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The dynamo effect/L'effet dynamo

Simple models and time scales in the dynamo effect

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Abstract

The dynamo effect has been studied for 50 years, starting from the simple homopolar dynamo of Bullard. Numerical calculations on simple models, such as the crossed dynamos of Rikitake, display large oscillations and brutal reversals. The present Note puts emphasis on time scales. One scale τ_1, purely electromagnetic, describes inductive decay (the frequency at which the skin depth is the size of the system), the other τ_2 is magnetohydrodynamic, controlling the exchanges of kinetic and magnetic energies. Usually $\tau_2 \ll \tau_1$: a clear separation of time scales makes the physics much more transparent. We show why sharp reversals must be expected, starting either from the slow or from the fast time scale. Such a comparison clarifies the issue. ***To cite this article: P. Nozières, C. R. Physique 9 (2008).***

Résumé

Modèles simples et échelles de temps dans l'effet dynamo. L'effet dynamo est une histoire vieille de 50 ans, qui commence avec la dynamo unipolaire de Bullard. L'étude numérique de modèles simples, par exemple les deux dynamos croisées de Rikitake, met en évidence des renversements brutaux et de grandes oscillations. La présente note met l'accent sur les échelles de temps. L'une d'elles, τ_1, décrit l'amortissement inductif des courants (c'est la fréquence à laquelle l'épaisseur de peau est la taille du système). L'autre τ_2 est magnétohydrodynamique, contrôlant les échanges d'énergies cinétique et magnétique. En général $\tau_2 \ll \tau 1$: une séparation claire des deux échelles de temps rend alors la physique beaucoup plus transparente. Nous montrons que les renversements brutaux sont prévisibles, et nous décrivons les aspects qualitatifs du problème en partant soit de l'échelle lente, soit de l'échelle rapide. La comparaison est instructive. ***Pour citer cet article : P. Nozières, C. R. Physique 9 (2008).***

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1. Models

The archetypal model of the dynamo effect is the homopolar dynamo of Bullard [1], sketched on Fig. 1. A conducting rotor rotates at an angular frequency Ω, driven by a torque Γ. Brushes at the periphery and axis feed a radial current I into the rotor, which closes on a loop around the axis. If $I = 0$ nothing happens, rotation being slowly damped by a friction torque $\Gamma_{\mathrm{f}} = -\beta\Omega$. If $I \neq 0$, current in the loop produces a magnetic field $\mathbf{B}$, which together with the radial current density $\mathbf{J}$ yields a tangential force density force density $\mathbf{J} \times \mathbf{B}$. For the appropriate relative orienta-

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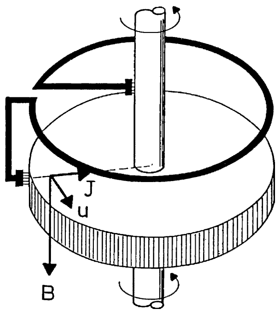

Fig. 1. A sketch of the Bullard homopolar dynamo.

tion of current loop and angular velocity the resulting electromagnetic torque Γ_{em} tends to accelerate the rotation. The instability appears beyond a threshold Ω_c, it saturates at a fixed point when Γ_{em} equilibrates Γ_{f}. The corresponding equations are elementary

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = A\Omega I$$
$$K\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \beta\Omega = \Gamma - AI^2$$

K is the moment of inertia of the rotor, A is a constant fixed by geometry. The first equation is electrical, basically Ohm's law with a resistance R, a self inductance L and the induction emf due to rotation of the conductor in the magnetic field B. The second equation is mechanical, blending inertia, friction and the magnetic torque. Note the symmetry that guarantees energy conservation: the time derivative of the total energy,

$$U = \frac{1}{2}K\Omega^2 + \frac{1}{2}LI^2$$

is the work of Γ minus heat dissipation (Joule heat + friction). Note that there are two fixed points: one goes from one to the other changing both the signs of Ω and I.

Such a model is simple only because it has few degrees of freedom. It enriches considerably if one adds more variables. The simplest example is the case of two identical coupled homopolar dynamos studied by Rikitake [2]. The dynamos are crossed, dynamo 1 feeding the loop of dynamo 2 and vice versa. The corresponding set of equations are a special case of the so called Lorenz model, first formulated in the frame of meteorology [3], subsequently as an example of a route to chaos (the corresponding literature in mathematical physics is enormous). The Lorenz model is only amenable to numerical work, which has yielded very spectacular results. The Rikitake dynamos, for instance, display very brutal reversals of the magnetic field, followed by wild oscillations. Because of these oscillations the reversals are somewhat erratic. The resemblance with the reversals of the earth magnetic field is striking: it triggered a lot of interest. Such a behaviour has two time scales, the time τ_1 *between* two reversals which is fairly long, and the fast oscillation period τ_2, also the *duration* of a reversal, which is much shorter. A similar difference of time scales is found in geophysics for the Earth's magnetic field (typically 100 000 years for τ_1, 100 years for τ_2). A numerical approach does not exploit that difference: the present work attempts to do so. Surprisingly, the physics behind the coupled Rikitake dynamos becomes fully transparent, the calculations becoming analytic as soon as the two time scales are separated. Of course such simplifications disappear if more than two variables are introduced for both the slow and fast motions: then we are back to numerics. But *separation of time scales* remains an essential asset. This work is thirty years old (it was done when the author visited Harvard in 1976). Published in 1978 [5] it remained mostly unnoticed (except by Ed. Bullard who died a few months later!).

The basic electromagnetic equations are Maxwell equations that are linear in **E**, **B**, **J**: the non-linearity comes from the Lorentz induction field $\mathbf{u} \times \mathbf{B}$ arising from the local convection velocity $\mathbf{u}(\mathbf{r})$. The hydrodynamic equations are Navier–Stokes equations (with genuine non-linear terms), an additional non-linearity arising from the Laplace force density on the fluid, $\mathbf{J} \times \mathbf{B}$. A possible route, proposed by Elsasser [4], is to find the eigenmodes of the linear problem and to project all the continuous variables on the corresponding basis, E_n, B_n, u_n: the equations become algebraic. If we are to truncate that system we may as well do it from the outset. In what follows we retain only *one* mechanical variable u: the geometry of convection is fixed and only the velocity scale remains. We retain *two* electrical variables x_1 and x_2, for instance a toroidal and a poloidal current distribution. The structure of the equations is

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = -\gamma_n x_n + \Lambda_{nm} x_m u$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = F - \beta u - \Lambda'_{nm} x_n x_m$$

As in the Bullard case F is the driving force of convection, γ_n and β describe Joule heat and viscous friction. Energy conservation relates Λ'_{nm} and Λ_{nm}. We thus recover the Lorenz model. In what follows we ignore β which is usually negligible. It is convenient to introduce polar coordinates

$$x_1 = r\cos\phi, \qquad x_2 = r\sin\phi$$

(a rotation π of the "phase" ϕ corresponds ro a reversal of the electromagnetic coordinates x_1 and x_2). The differential system takes a very simple form

$$\frac{\mathrm{d}r}{r\,\mathrm{d}t} = -a + mu, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = b + nu, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = F - mr^2$$

where a, b, m, n are all linear combinations of 1, $\cos 2\phi$ and $\sin 2\phi$ [5]. All the complexity of the model is hidden in these coefficients.

2. Time scales

The purely electric time scale is $\tau_1 \sim 1/\gamma_n$: $1/\tau_1$ is basically the frequency at which the skin depth is the size R of the system (vessel radius for a laboratory experiment, Earth radius for the earth dynamo):

$$\tau_1 \sim \frac{1}{\gamma_n} \sim \mu_o \sigma R^2$$

where μ_o is the vacuum permittivity and σ the fluid conductivity – for geophysics about 100 000 years. The purely mechanical time scale is $\tau_3 \sim 1/\beta$, limited by viscosity. In the geophysical case τ_3 is very long (10^8 years): we can safely ignore it. We will stick to that limit $\beta = 0$ for simplicity. The remaining time scale τ_2 is the MHD scale that describes exchange of kinetic and magnetic energies. It can be inferred from plain dimensional arguments

$$\tau_2 \sim \frac{1}{\sqrt{F\Lambda}}$$

Qualitatively it is the time it takes an Alfven wave to cross the system

$$\tau_2 \sim \frac{R}{B}\sqrt{\rho\mu_o}$$

where ρ is the mass density. We are interested in the limit $\tau_2 \ll \tau_1$, which is the case for the Earth dynamo. Such a limit corresponds to a large driving force F.

2.1. From slow to fast

Fast MHD oscillations are quenched if we assume mechanical equilibrium: we neglect inertia $\mathrm{d}u/\mathrm{d}t$ in the mechanical equation. The amplitude r is just $\sqrt{F/m}$. Carrying that result in the equation for $\mathrm{d}r/\mathrm{d}t$ we find

$$u = \frac{a}{m} - \frac{1}{2}\frac{m'}{m^2}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

In the end the equation of motion of the phase may be written

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{N(\phi)}{D(\phi)}$$

where the numerator N and denominator D are known linear functions of $\cos 2\phi$ and $\sin 2\phi$. Note the symmetry $\phi \to \phi + \pi$: the possibility of reversals is built in. The slow motion thus appears as a single degree of freedom $\phi(t)$.

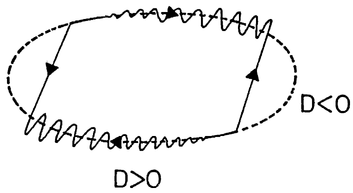

Fig. 2. The escape mechanism on the slow limit cycle.

Depending on coefficients N and D may or may not have zeroes. If neither has, the slow trajectory is a limit cycle which goes on for ever. Zeroes of N correspond to fixed points. There are four, two stable $\phi_1, \phi_1 + \pi$, two unstable $\phi_2, \phi_2 + \pi$. Motion stops at the first stable fixed point encountered. The interesting case is the one where D has zeroes while N does not. Such a singularity signals a change of the transverse stability of the slow orbit (motion perpendicular to the orbit in the (r, ϕ, u) space). In order to study that stability we keep ϕ fixed and we linearize the two coupled equations for r and u: we thus find for the transverse fluctuation δu the equation of motion

$$\frac{\mathrm{d}^2\delta u}{\mathrm{d}t^2} = -\frac{2FD}{m}\delta u = -\omega^2\delta u$$

Transverse motion is an oscillation, MHD in nature, with a period ω^{-1} which is nothing but the fast scale τ_2 described earlier. When $D > 0$ the slow orbit is *stable*. When D becomes negative the transverse motion is unstable: it grows exponentially, driving ϕ very quickly to the next stable region, on the opposite branch of the slow orbit (remember that ϕ has *two* stable fixed points). We thus have a very simple mechanism for the reversals of the magnetic field. Two remarks are in order:

(i) As usual the approach of an instability *accelerates* the drift along the slow orbit, as shown by the zero in the denominator $D(\phi)$: the instability "sucks" the drift;

(ii) When the magnetic field jumps it triggers a *large amplitude oscillation* around the new branch of the slow orbit, which slowly decays. The behaviour is sketched on Fig. 2. Such a damping is generic: it automatically appears as soon as one pushes the expansion to the next order in $1/F$, i.e. in τ_2/τ_1. We thus expect the damping time of "postjump" oscillations to be of order τ_1. It is a matter of numbers, but we may reasonably expect that oscillations are still there when the next reversal occurs. Such a *memory* effect between two reversals makes the behaviour very erratic, as shown for instance in numerical studies of coupled Rikitake dynamos.

2.2. *From fast to slow*

The preceding picture describes an helical trajectory around the slow limit cycle, interrupted by reversals that trigger a new large amplitude helix. The expansion we carried is of no avail if the amplitude of the helix is large. One must instead approach the problem from the opposite end, ignoring in a first step the pitch of the helix. We are left with a large amplitude oscillation of a *single* variable, with a period $\sim \tau_2$, which we can understand easily even for complicated potential wells. The next step is the *slow drift* of that oscillation along the slow orbit, on a time scale τ_1. Such a philosophy is familiar in MHD when the cyclotron frequency is much larger than other time scales: the cyclotron orbit is slowly "guided" along the trajectory of its center. Such an alternate picture provides a very clear picture of triggered oscillations and memory effects we just mentioned.

In order to suppress the drift we just need to eliminate Joule dissipation. Our set of differential equations becomes

$$\frac{\mathrm{d}r}{r\,\mathrm{d}t} = mu, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = nu, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = F - mr^2$$

An obvious constant of the motion follows from

$$r\frac{\mathrm{d}r}{\mathrm{d}t} + u\frac{\mathrm{d}u}{\mathrm{d}t} = Fu = \frac{F}{n}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

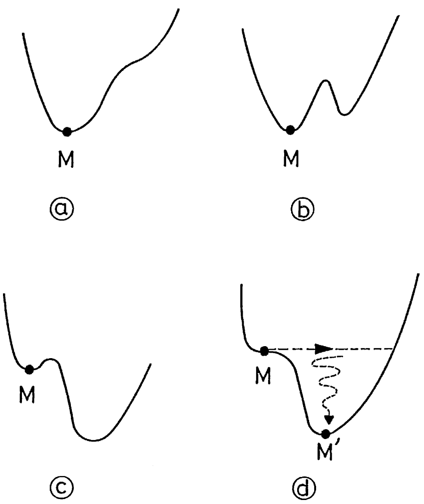

Fig. 3. Drift of the fast oscillation and the escape mechanism.

That just expresses energy conservation (we are using reduced units for simplicity: constant factors should be disregarded). The equation is readily integrated into

$$r^2 + u^2 = F \int_{\phi_o}^{\phi} \frac{\mathrm{d}\phi}{n} + A$$

Moreover we note that $\mathrm{d}r/r\,\mathrm{d}\phi = m/n$ where n and m are known functions of ϕ. We can also integrate this equation, thereby obtaining

$$r = \exp\left[\int_{\phi_o}^{\phi} \frac{m\,\mathrm{d}\phi}{n}\right]$$

A and ϕ_o are integration constants: ϕ_o yields the position of the guiding center along the slow trajectory, A measures the amplitude of the fast oscillation. Putting all of that together we extract u as function of ϕ.

$$u(\phi) = \left[A - G(\phi, \phi_o)\right]^{1/2} = \frac{1}{n}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Such an equation describes an oscillation in a potential well $G(\phi, \phi_o)$ which needs not be harmonic. The oscillation reverses when $G = A$, as usual.

The potential changes when ϕ_o turns around the slow orbit: all the physics is in the shape of G, which has a minimum at $\phi = \phi_o$. If G has two minima its evolution as a function of ϕ_o is sketched on Fig. 3. In situation (a) the system is at rest at the bottom of the potential well (on the slow limit cycle). As ϕ_o moves the potential distorts from (a) to (d), where the minimum M disappears, a standard bifurcation. The system *escapes to the next minimum* (the field reversal), starting a large amplitude oscillation: we recover the same picture, viewed from the other end. As ϕ_o moves further the new potential well distorts again and the process repeats, with one new feature: the oscillation triggered by the first jump may still be there when the next bifurcation approaches! The next jump then occurs before bifurcation, in the situation (b) when the residual amplitude exceeds the height of the potential barrier. Such *premature jumps* make the behaviour somewhat erratic.

In the preceding section we studied small fast oscillations close to the slow orbit. In much the same way we can study slow drift of the large oscillation: we restore Joule dissipation and treat is perturbatively. At small amplitude it is just our old calculation, at large amplitude it is more delicate: *oscillations modify the drift*. The formulation is more complicated (basically an "adiabatic" separation of time scales): a sketch of the method is given in the original article [5]. It does not affect the physical picture.

3. Conclusion

This brief Note is mostly qualitative: we did not attempt to give accurate calculations. Our goal was mostly to describe the structure of the calculation as simply as possible. Starting from the Bullard dynamo with only one electric and one rotation parameter, we found that adding a second electromagnetic degree of freedom changes the physical picture drastically. That is not new: it is just the Lorenz model! One important feature is the appearance of *two time scales*, a slow one describing the coupling of the two electromagnetic degrees of freedom, a fast one describing exchanges between magnetic and kinetic energies, a typical MHD issue. We believe that *separation of these two time scales* is a prerequisite of any description of dynamos. In this model with only two slow variables, conservation laws restrict the slow trajectory to a limit cycle: it is clear that additional variables will complicate the behaviour dramatically. The issue is well known in mechanical oscillators: when the dimension exceeds 2 trajectories are no longer closed and chaos quickly ensues. Note that adding new variables will suppress the periodicity of reversals in a limit cycle. That is yet another reason (on top of premature jumps) for the erratic behaviours often observed.

The only virtue of the present discussion is to emphasize simply the physics behind slow and fast processes: realistic calculations will necessarily be numerical. We find that field reversals are in no way surprising: they are one of the standard scenarios that follow from time scale separations. These reversals trigger large oscillations: a crucial issue is their damping rate, slower or faster than the time between two reversals? Only specific models can answer such a question. One technical feature might be improved: the formulation of slow drift for large amplitude oscillations – at the loss of simplicity unfortunately. One thing is sure: dynamo effect is a fascinating world!

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