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On invariants of quark and lepton mass matrices in the Standard Model

Une approche basée sur les invariants construits à partir des masses (quarks et leptons), dans le cadre du Modèle Standard

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ABSTRACT

In this article I present the motivation for introducing the invariant functions of mass matrices, based on my own work, and give some examples. Since their introduction in 1985, in the framework of the Standard Electroweak Model, they have been used by many authors. Some authors have gone further along this path and have studied the extensions of this concept to frameworks beyond the Standard Model.

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R É S U M É

Cet article présente une formulation basée sur des invariants construits à partir des matrices de masses, selon l'approche développée par l'auteur, et fournit des exemples d'applications. Depuis leur introduction en 1985 dans le cadre du Modèle Standard des Interactions Electro-faibles, ces invariants ont été utilisés par de nombreux auteurs, dont certains ont étendu l'approche à un cadre plus large que le Modèle Standard.

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1. Introduction

Since many decades the “flavor puzzle” has been on the minds of many physicists. However, through the course of history, the language has changed and the emphasis has shifted. Nonetheless, the basic issues keep on returning as rephrased questions.

In 1983 I had the privilege of attending the annual meeting of the Norwegian Physical Society. One of the plenary speakers was the great Dutch physicist Hendrik Casimir (1909–2000). His name is familiar to most physicists from the “Casimir effect”. In group theory and particle physics the “Casimir invariants” play a central role. Casimir, as a young man, had been at centers of “action” in theoretical physics, such as with Niels Bohr in Copenhagen, and with Wolfgang Pauli, as his assistant, in Zürich. Later on, in 1946, he had left physics and gone to industry. Therefore, it was particularly interesting to hear what this powerful voice from the past had to say. The most surprising statement he made was that the greatest puzzle in physics is why the ratio of masses of the proton and the electron is 1836! Casimir was thus telling us that already in the “old days”, i.e., before the second world war, physicists had been concerned with what we now call the flavor problem, albeit in a much simplified version. Richard Feynman (1918–1988) was also puzzled by the flavor problem. He considered the question “why the muon weighs” to be one of the most important ones in physics. By now, every particle physicist has

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heard of the famous statement “who ordered that?” by Isidor Rabi (1898–1988) when the muon was discovered. The flavor problem has always been considered to be “super-important”. However, in spite of a huge amount of collected knowledge about this matter, the basic questions have not yet been answered.

2. Models of mass matrices

The question of quark masses and mixings plays a leading role in the flavor puzzle. During several decades theorists have been inventing models for mass matrices hoping to gain an insight into the “underlying principles”. The advent of grand unified theories, especially the Georgi–Glashow model, made it plausible that there may be a connection between the lepton and quark masses. Since then many models have been proposed. In order to make the mass matrices as simple and predictive as possible, many authors have tried to put in as many zeros as possible into them. These zeros give what is called the “texture” of the mass matrices. These textures have been studied and even tabulated. However, it is always good to have some measure of reliability of a proposed model. In the case of textures an obvious question is: what is the significance of these zeros? I shall return to this question at the end of this article.

In discussing the question of mass matrices it is quite appropriate to remind ourselves of the lesson from special relativity. Consider, for example, the collision of two particles A and B. In the rest frame of B, the momentum of B is zero while that of A is nonzero. By going to the rest frame of A, the zero in the momentum of B evaporates and moves to the momentum of A. Furthermore, in their center of mass system both zeros evaporate. Obviously, there is nothing special about a zero in this case. It has no deeper “meaning”. The meaningful quantities are kinematical invariants such as $s = (p_A + p_B)^2$, where the p 's stand for four-momenta. The masses are, of course, frame-independent quantities, $M_A^2 = p_A^2$, etc. Similarly, in the case of quark and lepton masses and mixings there are frame-dependent and frame-independent quantities, but now in the flavor space. Only frame-independent results can be trusted.

In the case of special relativity we have learned what is meant by (inertial) frames and know the transformation rules. But what are the frames in flavor physics? The answer is that these frames depend on the model used and the invariants within it. Let us now turn to the case of quarks in the Standard Model.

3. The invariants in the Standard Model

In this short article, all I wish to remind the reader about the Standard Model with n families of quarks and leptons, is that the Higgs sector produces what is called mass matrices for the up-type and down-type quarks as well as for leptons. These are all n -by- n matrices. For the neutrinos there is an additional “complication” because the right-handed neutrinos are gauge-singlets and therefore can couple to themselves. Thus, there could be additional (Majorana) mass terms in the theory. However, within the framework of the Standard Model, these terms are not considered to be “nice”, because one would like all masses to originate from the phenomenon of spontaneous symmetry breaking.

Let us consider the case of the quarks and denote the mass matrices generated for the up-type and down-type quarks by M_u and M_d respectively. In the Standard Model with n families these are general n -by- n matrices that need not even be Hermitian.

The crucial observation that leads to the concept of invariant functions of mass matrices is that these matrices are not uniquely defined but are frame-dependent in the sense that given any such pair M_u and M_d , one can obtain an infinite number of other equivalent pairs by unitary rotations in the flavor space. The measurable quantities must be “invariant functions” under such rotations. These invariants were first introduced in [1] and studied in more detail in [2]. Actually, what enters, in the Standard Model, is the pair¹

$$S_u \equiv M_u M_u^\dagger, \quad S_d \equiv M_d M_d^\dagger \quad (1)$$

In the “frame” X these matrices are replaced by $X S_u X^\dagger$ and $X S_d X^\dagger$, where X is unitary. Thus the invariant functions of mass matrices, $f(S_u, S_d)$, satisfy the condition

$$f(S_u, S_d) = f(X S_u X^\dagger, X S_d X^\dagger) \quad (2)$$

Trivial examples of such invariant functions are traces over chains of powers of the mass matrices, i.e., $\text{tr}(S_u^i S_d^j S_u^k S_d^l \dots)$. As is well known, one needs to diagonalize the mass matrices in order to identify the physical states. Let us consider the case that nature seems to have chosen, i.e., $n = 3$. First we note that there are two “extreme frames”, one in which the up-type quark mass matrix is diagonal, i.e.,

¹ We know from theory of matrices that an arbitrary n -by- n matrix M may be diagonalized through the operation $V_L M V_R^\dagger$ where the two V 's are unitary matrices and the index L (R) tells us that the unitary matrix in question multiplies M from the left (right). Evidently $V_R M^\dagger V_L^\dagger$ is also diagonal. Moreover, the matrices $V_L M M^\dagger V_L^\dagger$ and $V_R M^\dagger M V_R^\dagger$ are both diagonal with non-negative eigenvalues which we identify with the squares of the appropriate quark masses. However, due to the fact that there are no right-handed currents (in the Standard Model) the matrix V_R may be absorbed into the definition of the right-handed quarks and thus “disappears”, while the V_L 's from the up and the down quark sectors build up the quark mixing matrix. As this matter is discussed in many textbooks we abstain from giving more details here.

$$S_u = \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix}, \quad S_d = V \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix} V^\dagger \quad (3)$$

where the m 's refer to the quark masses and V is the quark mixing matrix. The other extreme frame is one in which the down-type quark mass matrix is diagonal, i.e.,

$$S_d = \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix}, \quad S_u = V^\dagger \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix} V \quad (4)$$

These constructions are analogs of going from one extreme kinematical frame, where particle B is at rest to the other when the particle A is at rest.

The reader wonders why the invariant functions of mass matrices were introduced to begin with. The original reason was that in 1980s one was looking for a measure of CP violation in the Standard Model. The question asked was: could CP be maximally violated in the quark sector of the Standard Model? After all parity is maximally violated in interactions mediated by the W -bosons. As is often the case, there were conflicting opinions on what was meant by maximal CP violation. Some authors were advocating that CP is maximally violated if the CP phase in the quark mixing matrix is 90 degrees. However, such a definition makes no sense because there is no such unique CP phase. This phase is convention-dependent: your CP phase is in general a function of my CP phase and mixing angles. The point raised by the present author was that such a measure can only make sense if it is frame-independent, i.e., it has to be an invariant function of the quark mass matrices.

4. The invariant function for CP violation with three families

In spite of the fact that, as mentioned above, there is no unique definition of the CP phase in the Standard Model with three families, one may show that there exists a unique quantity (up to an overall constant) related to CP violation [1]. This quantity is the determinant of the commutator of the quark mass matrices, $\det[S_u, S_d]$,

$$\det[S_u, S_d] = 2iJ \cdot v(S_u) \cdot v(S_d) \quad (5)$$

where J is an invariant whose magnitude equals twice the area of any of the six by now well-known unitarity triangles [3]. The quantities $v(S_u)$ and $v(S_d)$ are (Vandermonde determinants) given by

$$v(S_u) = (m_u^2 - m_c^2)(m_c^2 - m_t^2)(m_t^2 - m_u^2) \quad (6)$$

$$v(S_d) = (m_d^2 - m_s^2)(m_s^2 - m_b^2)(m_b^2 - m_d^2) \quad (7)$$

Looking into literature, one sees that the determinant in Eq. (5) appears, albeit sometimes in disguise, in every computation involving CP violation, in the three-family version of the Standard Model. It appears, in all its glory, when all the six quarks enter in the calculation on equal footing but otherwise in a well-defined truncated form, where some factors in the determinant are missing due to assumptions made [2]. Examples of the first kind are the renormalization of the θ -parameter of QCD by the electroweak interactions and the calculation of the baryon asymmetry of the universe in the Standard Model. In the case QCD all the six quarks are "running" in the loops in an equivalent manner. An example of the second kind is the computation of the electric dipole moment of the neutron in the quark model where one computes the contribution of the up and down quarks. Consider the contribution of the down quark. Since in such a computation, the down quark appears in the external legs of the relevant Feynman diagrams, it is tacitly assumed that we know the identity of this quark. In other words, $m_d \neq m_s$ and $m_d \neq m_b$. Of course, if say m_d were equal to m_s there would have been no way to distinguish the d and the s quarks. Therefore, in the calculation of the electric dipole moment of d quark, the factors $(m_d^2 - m_s^2)$ and $(m_b^2 - m_d^2)$ do not appear but all the other factors in the determinant are present.

It should also be mentioned that the absolute values of the elements of the quark mixing matrix are measurable quantities and thus can be expressed as invariant functions. These functions were constructed in [2] (see also [4]).

The above commutator is the simplest in a family of commutators of functions of mass matrices (see the first paper in [1]),

$$C(f, g) \equiv [f(S_u), g(S_d)] \quad (8)$$

f and g being functions that are diagonalized with the same unitary matrices that diagonalize S_u and S_d respectively. The determinants of these commutators, which are also invariant functions, are given by

$$\det[f, g] = 2iJ \cdot v(f) \cdot v(g) \quad (9)$$

where $v(f) \equiv v(f(S_u))$, and $v(g) \equiv v(g(S_d))$. More explicitly

$$v(f) = \sum_{i,j,k} \epsilon_{ijk} f_j f_k^2 = (f_1 - f_2)(f_2 - f_3)(f_3 - f_1) \quad (10)$$

The f_j denote the three eigenvalues of the matrix $f(S_d)$ and the quantities related to the down-type quarks are defined similarly. An essential point is that Eq. (9) holds irrespectively of whether f and g are Hermitian or not. This property makes the above formalism applicable to neutrino oscillations.

5. Other applications

The space allocated to this paper allows me only to quote a few other results. Similar invariants enter when one studies *CP violation in the neutrino sector* of the Standard Model [5]. For this case, the neutrino Majorana mass matrix is largely irrelevant and we may introduce for the leptons the analogs of the pair in Eq. (1)

$$S_\nu \equiv M_\nu M_\nu^\dagger, \quad S_l \equiv M_l M_l^\dagger$$

and the commutators

$$\Delta^\pm \equiv [e^{\pm 2i\xi S_\nu}, S_l] \quad (11)$$

where ξ is a kinematical parameter (related to oscillation length). The determinants of these commutators are invariant functions of lepton mass matrices related to CP violation in the leptonic sector. Using Eq. (9) we have

$$\det \Delta^\pm = 2i J_{\nu \cdot \nu}(S_l) \cdot \nu(e^{\pm 2i\xi S_\nu}) \quad (12)$$

Here J_ν is the leptonic analog of the CP invariant of the quark mixing matrix. Thus J_ν is simply twice the area of any of the six leptonic unitarity triangles. The two ν 's are Vandermonde determinants. Because of lack of space, I don't give the details here. It is indeed (at least intellectually) gratifying that such invariants can be constructed. One can also construct the corresponding invariants for the case of neutrino oscillations in matter (see [5]).

It is now appropriate to return to the question of textures that was briefly mentioned in Section 2. Indeed theorists love zeros in mass matrices as they make life simpler for them. However, as we have seen such zeros are not invariants! They will in general evaporate by going to a different frame. Therefore, they don't mean anything. Also the so called the quark-lepton complementarity relations suffer from not being relations among invariants and are thus ill-defined. Some aspects of this matter are discussed in [6].

6. Outlook

The issue of masses and mixings remains an unsolved problem that deserves our attention. However, we should always keep in mind that important results should be frame-independent. So, if you have an important message about masses and mixing to transmit, you should be able to formulate it in an invariant form.

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