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Ultra deep nulling interferometry using fractal interferometers

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Abstract

The difficult goal of directly detecting a planet around a star requires the cancellation of, as far as possible, the stellar light and *nulling interferometry* is one way to do so: the star is put on a central dark fringe while the planet is supposed to be on a bright fringe. One problem is, however, leaks due to the finite angular dimension of the stellar disk, resolved by the interferometer. The solution is to increase the exponent of the term θ^n which describes the cancellation efficiency with respect to the angular distance to the axis of the central dark fringe. Efficient configurations have been found, using basically *guess and check* methods until recently. I present here one method to define configurations of telescopes that achieve any given power of θ. The principle is based on a peculiar property of a partition into two sets of the first 2^N integers; the partition is built using the Prouhet–Thué–Morse sequence which presents some *fractal* properties. A phase shift (0 or π) between 2^N telescopes is applied according to this partition. I first examine 1-D pattern of identical telescopes, then extend the method to 2-D configurations of identical telescopes, to 1-D arrays and 2-D arrays of non-identical telescopes and finally to arrays where the phase shift between n groups of telescopes is $2k\pi/n$. I examine then how a non-perfect fractal interferometer behaves and show that its robustness with respect to nulling stability is an important advantage. ***To cite this article: D. Rouan, C. R. Physique 8 (2007).***

Résumé

Interférométrie annulante à grande efficacité. Le but difficile de détecter directement une planète autour d'une étoile exige de réussir à éteindre autant que possible la lumière stellaire et *l'interférométrie annulante* est l'une des voies explorées. À cette fin, l'étoile est maintenue sur une frange centrale sombre tandis que la planète est censée être sur une frange brillante. Un problème sérieux est cependant associé aux fuites de lumière dues à la dimeètre résout. Une solution est alors de chercher à augmenter l'exposant de l'expression θ^n qui décrit l'efficacité d'annulation en fonction de la distance angulaire à l'axe de la frange sombre. Les configurations efficaces qui ont été trouvées l'ont été jusqu'ici par essais. Je présente ici une méthode pour définir des configurations de télescopes qui permettent d'obtenir n'importe quelle puissance donnée de θ. Le principe est basé sur une propriété particulière d'une partition en deux ensembles des 2^N premiers nombres entiers ; la partition est construite en utilisant la séquence de Prouhet–Thué–Morse qui a des propriétés *fractales*. Un déphasage (0 ou π) entre 2^N télescopes est appliqué selon cette séquence. J'examine d'abord un arrangement 1-D de télescopes identiques, puis étend la méthode aux configurations 2-D de télescopes identiques, à des réseaux 1-D et 2-D de télescopes non-identiques et à des réseaux où le déphasage entre n groupes de télescopes est $2k\pi/n$. Dans une dernière partie j'examine le comportement d'un interféromètre fractal imparfait en montrant qu'ils est robuste en terme de stabilité de l'annulation. ***Pour citer cet article : D. Rouan, C. R. Physique 8 (2007).***

Keywords: Exoplanet; Interferometry; Direct detection

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1. Introduction

Because of the huge contrast (10^{6-10}) between a star and an orbiting planet, the difficult goal of directly detecting the photons of the planet requires he cancellation of as much as possible of the light from the star. Bracewell [1] was the first to propose the concept of a nulling interferometer, where the light collected by two telescopes is coherently combined with, however, the trick that a π phase shift is applied on one of the optical paths. As a result, a system of fringes is projected onto the sky with a central fringe which is a dark one. If the star image is put on this central dark fringe, it disappears (actually, the stellar photons are all sent to the second output of the interferometer), and if the planet is on a bright fringe, then it can be more easily detected, in principle. The principles are detailed in the paper by Ollivier [2] in this same volume. The modulation of the fringes with respect to the angular distance varies according to a $(1 - \cos\theta)$ function, so that close to the axis the nulling is $\propto \theta^2$. If the interferometer has enough resolution (equivalently, a long basis), as it should have to separate a planet, then the angular diameter of the stellar disk is no longer small with respect to the fringe period and leaks of stellar light will contaminate the planet flux and severely limit the detection capability.

Several interferometric configurations have been presented in order to improve the quality of the rejection, especially by increasing the exponent of the term θ^n which gives the cancellation efficiency with respect to angular distance to the axis of the central fringe Leger [3]. The Angel's cross [4], the Oases configuration [5] or Mariotti/Laurance (the ESA Darwin project) configuration do fulfill this condition of a nulling with an exponent >2, as illustrated on Fig. 1. In general, these configurations have been found through some trial method, but, as far as I know, no systematic method has been proposed to reach any given power of θ, until my first proposal, Rouan [6,7].

I present in the following the method to define configurations of telescopes that fulfill the condition, at least theoretically. The principle is based on a remarkable property of a peculiar partition in two sets of the $N = 2^L$ first integers, which is built according to the Prouhet–Thué–Morse sequence. This property is such that, provided that to half of the N telescopes' output a π phase shift is applied, it is possible to cancel the $(L - 1)$ first terms of the development in θ of the recombined amplitude, thus leading to a θ^{2L} behaviour of the intensity. Because the Prouhet–Thué–Morse sequence has some fractal properties, I propose to call those configurations *fractal interferometers*. In the following, I shall also use the acronym VDNI which stands for Very Deep Nulling Interferometer.

In this article, we examine first 1-D configurations of identical telescopes and then show how one can generalise the method of construction to 2-D arrays—of still identical telescopes. Then, it is shown that the 2-D arrays can be compressed (in fact projected) to build 1-D arrays of non-identical telescopes. In turn, those 1-D arrays can be the basis for building 2-D arrays of non-identical telescopes. We also show that the Prouhet property can be generalised to partitions of the k^L first integers in k subsets, so that ultra-nulling arrays can also be built with, this time a phase

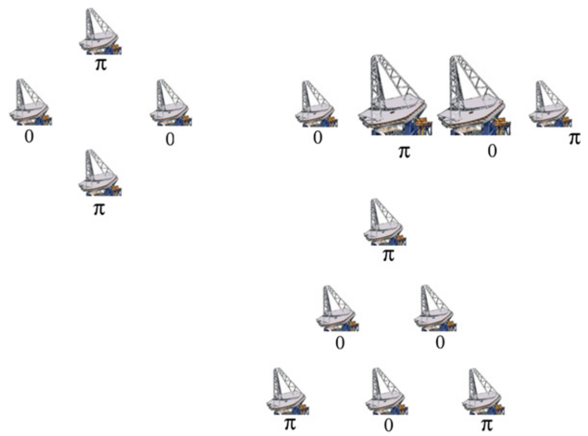

Fig. 1. Different configurations of interferometric nullers that have been proposed in order to improve the starlight rejection; the expression within parenthesis gives the power-law nulling function of each configuration.

shift which is no longer π but rather $2\pi/k$. The last section is devoted to the evaluation of the actual efficiency that such arrays could have with respect to the primary goal of detecting extrasolar planets when non-ideal conditions are considered, such as distances between telescopes or imperfect phase shift.

2. Prouhet–Thué–Morse sequence and Prouhet's partition of the integers

The Prouhet–Thué–Morse sequence is very easily built and can be considered as one of the simplest fractal objects. Let us start from the value 1. This is the first term of the series. Now take its binary complement, 0, and concatenate it to the first term: this produces the second term of the series, i.e. 10. Let us reproduce the operation: we take the last element, 10, and concatenate to it its binary complement, 01: this leads to the third term, 1001. The following terms are built the same way by concatenating the binary complement of the previous term:

10010110

1001011001101001

10010110011010010110100110010110

etc.

One can note that: (a) no term of the series ever contains more than two identical digits (1 or 0) that are juxtaposed (e.g. no 111 or 0000); (b) periodic patterns never appear; (c) the Lth term of the series contains 2^L digits (considering that the index of the first term is 0).

Prouhet [8] had the idea to distribute the first 2^L integers into two sets according to the Prouhet–Thué–Morse sequence (actually, he did not call it this, since Thué and Morse were not yet born ...): the first set corresponds to integers that have the same rank as the (1) in the sequence and the second set to those that have same rank as the (0). This reads:

$$\mathcal{O} = [1, 4, 6, 7, 10, 11, 13, 16, \ldots, o_k, \ldots] \quad \text{and} \quad \mathcal{E} = [2, 3, 5, 8, 9, 12, 14, 15, \ldots, e_k, \ldots]$$

where $\mathcal{O}$ stands for *Prouhet odd* and $\mathcal{E}$ for *Prouhet even*, as I propose to call them. The rather surprising property established by Prouhet is that the sum of the elements of the two series are equal, but much more remarkably, the sum of the squares, of the cubes, etc. are also equal, and this up to the power $L-1$ when considering the 2^L first integers. In a condensed form this can be written:

$$\sum_{k=0}^{2^{L-1}} (o_k)^p = \sum_{k=0}^{2^{L-1}} (e_k)^p, \quad \forall p < L$$

For instance, if $L = 4$ (the 16 first integers), then one can check that:

$$\begin{aligned} 1+4+6+7+10+11+13+16 &= 2+3+5+8+9+12+14+15 \\ 1^2+4^2+6^2+7^2+10^2+11^2+13^2+16^2 &= 2^2+3^2+5^2+8^2+9^2+12^2+14^2+15^2 \\ 1^3+4^3+6^3+7^3+10^3+11^3+13^3+16^3 &= 2^3+3^3+5^3+8^3+9^3+12^3+14^3+15^3 \end{aligned}$$

but that

$$1^4+4^4+6^4+7^4+10^4+11^4+13^4+16^4 \neq 2^4+3^4+5^4+8^4+9^4+12^4+14^4+15^4$$

It is this noticeable property that is exploited in the following.

3. 1-D *fractal* nulling interferometers

Let us consider an interferometer made with a set of $N = 2^L$ identical telescopes regularly aligned, with a distance d between two adjacent telescopes, as illustrated on Fig. 2. If the telescopes are pointing in a direction θ (measured from the normal to the baseline), then the amplitude a of the wave at the output of the recombiner is:

$$a = \sum_{k=0}^{N} \exp(jk\phi)$$

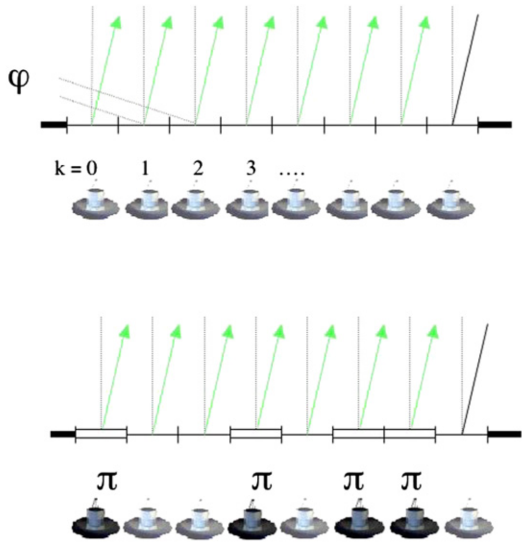

Fig. 2. Top: configuration of a linear interferometer with equally spaced telescopes. Bottom: the *fractal* nulling interferometer, where some telescopes (the darker ones) experience a π phase shift.

where ϕ is the phase difference between two consecutive telescopes: $\phi = 2\pi\theta d/\lambda$. Let us now add a π phase shift on all *Prouhet even* telescopes; in other words we change the sign of the amplitude. The previous relation becomes:

$$a = a_+ - a_- = \sum_{k\in\mathcal{O}} \exp(jk\phi) - \sum_{l\in\mathcal{E}} \exp(jl\phi)$$

where $\mathcal{O}$ and $\mathcal{E}$ stand respectively for the *Prouhet odd* and *Prouhet even* sets. Considering relatively small values of ϕ, i.e. of θ, one can develop each exponential in series of $(k\phi)^n$, so that the two terms of the above expression become:

$$a_+ = \sum_{k\in\mathcal{O}} 1 + j\phi \times \sum_{k\in\mathcal{O}} k - \phi^2 \times \sum_{k\in\mathcal{O}} k^2 - j\phi^3 \times \sum_{k\in\mathcal{O}} k^3 + \cdots$$

and

$$a_- = \sum_{k\in\mathcal{E}} 1 + j\phi \times \sum_{k\in\mathcal{E}} k - \phi^2 \times \sum_{k\in\mathcal{E}} k^2 - j\phi^3 \times \sum_{k\in\mathcal{E}} k^3 + \cdots$$

Thanks to the remarkable property found by Prouhet, i.e. $\sum_{k\in\mathcal{O}} k^p = \sum_{k\in\mathcal{E}} k^p$, the $L - 1$ first terms of each series cancel out mutually, so that the desired property is reached: $a \propto \phi^L \propto \theta^L$, and the intensity has the behaviour: $I \propto \theta^{2L}$. For instance, the interferometer shown in the lower part of Fig. 2 would produce a nulling function $\propto \theta^6$.

One shows in addition, that the residual terms of the amplitude follows the expression: $a \propto 2^{n(n-1)/2}\phi^L + \cdots$

Basically, one can interpret the result of the *ultra-nulling* performance as due to a recursive application of a nulling operation to a pair of interferometers where already, each is a nulling one, as illustrated on Fig. 3. The fact that the order of the nulling increases at each step, is thus easier to understand.

4. 2-D *fractal* nulling interferometers

Let us consider now a square grid of telescopes where on any row *and* any column the sign of the output of each telescope is set following a Prouhet–Thué–Morse sequence—or its complement—. For instance, the grid of Table 1 would be valid for a square grid of 8 × 8 telescopes; note that this array would be as populated in telescopes as the first version of the future ALMA millimetric interferometer.

Let us consider a plane wave coming from a direction $[\theta, \psi]$, where ψ is the azimuthal angle and, as previously, θ is measured from the normal to the plane of the interferometer. The amplitude at the output of the recombiner is given by

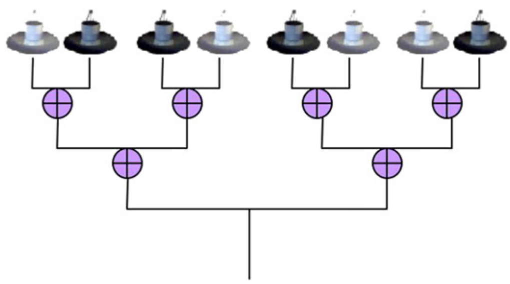

Fig. 3. This cartoon illustrates why when building a fractal interferometer of 2^L telescopes, in fact, one recombines successively, and each time in a nulling way, pairs of interferometers where each is already a nulling one.

Table 1
Grid depicting the sign applied to the output of each telescope of an interferometer with 64 equally spaced telescopes ($L = 3$). Telescopes with a $-$ sign are those where a π phase shift is applied. Here, the nulling function is $\propto \theta^{12}$

+	−	−	+	−	+	+	−
−	+	+	−	+	−	−	+
−	+	+	−	+	−	−	+
+	−	−	+	−	+	+	−
−	+	+	−	+	−	−	+
+	−	−	+	−	+	+	−
+	−	−	+	−	+	+	−
−	+	+	−	+	−	−	+

$$
\begin{aligned}
a &= a_+ - a_- \\
&= \sum_{k\in\mathcal{O}}\sum_{k'\in\mathcal{O}} \exp\big[j(ku+k'v)\phi\big] + \sum_{k\in\mathcal{E}}\sum_{k'\in\mathcal{E}} \exp\big[j(ku+k'v)\phi\big] \\
&\quad - \sum_{k\in\mathcal{O}}\sum_{k'\in\mathcal{E}} \exp\big[j(ku+k'v)\phi\big] - \sum_{k\in\mathcal{E}}\sum_{k'\in\mathcal{O}} \exp\big[j(ku+k'v)\phi\big]
\end{aligned}
$$

where $u = \cos\psi$ and $v = \sin\psi$ and k and k' are the indices for the line and row of each telescope on the (x, y) grid. When developing the exponential terms, one find expressions of the type:

$$
\begin{aligned}
&\Big[\sum_{k,k'\in\mathcal{E}*} - \sum_{k,k'\in\mathcal{O}*}\Big](ku+k'v)^n \\
&\quad = \Big[\sum_{k,k'\in\mathcal{E}*} - \sum_{k,k'\in\mathcal{O}*}\Big]\big\{k^n + (ku)^{n-1}(k'v) + (ku)^{n-2}(k'v)^2 + \cdots + (ku)^{n-m}(k'v)^m + \cdots\big\}
\end{aligned}
$$

where the condensed expression $\sum_{k,k'\in\mathcal{E}*}$ stands for $\sum_{k\in\mathcal{O}}\sum_{k'\in\mathcal{O}} + \sum_{k\in\mathcal{E}}\sum_{k'\in\mathcal{E}}$ and $\sum_{k,k'\in\mathcal{O}*}$ stands for $\sum_{k\in\mathcal{O}}\sum_{k'\in\mathcal{E}} + \sum_{k\in\mathcal{E}}\sum_{k'\in\mathcal{O}}$.

As long as one of the exponents m and m' in the term $(kv)^m(k'v)^{m'}$ is $<L$, there is cancellation of the sum $[\sum_{k,k'\in\mathcal{E}*} - \sum_{k,k'\in\mathcal{O}*}]$, because of the property of the Prouhet's partition. This leaves as the first non-vanishing term of the development of the amplitude a term $\propto (uv\phi^2)^L$, or equivalently for the intensity, a term varying as θ^{4L}. The nulling interferometer of Table 1 would thus have a nulling capability in θ^{12}! By the way, one can note that the case $L = 1$ (2×2 telescopes) corresponds indeed to the Angel's cross [4] which is also the baseline of the X-array, recently retained as the basis of the new TPF-I NASA project.

To illustrate the increase in nulling capability with respect to a resolved stellar disk, Fig. 4 shows the log and linear cuts of a 2×2, a 4×4, a 8×8 and a 16×16 nulling 2-D arrays just as we have defined them.

As concerns the number of telescopes required to reach a given nulling exponent α (on the intensity), the 2-D array and the 1-D array are equivalent, since this number is, in both cases, equal to $2^{\alpha/2}$. However, in order to cover a larger surface of the sky when looking for planets, it is clear that a 2-D array is more efficient for a given number of telescopes.

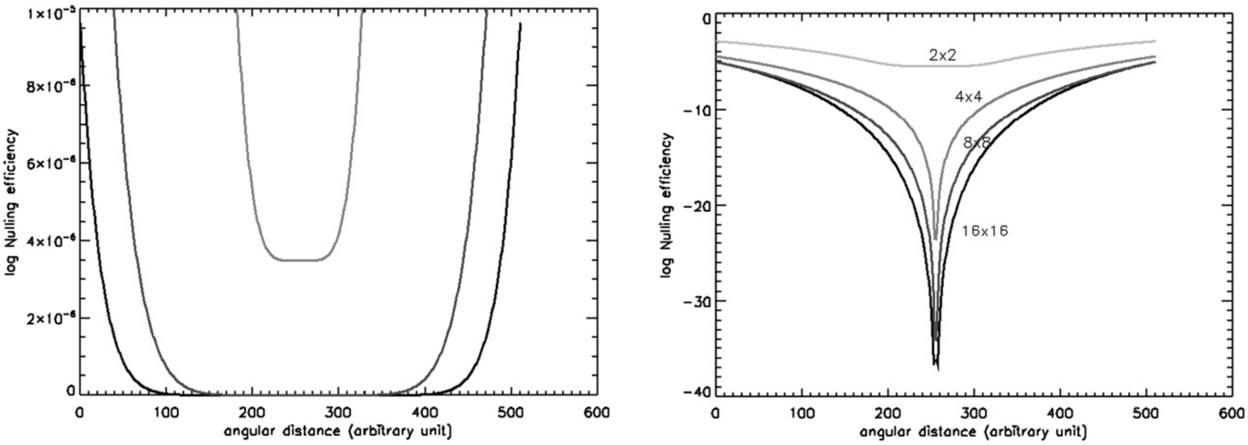

Fig. 4. Left: cut of the transmission map of the four first 2D square VDNI; the very characteristic trough shape is clearly seen; Right: same, with a log scale.

5. 1-D and 2-D nulling interferometers with telescopes of different sizes

Of course the solutions proposed above are not the unique ones to build nulling interferometers, as illustrated by the different configurations shown on the right of Fig. 1 that do not belong to the family of interferometers defined until now. This raises a new question: is there some systematic way to construct 1-D nulling interferometers of any given dimension (in terms of the number of telescopes)? The answer is again yes.

One first interesting possibility is to project any of the 2-D arrays we described above, along a direction parallel to one diagonal. Of course, the properties of nulling must be preserved in the projection. By projection we mean performing the algebraic summation of all telescopes belonging to a same column perpendicular to the diagonal, as illustrated on Fig. 5. We then obtain a linear interferometer, but with telescopes of different sizes, the surface of a resulting telescope being the algebraic sum of the surfaces of the unit telescopes along the column. When the sum is zero, there is no telescope at this position. For instance the simple Angel cross pattern leads to the known configuration $-1, 2, -1$: the minus sign means that a π phase shift is applied and the numbers give the relative collecting surface. Fig. 5 illustrates another example of 1-D nulling interferometer configuration $(+1, -2, -1, +4, -1, -2, +1)$, obtained for $L = 2$. The reason why the nulling is preserved is that the projection is equivalent to a shear of the array

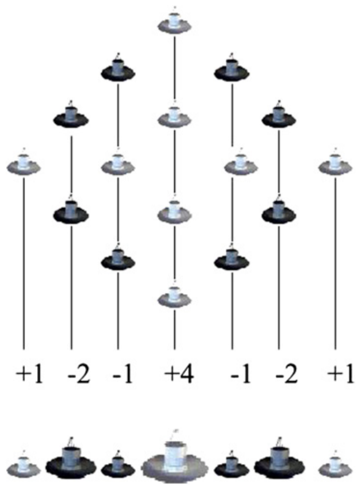

Fig. 5. Example of how to obtain a linear interferometer from a *fractal* nulling 2-D interferometer. The algebraic sum of telescopes surfaces is done along all sub-diagonal of the grid. A minus sign is applied when the telescope has a π phase shift (darker telescopes). The resulting 1-D interferometer is shown at the bottom.

followed by a sum on one row. The shear is obtained by shifting each row with respect to the previous one by one step: clearly the nulling capability of any row is kept and, since in the other direction the centres of all rows are still aligned (on a straight line which is no longer along the column direction), the nulling effect is also maintained. Crushing now the rows on one single row does not change the property.

This can be generalised by making a more pronounced shear or, in other words, by shifting the rows by more than one step. For instance, with a step of 2, the 4×4 array would become:

$$\begin{array}{cccccccccc} +1 & -1 & -1 & +1 & & & & & & \\ & & -1 & +1 & +1 & -1 & & & & \\ & & & & -1 & +1 & +1 & -1 & & \\ & & & & & & +1 & -1 & -1 & +1 \end{array}$$

leading to the 1-D configuration: $+1, -1, -2, +2, 0, 0, +2, -2, -1, +1$ (0 means no telescope).

It is possible to extend the idea further and construct linear arrays from other types of 2-D nulling arrays. A first idea is to build 2-D arrays from rectangular *fractal* arrays which are subsets of the square arrays of Section 2. For instance a 4×8 array could be the upper half of the 8×8 array presented on Table 1.

Another possibility is to build 2-D arrays by replicating on 2^L lines one 1-D multi-size interferometers obtained in the first part of the present section, and weighting with a minus sign the lines that have a rank determined by the (1) in the Prouhet–Thué–Morse sequence. For instance the configuration

$$\begin{array}{ccc} +1 & -2 & +1 \\ -1 & +2 & -1 \end{array}$$

is a viable one. It would lead to the 1-D pattern (1 step shift): $+1, -3, +3, -1$, which is indeed the OASES configuration of Fig. 1.

If the 2-D array dimension is $M \times M'$, the number of telescopes in the 1-D array is given by $M + (M' - 1) \times n_{\text{step}}$. If, as it is prescribed, $M = 2^L$ and $M' = 2^{L'}$, then $N_{\text{tel}} = 2^L + (2^{L'} - 1) \times n_{\text{step}}$. One can easily convince oneself that there is always at least one set of the three parameters L', L' and n_{step} that can give any integer N_{tel}. The affirmation that there is a systematic way to construct 1-D nulling interferometers of any given number of telescopes is thus demonstrated. The final exponent of the nulling function is always $\alpha = 2 \ln N / \ln 2$ where N is the number of telescopes used at the beginning of the process. Since in the algebraic sum some telescopes 'vanish', there is indeed a gain in terms of reduction of the collecting surface for a given nulling efficiency. However, from a practical point of view, the development of an instrument with telescopes of different sizes may not be cost effective and a trade-off must be found.

6. Generalisation to arrays with phase shift different from π

The remarkable property of the two subsets of integers defined according to the PTM series can be in fact extended to any given number n of subsets, as desired, provided that one considers the first n^L integers.

The way the subsets are built has some similarities with the case $n = 2$. If we start from an initial sequence *abcd*..., then the same rule defined above applies: replicate, permute circularly, concatenate. For instance, let us consider the case $n = 3$ and an initial sequence

$$S_0 = abc$$

the second term of the series will then be:

$$S_1 = abc\ bca\ cab$$

where each subgroup of three elements following the first one is obtained by a circular permutation of this first subgroup. The next term will then read:

$$S_2 = abc\ bca\ cab\ \ bca\ cab\ abc\ \ cab\ abc\ bca$$

We have simply permuted twice the 3 sub-groups—of 3 elements each—of S_1 and concatenated the result to S_1.

The length of the current term of the series obviously increases rapidly, in fact as k^L. Now let us associate the first k^L integers to the elements of the Lth term of the series; we thus define k partitions of the integers: a first one where

Fig. 6. The cartoon shows how the phase shift 0 (symbolised by a blue telescope), $2\pi/3$ (in red) and $4\pi/3$ (in green), are distributed on the telescopes of a linear array of 3^2 telescopes according to the Prouhet's rule.

the integers are the ranks of the 'a', a second one where the ranks are those of the 'b', etc. For instance, if we consider the term S_2 of the previous example, we would obtain the three subsets:

$$[1, 6, 8, 12, 14, 16, 20, 22, 27] - [2, 4, 9, 10, 15, 17, 21, 23, 25] - [3, 5, 7, 11, 13, 18, 19, 24, 26]$$

A rather surprising fact is that the Prouhet's property on the equality of the sums of the elements, their square, their cube, etc., is still valid for this partition. This property reads:

$$\sum_{i=0}^{k^{L-1}} (a_i)^p = \sum_{i=0}^{k^{L-1}} (b_i)^p = \sum_{i=0}^{k^{L-1}} (c_i)^p = \cdots, \quad \forall p < L$$

In this relation, a_i are the elements of the first subset (A) of the partition, b_i the elements of the second subset (B), etc.

Now let us consider a linear array of k^L telescope pointing in a same direction. If we apply a phase shift of $\mathrm{i}2\pi/k$ to all telescopes of the ith subset, as illustrated on Fig. 6, then an on-axis nulling will be obtained, with a θ^L power-law. I will not give the full demonstration of this result, since it is based on the same techniques as used before in the case $k = 2$, i.e. performing a Taylor's development of the complex amplitude corresponding to each telescope. One shows then that there is again a perfect cancellation of the first terms of the development. In short, this comes from the classical relation:

$$\left(1 + \mathrm{e}^{j\pi/N} + \mathrm{e}^{j2\pi/N} + \mathrm{e}^{j3\pi/N} + \cdots + \mathrm{e}^{j(N-1)\pi/N}\right) = 0$$

using the additional remark that any power of this relation is also zero.

Obviously, building a 2-D array can be done using the same rules as in the case $k = 2$. I did not explore all the possibilities of constructing 1-D and 2-D arrays of non-identical telescopes, but obviously there must be an as rich reservoir of potential nulling interferometers.

7. Simulation: VDNI interferometers in the real world

To be fair, it is worth mentioning that this work belongs more to the brain-storming class than to the instrumental engineering one. It was first motivated by the wish to mathematically solve the θ^n problem in nulling interferometers, that appeared soon after the launch of the concept of Darwin/TPF type mission at the end of the 1990s. The richness of the solution I found, in terms of possible configurations of nulling interferometers, was rather unexpected and contributed to the excitement in this exercise. Now, it is clear that a good efficiency requires a rather large number of unit telescopes with peculiar configurations, so that this architecture is certainly not relevant for existing ground-based interferometers, but rather could possibly concern a dedicated space experiment. With its 50 telescopes, the ALMA project could, however, be a noticeable exception that may deserve some thought. Now it is possible that unexpected properties be associated to this peculiar architecture when actual, i.e. non-perfect, interferometers are considered. This is what is explored in this section.

In the real world, there are uncertainties in the parameters defining an interferometer, and we explore in this section how such ultra-nulling interferometers are affected by those imperfections. We compare here the nulling properties of a non-perfect VDNI, when distances between telescope and phase shift are not nominal, so that the nulling performance is degraded. Indeed the question of the nulling stability is an issue since the associated noise tends to be the most important contributor in the present studies of space interferometers (Lay [9], Chazelas et al. [10]).

For this comparison, I consider only 2D square arrays of the basic type: 2×2, 4×4, 8×8, 16×16. I assume that the total surface of telescopes is the same: there are either few large telescopes or many small telescopes. With this hypothesis, I have in mind the assumption that in terms of mission cost, this could be equivalent because the production in series of identical small telescopes should be rather cheap (e.g. 64×25 cm telescopes versus 4×1 m telescopes): of course this assumption would deserve to be confirmed.

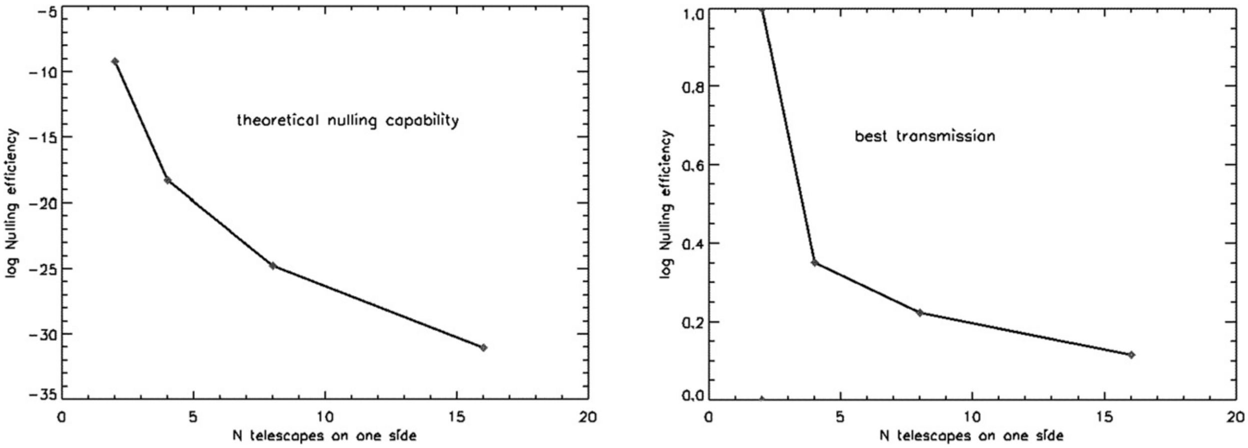

Fig. 7. Left: theoretical nulling capability of the four first 2D square VDNI vs the number of telescopes on one side ; Right: best transmission of the VDNI.

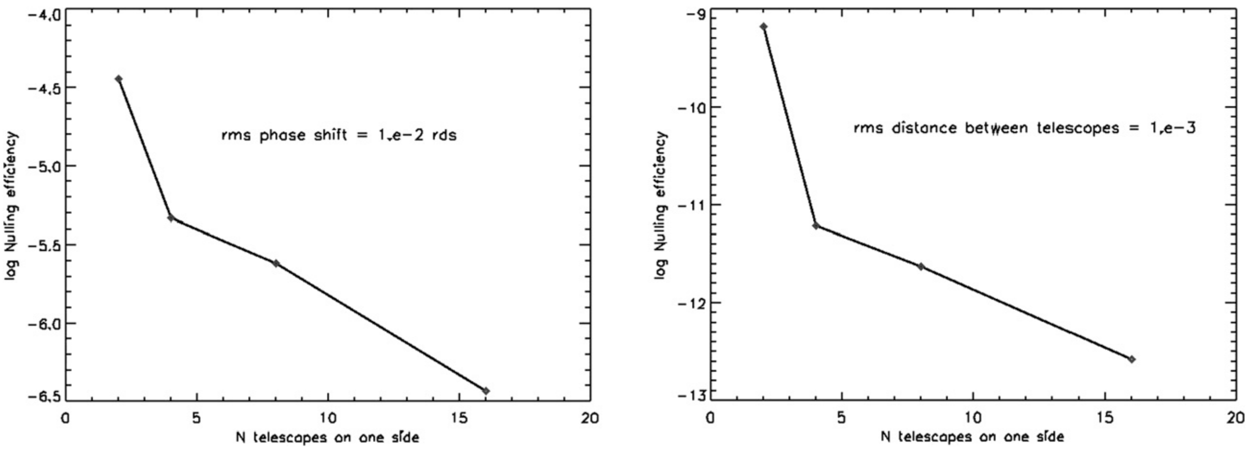

Fig. 8. Left: nulling capability of the four first 2D square VDNI when distance between telescopes fluctuates, with rms $\delta d/d = 10^{-3}$; Right: same but when π phase shift fluctuates, with $\sigma_\phi = 1$ deg.

Simulations were done for an Earth–Sun system: this means that the distance between telescopes is such that the projected first maximum of transmission is at 1 AU and that the star diameter to consider for leaks calculation is 0.01 AU. Two causes of nulling degradation were considered in this exercise:

(a) the distances between telescopes are not constant and the rms relative variation is $\delta d/d = 10^{-3}$;
(b) there is an imperfect π phase shift with an rms fluctuation of 0.01 radians.

Figs. 7, 8, and 9 summarize the results of this simple study.

First, in terms of absolute performances (perfect nulling configuration), it is confirmed that the nulling capability of VDNI is indeed excellent and far beyond the required factor of 10^{10} for an Earth detection, since, for instance, it reaches 10^{-32} for the 16×16 array (Fig. 7(a)). However, the maximum transmission of the planet is decreasing with order; it is only 12% for the 16×16 pattern, while it is 100% for the 2×2 one (Fig. 7(b)). This represents clearly a first limitation to this type of nulling interferometer: going beyond 8×8 would probably be useless.

As regards the nulling performance when fluctuations of distances and of phases are introduced, Fig. 8(a), (b) shows that there is a clear improvement when using, for a given level of fluctuation, larger arrays: for instance, even with a very severe phase fluctuation of 0.5° rms, the nulling capability can be maintained at a fairly good level with a 16×16 array, with a gain of 85 over the 2×2 array. This is a very significant improvement.

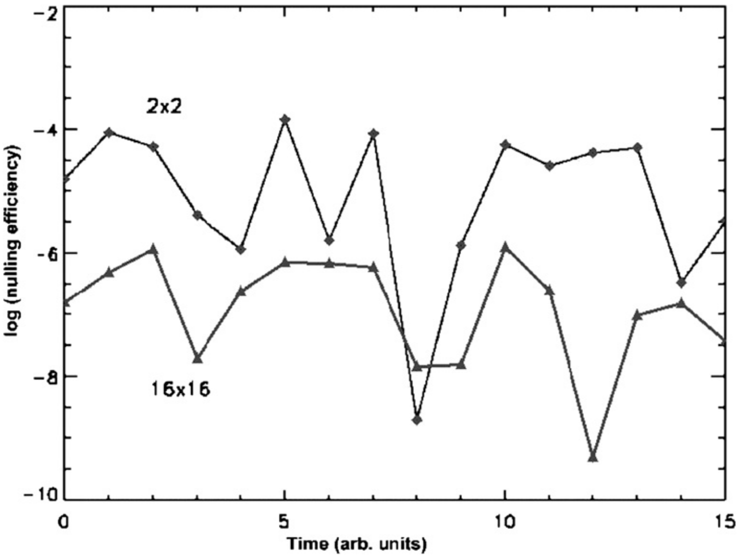

Fig. 9. Example of nulling stability as a function of time for 2×2 and 16×16 patterns.

If one looks at the detail of the behaviour with time of the nulling performance (Fig. 9), it clearly appears that the larger the array, the lesser the amplitudes of fluctuations, and especially the big ones (note the logarithmic scale of Fig. 9): for instance, in the case of the 2×2 array, the nulling factor can vary by as much as a factor of 2000, while it is only a factor of 370 for the 16×16 array. One can logically suspect some averaging effect to explain this robustness of nulling efficiency for the VDNI of highest order.

8. Conclusions

I showed in this article that there is a systematic method to designing a nulling interferometer that provides, in principle, a central null depth of any given power of θ. The solution is based on a singular property of a partition of the first 2^L integers, discovered by Prouhet. It consists in defining the interferometer with 2^L regularly aligned identical telescopes and to apply a π phase shift to telescopes whose rank is defined by the Prouhet–Thué–Morse sequence. From this basic pattern, one can then build an *algebra of nulling interferometers*: 2-D nulling patterns by combining those building bricks in two dimensions, then 1-D or 2-D nulling arrays of non-identical telescopes and finally a generalisation permits the building of nulling interferometers with phase shifts different from 0 or π.

When exploring how this concept of fractal interferometers would behave in the real world, i.e. when for instance the phase shift or the distances between telescopes are no longer at their nominal values, it appears that the higher order arrays are the more robust with respect to the loss of nulling performance, including the stability with time. However when the order increases, the transmission map begins to degrade, putting a reasonable limit around 8×8 telescopes on the size of an efficient array. Since the stability of the nulling efficiency is clearly a concern today in the design of a nulling interferometer in space, it seems thus that the VDNI concept which improves significantly this performance deserves some attention. More detailed estimates are obviously required in order to confirm this property. If we consider that in the cost evaluation, the increase due to a large number of telescopes may be balanced by a much smaller and cheaper unit telescope and the fact that there is a lower recurrent cost when producing systems in series, one cannot exclude that such interferometric arrays may be an alternative.

Finally, I wish to mention that those results are based on the remarkable property discovered by Prouhet, one and half centuries ago, but which is today not as well known and used as it deserves, in my opinion. I am actually convinced that this property could be exploited in many other situations of physics or technology where a combination of complex amplitudes is involved. Many more applications are likely to come and, for instance, a new concept of a quasi-achromatic phase shifter will be proposed in the very near future, using a variant of this property (Rouan and Pelat [11]).

References

[1] R.N. Bracewell, Nature 274 (1978) 780.
[2] M. Ollivier, C. R. Physique 8 (2007) doi:10.1016/j.crhy.2007.02.0008 (this issue).
[3] A. Leger, J.M. Mariotti, B. Mennesson, et al., Icarus 123 (1996) 249.
[4] R. Angel, in: P.-Y. Bely, C.J. Burrows, G.D. Illingworth (Eds.), The Next Generation Space Telescope, 1989, p. 81.
[5] N.J. Woolf, J.R.P. Angel, in: D. Soderblom (Ed.), ASP Conf. Ser. 119: Planets Beyond the Solar System and the Next Generation of Space Missions, 1997, p. 285.
[6] D. Rouan, in: M. Fridlund, T. Henning, H. Lacoste (Eds.), ESA SP-539: Earths: DARWIN/TPF and the Search for Extrasolar Terrestrial Planets, 2003, pp. 565–568.
[7] D. Rouan, in: C. Aime, F. Vakili (Eds.), IAU Colloq. 200: Direct Imaging of Exoplanets: Science and Techniques, 2006, pp. 213–220.
[8] E. Prouhet, Comptes Rendus des Séances de l'Académie des Sciences 33 (1851) 225.
[9] O.P. Lay, Appl. Opt. 43 (2004) 6100.
[10] B. Chazelas, F. Brachet, P. Bordé, et al., Appl. Opt. 45 (2006) 984.
[11] D. Rouan, D. Pelat, in: The Direct Detection of Planets and Circumstellar Disks in the 21st Century, 2007, in preparation.