R_0^2 c^4}\left(\frac{117}{4} + \frac{43}{8}\nu + \nu^2\right)\right.$$
$$\left. + \frac{m^3}{R_0^3 c^6}\left(-61 + \left[\frac{4777}{48} - \frac{325}{64}\pi^2\right]\nu - \frac{31}{8}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\} \tag{26}$$

This result does not look the same as our previous result (15), but this is simply due to the fact that it depends on the ADM radial separation R_0 instead of the harmonic one r_0. Fortunately all the material needed to connect R_0 to r_0 with the 3PN accuracy is known [28]. In the case of circular orbits we readily find

$$R_0 = r_0\left\{1 + \frac{m^2}{r_0^2 c^4}\left(-\frac{1}{4} - \frac{29}{8}\nu\right) + \frac{m^3}{r_0^3 c^6}\left(\left[\frac{3163}{1680} + \frac{21}{32}\pi^2 - \frac{22}{3}\ln\left(\frac{r_0}{r_0'}\right)\right]\nu + \frac{3}{8}\nu^2\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\} \tag{27}$$

The difference between R_0 and r_0 is made out of 2PN and 3PN terms only. Inserting Eq. (27) into Eq. (26) and re-expanding to 3PN order we find that indeed our basic stability-criterion function $\widehat{C}_0$ comes out the same with our two methods.

Finally let us give to the function $\widehat{C}_0$ an invariant meaning by expressing it with the help of the orbital frequency ω_0 of the circular orbit, or, more conveniently, of the frequency-related parameter

$$x_0 = \left(\frac{m\omega_0}{c^3}\right)^{2/3} \tag{28}$$

From the inverse of Eq. (10) we readily obtain r_0 as a function of x_0. This allows us to write the criterion for stability as $C_0 > 0$, where $C_0 = \frac{m^2}{c^6 x_0^3}\widehat{C}_0$ admits the gauge-invariant form, which will be the same in all coordinate systems,

$$C_0 = 1 - 6x_0 + 14\nu x_0^2 + \left(\left[\frac{397}{2} - \frac{123}{16}\pi^2\right]\nu - 14\nu^2\right)x_0^3 + \mathcal{O}(x_0^4) \tag{29}$$

This form is more interesting than the coordinate-dependent expressions (15) or (26), not only because of its invariant form, but also because as we see the 1PN term yields exactly the Schwarzschild result that the innermost stable circular orbit or ISCO of a test particle (i.e. in the limit $\nu \to 0$) is located at $x_{\text{ISCO}} = 1/6$. Thus we find that, at the 1PN order, but for *any* mass ratio ν,

$$x_{\text{ISCO}}^{\text{1PN}} = \frac{1}{6} \tag{30}$$

One could have expected that some deviations of the order of ν already occur at the 1PN order, but it turns out that only from the 2PN order does one find the occurrence of some non-Schwarzschildian corrections proportional to ν. At the 2PN order we obtain

$$x_{\text{ISCO}}^{\text{2PN}} = \frac{3}{14\nu}\left(1 - \sqrt{1 - \frac{14\nu}{9}}\right) \tag{31}$$

For equal masses this gives $x_{\text{ISCO}}^{\text{2PN}} \simeq 0.187$. Notice also that the effect of the finite mass corrections is to increase the frequency of the ISCO with respect to the Schwarzschild result (i.e. to make it more inward), and we find $x_{\text{ISCO}}^{\text{2PN}} = \frac{1}{6}[1 + \frac{7}{18}\nu + \mathcal{O}(\nu^2)]$. Finally, at the 3PN order and for equal masses $\nu = 1/4$, we find that according to our criterion all the circular orbits are stable, and there is no ISCO. More generally, we find that at the 3PN order all orbits are stable when the mass ratio is $\nu > \nu_c$ where $\nu_c \simeq 0.183$.

Note that the above stability criterion C_0 gives an innermost stable circular orbit, when it exists, that is not necessarily the same as—and actually differs from—the innermost circular orbit or ICO, which is defined by the point at which the center-of-mass binding energy of the binary for circular orbits reaches its minimum value [41]. In this respect the present formalism, which is based on systematic post-Newtonian expansions (without using post-Newtonian resummation techniques like Padé approximants [42]), differs from some 'Schwarzschild-like' methods such as the effective-one-body approach [43] in which the ICO happens to be also an innermost stable circular orbit or ISCO.

As a final comment, let us note that the use of a *truncated* post-Newtonian series such as Eq. (29) to determine the ISCO is a priori meaningful only if we are able to bound the neglected error terms. Furthermore, since we are dealing with a stability criterion, it is not completely clear that the higher-order post-Newtonian correction terms, even if they are numerically small, will not change qualitatively the response of the orbit to the dynamical perturbation. This is maybe a problem, and which cannot be answered rigorously with the present formalism. However, in the regime of the ISCO (when it exists), we have seen that x_0 is rather small, $x_0 \simeq 0.2$ (this is also approximately the value for the ICO computed in Ref. [41]), which indicates that the neglected terms in the truncated series (29) should not contribute very much, because they involve at least a factor $x_0^4 \simeq 0.002$. On the other hand, we pointed out that in the limit $\nu \to 0$ the criterion C_0 gives back the correct *exact* result, $x_{\text{ISCO}}^{\nu \to 0} = 1/6$. This contrasts with the gauge-dependent power series (15) or (26) which give only some approximate results. Based on these observations, we feel that it is reasonable to expect that the gauge-invariant stability criterion defined by Eq. (29) is physically meaningful.

Acknowledgements

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