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Some basic principles of a "LISA"

Quelques principes de base d'un « Lisa »

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ABSTRACT

A Laser Interferometer Space Antenna (LISA) is a concept studied and developed since a few decades both by European and American teams. Its aim is to study the gravitational wave signals emitted by astrophysical sources such as supermassive black hole (SMBH) coalescences, captures of compact objects by SMBHs, compact galactic binaries, etc. The LISA mission has been first an ESA/NASA mission (1998–2011), then became an ESA mission under the name of NGO (2012): it could hopefully be proposed for selection in 2013. The very basic principles of such a mission still deserve a presentation, being quite generic: this is the aim of the present article.

RÉSUMÉ

Une antenne interférométrique spatiale (Lisa) est une idée étudiée et développée depuis quelques décennies en Europe et aux États-Unis. Son but est d'étudier les signaux d'ondes gravitationnelles émis par des sources astrophysiques comme des coalescences de trous noirs supermassifs, par des captures d'objets compacts par des trous noirs supermassifs, par des systèmes binaires compacts galactiques etc. La mission Lisa a été successivement une mission ESA/Nasa (1998–2011), puis devint une mission ESA sous le nom de NGO en 2012, en compétition avec deux autres missions de classe L1, et n'a pas été retenue par le conseil « ESA Space Science Advisory Committee » pour une sélection par le « Science Program Committee ». Cependant, on attend un nouvel appel à projets de l'ESA en 2013 où une mission de type LISA pourra être proposée à nouveau. Les principes d'une telle mission méritent donc d'être présentés ici.

1. Introduction

It is known since forty years that terrestrial gravitational wave (GW) antennas cannot operate below a few Hz, due to seismic excitation of suspended mirrors (test masses) by direct Newtonian coupling with moving masses underground. The very low-frequency band (below 1 Hz) is however extremely interesting for astrophysics. Evidences of supermassive black holes (SMBH) in galaxy bulges are regularly reported, and even collisions of galaxies involving such SMBH. On the other hand, the GW amplitude generated by SMBH interactions ending with inspirals and merging is so large that such events could be detected at cosmological distances by an antenna of sensitivity comparable to that of present ground-based antennas, but shifted in the very low frequency domain. This is why space antennas, escaping terrestrial issues, have been devised,

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designed and planned since the beginning of the long-term search for direct GW detection. Several projects were proposed by Peter Bender (LAGOS, ∼1981), by Ronald Hellings (SAGITTARIUS, ∼1990). In December 1993, the project of a large spatial antenna named LISA (Laser Interferometer Space Antenna) was proposed as a "cornerstone mission" to the European Space Agency (ESA) in the framework of the "Horizon 2000 plus" program. It became then a common project shared by the ESA and the National Aeronautic and Space Administration (NASA) until 2011 [1]. The National Research Council of the USA, at the end of the "Astro2010" decadal survey (Astronomy & Astrophysics), recommended LISA as the second priority among large space missions for the 2010–2020 decade. Then, the cuts in the NASA budget lead to a redefinition of LISA as an ESA-only mission, with a corresponding downsizing. Under the name NGO/eLISA, the new project was in competition with two other class L1 missions, namely ATHENA (X-ray observatory) and JUICE (exploration of the moons of Jupiter). Recently (April 2012), the Space Science Advisory Committee (SSAC) of the ESA has chosen JUICE as the mission to be proposed to the Science Program Committee (SPC). The elimination of LISA from the landscape of the fundamental science missions seems not irreversible, because a new call for proposals will likely be emitted by the ESA in 2013. Anyway, the principles of a mission studied and developed since more than twenty years by an international collaboration still deserve presentations and discussions.

If in General Relativity (GR) we trust, frequent events of high interest could be detected and analyzed during the few years of operation of a LISA-like mission. We refer to such a mission not as "LISA" but as "a LISA" to emphasize the generic character of the discussion. We do not address here the issues of the payload structure nor the technical problems related to drag-free operations apart from a brief description. We briefly summarize the astrophysical interest of such a mission, which has been widely presented in a number of papers (see, for instance, [2]), and we focus on less frequently discussed mathematical and physical points regarding orbital motion, optical links, phase noise cancellation and the effects of simple gravitational signals.

2. Physics at low gravitational frequency

The mechanisms responsible for gravitational emission are present in the Universe at different scales and different occurrence rates. Apart from violent explosions in supernovae events, all are related to binary systems. Binary systems may involve "light" objects like neutron stars or stellar-class black holes, i.e. objects having masses of the order of a fraction of a solar mass up to a few dozen of solar masses. This corresponds the high frequency part of the GW spectrum, and roughly to the detection band (10 Hz to 1 kHz) of the terrestrial antennas (LIGO, Virgo, etc.). Above that class of mass, all situations may be imagined, up to binary systems involving massive or supermassive black holes (above 10^4 solar masses), corresponding to the low frequency part of the GW spectrum. LISA was designed to address the frequency band roughly from 0.1 mHz to 0.1 Hz. The detection philosophy is very different for LISA, compared to terrestrial antennas. In terrestrial antennas, targets are light objects, thus weakly emitting, being consequently detectable at short distances (tens of Mpc) and thus statistically rare. The detection process consists thus in searching for rare events in a stream of pure instrumental noise. In a low frequency antenna like LISA, some targets are associated with very massive objects, thus strongly emitting, being consequently detectable at cosmological distances, and thus statistically frequent. In fact, we expect several different sources to be present simultaneously in the data flow.

These two extreme domains of GW physics are therefore complementary. Present terrestrial antennas have reached their nominal sensitivity and in accordance with plans, advanced versions are now being developed, allowing us to transform sophisticated optomechanical systems into real GW observatories within a few years. Meanwhile, we can hope that LISA will benefit from an international support, and will complete our observational GW spectrum.

2.1. Physics around supermassive black holes

Most of galaxies are known to host supermassive black holes (SMBH) in their bulges. On the other hand, collisions of galaxies are currently observed. We expect therefore several SMBH interacting at some degree presently in the volume of universe potentially observable by a LISA. Several mechanisms have been proposed for explaining the presence of such SMBH's in galaxies: among those, capture of nearby objects, and in particular, other smaller black holes. If the assumption is correct, it means that such events continuously happen in a large volume of universe. involving millions of galaxies. Coalescences of SMBH could be detected and analyzed by a LISA at the rate of several per year. This would be a unique opportunity to test General Relativity in the regime of strong gravitational field. Capture of smaller compact objects (neutron stars or stellar black holes) results in complex orbits determined by their masses, but also by the spins of the interacting bodies. These Extreme Mass Ratio Inspirals (EMRI) would occur with a high signal-to-noise ratio (SNR). A detailed analysis of the signal would allow us to accurately test the Kerr geometry supposed to describe the field of a rotating SMBH.

2.2. Galactic binaries

Compact binaries are a large population in our galaxy, weakly emitting, but at short distances. They are expected to generate a continuum of waves in the very low frequency range (roughly below 1 mHz). In this continuum, some are even nearer and among those, some are optically known as well as their coordinates in directories, so that the emitted radiation can be modeled in detail. It is thus of the greatest interest to detect and characterize the corresponding signals in order to check the transfer functions of LISA that are known as "verification" binaries.

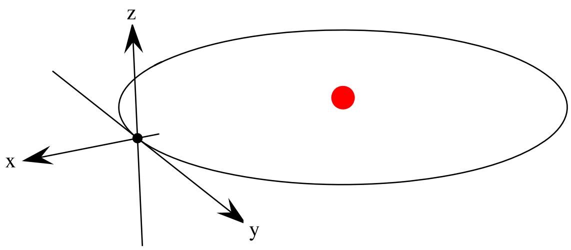

Fig. 1. CW co-moving frame.

2.3. Cosmological background

The equivalent of the Cosmological Microwave Background is predicted by various cosmological models based on GR, describing the early Universe. In certain configurations, a LISA could allow us to put constraints on the spectral density of such a background.

3. How to make a stable constellation of orbiting spacecraft

The principle of LISA is to read the GW signature in the perturbation undergone by six infrared optical beams linking three spacecraft in triangular formation orbiting the Sun. More precisely, the 5 Mkm distance foreseen between spacecraft allows an interaction with a GW to cause a modulated frequency shift of the sent light wave with respect to the stable frequency of the receiver's local oscillator. The first questions to be examined are thus relative to the spacecraft orbits and to propagation of light over long distances.

3.1. Orbitography

LISA involves three spacecraft in heliocentric orbits with their respective orbit parameters chosen in order to insure a stable triangular constellation (i.e. in which the inter-spacecraft distances are nearly constant). This is a nontrivial question, and the theory leading to such a result deserves a short discussion.

3.1.1. The Clohessy and Wiltshire equations

The three spacecraft are at large distances from each other, but small compared to the orbit of their center of mass, assumed to orbit the Sun on a 1-AU radius orbit. If namely $R \sim 150$ Mkm is the order of magnitude of the Earth's orbit radius, and if we denote by L the common inter-spacecraft distance, we have $\alpha = L/2R \sim 1/60$. We may consider the center of mass as a reference point and see the spacecraft as moving objects with respect to that reference point. In this context, we can use the Clohessy & Wiltshire (CW) equations [3] obtained by expanding the Sun's gravitational potential in the neighborhood of an orbiting reference point, introducing a comoving reference frame (see Fig. 1) having a x axis normal to the orbit in the orbital plane, a y axis tangential to the orbit, still in the orbital plane, and a z axis orthogonal to the two preceding ones (i.e. normal to the orbital plane). The CW equations are as follows:

$$\begin{cases} \ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = 0 \\ \ddot{y} + 2\Omega\dot{x} = 0 \\ \ddot{z} + \Omega^2 z = 0 \end{cases} \tag{1}$$

where $\Omega = 2\pi/1$ year. A solution excluding runaway terms and depending on four arbitrary constant x_0, y_0, z_0, $\dot{z}_0$ is:

$$\begin{cases} x(t) = \dfrac{1}{2} y_0 \sin\Omega t + x_0 \cos\Omega t \\ y(t) = y_0 \cos\Omega t - 2x_0 \sin\Omega t \\ z(t) = z_0 \cos\Omega t + \dfrac{\dot{z}_0}{\Omega} \sin\Omega t \end{cases} \tag{2}$$

The distance between the current point to the reference one is:

$$d^2 = x(t)^2 + y(t)^2 + z(t)^2 = \left(\frac{y_0^2}{4} + 4x_0^2 + \frac{\dot{z}_0^2}{\Omega^2} \right) \sin^2\Omega t + \left(x_0^2 + y_0^2 + z_0^2\right) \cos^2\Omega t + \left(\frac{2z_0\dot{z}_0}{\Omega} \right) \sin\Omega t \cos\Omega t$$

One can obtain $d = cst$ by requiring that the terms of frequency Ω et 2Ω in phase and in quadrature identically vanish. This is equivalent to:

$$z_0^2 - \frac{\dot{z}_0^2}{\Omega^2} = 3\left(x_0^2 - \frac{y_0^2}{4} \right)$$

and

$$\frac{2z_0\dot{z}_0}{\Omega} = 3x_0 y_0$$

The solution of the problem is found by adding the first equation to i times the second, giving one complex equation:

$$\left(z_0 + i\,\frac{\dot{z}_0}{\Omega}\right)^2 = 3\left(x_0 + i\,\frac{y_0}{2}\right)^2$$

which yields immediately:

$$z_0 = \mu\sqrt{3}x_0, \qquad \frac{\dot{z}_0}{\Omega} = \frac{1}{2}\mu\sqrt{3}y_0$$

with $\mu \equiv \pm 1$, so that the motion equations reduce again:

$$\begin{cases} x(t) = \dfrac{1}{2}\rho_0 \cos(\Omega t - \phi_0) \\ y(t) = -\rho_0 \sin(\Omega t - \phi_0) \\ z(t) = \mu\rho_0 \dfrac{\sqrt{3}}{2}\cos(\Omega t - \phi_0) \end{cases} \tag{3}$$

where:

$$\rho_0 = \sqrt{4x_0^2 + y_0^2}, \qquad \tan\phi_0 = \frac{y_0}{2x_0}$$

After a rotation of angle $\mu\pi/3$ around the y axis, then a rotation of angle $-\Omega t$ around the new z axis, we get in the new coordinates:

$$x'(t) = \rho_0\cos\phi_0, \qquad y'(t) = \rho_0\sin\phi_0, \qquad z'(t) = 0$$

We conclude that requiring the stability of the distance to the reference point leads to install the body in such a way that it is at rest with respect to a plane inclined at $\mu\pi/3$ on the ecliptic and counter-rotating with a period of 1 year. The final conclusion is that any constellation having such nodes is stable (at least in this approximate theory).

3.1.2. A simple model of orbits

Knowing that the LISA plane must be inclined by $\pi/3$ with respect to the ecliptic, we can place one spacecraft on an orbit in such a way that its location at perihelion joined to the center of mass makes an angle of $\pi/3$ with respect to the ecliptic. Recall that a Keplerian orbit of eccentricity e can be parameterized in its own plane (a being the semi-major axis) by:

$$\begin{cases} x = a(\cos E - e) \\ y = a\sqrt{1-e^2}\sin E \\ z = 0 \end{cases} \tag{4}$$

where $E(t)$ is the eccentric anomaly, related to time through the implicit equation:

$$E(t) - e\sin E(t) = \Omega t \quad (\Omega \equiv 2\pi/1\ \text{year})$$

If we give an inclination ϵ of the orbital plane with respect to the ecliptic, and a semi-major axis equal to 1 AU, this becomes (R being the radius of the Earth's orbit, 1 AU):

$$\begin{cases} x(t) = R(\cos E - e)\cos\epsilon \\ y(t) = R\sqrt{1-e^2}\sin E \\ z(t) = -R(\cos E - e)\sin\epsilon \end{cases} \tag{5}$$

Let us denote by L the desired inter-spacecraft distance, and introduce the small parameter $\alpha = L/2R$. Then, the preceding condition (i.e. the magic $\pi/3$ angle) is fulfilled if:

$$\epsilon = \arctan\left[\frac{\alpha}{1 + \alpha/\sqrt{3}}\right] \tag{6}$$

and

$$e = \sqrt{1 + \frac{2\alpha}{\sqrt{3}} + \frac{4\alpha^2}{3}} - 1 \tag{7}$$

Keeping in mind that LISA is rotating in its own plane with a period of 1 year, we see that a second spacecraft will occupy the location of the first one four months later. In the same way, the third spacecraft will arrive at the same place eight months later. The orbit of spacecraft #i ($i = 1, 2, 3$) will therefore be deduced from (5) by:

$$\begin{cases} X_i(t) = x(t)\cos\theta_i - y(t)\sin\theta_i \\ X_i(t) = x(t)\sin\theta_i + y(t)\cos\theta_i \\ Z_i(t) = z(t) \end{cases} \tag{8}$$

where $\theta_i \equiv 2(i-1)\pi/3$, and where, moreover, the eccentric anomaly is E_i, such that:

$$E_i(t) - e\sin E_i(t) = \Omega t - 2(i-1)\pi/3$$

If we restrict the calculation at first order in e, ϵ, α, we get for instance for spacecraft #1,2:

$$\begin{cases} X_1 = R\big[\cos\phi - e\big(1+\sin^2\phi\big)\big] + \mathcal{O}\big(e^2\big) \\ Y_1 = R[\sin\phi + e\sin\phi\cos\phi] + \mathcal{O}\big(e^2\big) \\ Z_1 = -R\sqrt{3}e\cos\phi + \mathcal{O}\big(e^2\big) \end{cases}$$

then

$$\begin{cases} X_2 = R\left[\cos\phi + e\left(\frac{3}{4} - \frac{1}{4}\cos 2\phi + \frac{\sqrt{3}}{4}\sin 2\phi\right)\right] + \mathcal{O}\big(e^2\big) \\ Y_2 = R\left[\sin\phi - e\left(\frac{3\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\cos 2\phi + \frac{1}{4}\sin 2\phi\right)\right] + \mathcal{O}\big(e^2\big) \\ Z_2 = -R\sqrt{3}e\left[-\frac{1}{2}\cos\phi + \frac{\sqrt{3}}{2}\sin\phi\right] + \mathcal{O}\big(e^2\big) \end{cases}$$

The distance between the two spacecraft is obviously:

$$D^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 = 12e^2R^2 + \mathcal{O}\big(e^3\big)$$

But at the first order, we have:

$$e = \frac{\alpha}{\sqrt{3}}$$

so that, after some algebra, we find:

$$D^2 = 12e^2R^2 \quad \Rightarrow \quad D = \sqrt{12}\frac{\alpha}{\sqrt{3}}R = L$$

showing, at this level of approximation, the stability of the triangular formation.

3.1.3. Orbital evolution of LISA

In the barycentric frame of the Solar System, we can, after the preceding calculations, represent the motion of LISA by combination of a circular orbital motion of the center of mass:

$$\vec{r}_0(t) = \begin{pmatrix} R\cos\Phi \\ R\sin\Phi \\ 0 \end{pmatrix} \quad (\Phi \equiv \Omega t) \tag{9}$$

with a first-order (in e) correction, so that the whole orbital motion of spacecraft #i is parameterized by:

$$\vec{X}_i = \vec{r}_0 + 2eR\vec{u}_i = \vec{r}_0 + \frac{L}{\sqrt{3}}\vec{u}_i \tag{10}$$

where the $\vec{u}_i(t)$ are unit vectors given by:

$$\vec{u}_1(t) = \frac{1}{4}\begin{pmatrix} \cos 2\Phi - 3 \\ \sin 2\Phi \\ -2\sqrt{3}\cos\Phi \end{pmatrix}, \qquad \vec{u}_2(t) = \frac{1}{4}\begin{pmatrix} \cos(2\Phi - 2\pi/3) + 3/2 \\ \sin(2\Phi - 2\pi/3) - 3\sqrt{3}/2 \\ -2\sqrt{3}\cos(\Phi - 2\pi/3) \end{pmatrix}$$

$$\vec{u}_3(t) = \frac{1}{4}\begin{pmatrix} \cos(2\Phi - 4\pi/3) + 3/2 \\ \sin(2\Phi - 4\pi/3) + 3\sqrt{3}/2 \\ -2\sqrt{3}\cos(\Phi - 4\pi/3) \end{pmatrix} \tag{11}$$

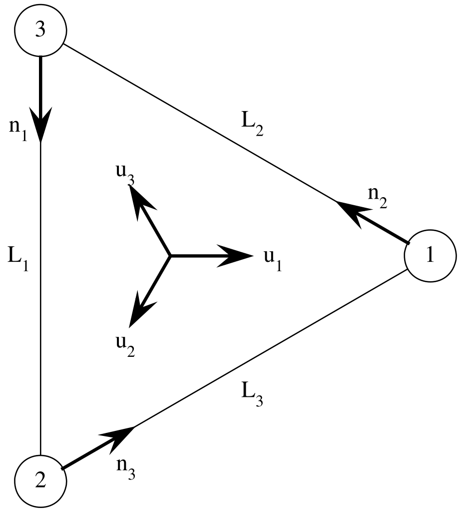

Fig. 2. Notation for the links between spacecraft.

It will be useful to have the unit vectors $\vec{n}_i(t)$ along the three sides of the triangle (see Fig. 2). If we assume, for instance, n_1 pointing from spacecraft #3 to spacecraft #2, we have:

$$n_1(t) = \frac{\vec{u}_2 - \vec{u}_3}{\|\vec{u}_2 - \vec{u}_3\|}$$

and others by cyclic permutation of indices, so that:

$$\vec{n}_1(t) = \frac{1}{4}\begin{pmatrix} \sin 2\Phi \\ -\cos 2\Phi - 3 \\ -2\sqrt{3}\sin\Phi \end{pmatrix}, \qquad \vec{n}_2(t) = \frac{1}{4}\begin{pmatrix} \sin(2\Phi - 2\pi/3) + 3\sqrt{3}/2 \\ -\cos(2\Phi - 2\pi/3) + 3/2 \\ -2\sqrt{3}\sin(\Phi - 2\pi/3) \end{pmatrix}$$

$$\vec{n}_3(t) = \frac{1}{4}\begin{pmatrix} \sin(2\Phi + 2\pi/3) - 3\sqrt{3}/2 \\ -\cos(2\Phi + 2\pi/3) + 3/2 \\ -2\sqrt{3}\sin(\Phi + 2\pi/3) \end{pmatrix} \tag{12}$$

3.2. Laser links

The three spacecrafts are joined by infrared laser beams over $L = 5$ Mkm. We shortly discuss the questions of light propagation over long distances, and of the time delay between emission and detection of a photon (see [10]).

3.2.1. Long-range diffraction

The light beam is produced by a solid state Nd:YAG laser developed at the University of Hanover (Germany). It is planned to deliver 1 W of light stabilized in power and frequency. The beam is expanded by a magnification system (a telescope) up to a diameter of 30 cm. If, for the sake of simplicity, we assume a laser beam at its waist w_0 starting from the telescope, we have a beam width $w_1 = \lambda L/\pi w_0$ after a propagation $L = 5$ Mkm between the two spacecraft. If we assume further a light collector (a telescope identical to the preceding) having a cross section of the same order of magnitude, the ratio of received to emitted power is:

$$\frac{P_1}{P_0} = \left(\frac{\pi w_0^2}{\lambda L}\right)^2$$

so that, for $P_0 \sim 1$ W, we obtain about almost 200 pW after propagation.

3.2.2. Propagation delays

It is interesting to note that the propagation delay from spacecraft A to spacecraft B, separated at the time-coordinate t_0 by a distance L is non-reciprocal. This plays a major role in the detection scheme to be presented in a foregoing section. Assume $\vec{v}_A$ and $\vec{v}_B$ the velocities of the two spacecraft at time t_0. The trajectory of spacecraft B may be parameterized during a short time (of the order of 17 s) by:

$$\vec{x}(t) = \vec{x}_B + (t - t_0)\vec{v}_B$$

The photon emitted at spacecraft A is launched along a direction represented by the (unknown) unit vector $\vec{w}$, so that its path may be parameterized by:

$$\vec{x}(t) = \vec{x}_A + c(t - t_0)\vec{w}$$

and denoting by $\delta t \equiv t_1 - t_0$ the propagation delay, t_1 being the time at which the photon reaches spacecraft B, we get:

$$\vec{w} = \frac{\tau}{\delta t}\vec{n} + \vec{\beta}_B$$

with $\vec{n} \equiv (\vec{x}_B - \vec{x}_A)/L$, $\tau \equiv L/c$ and $\vec{\beta}_B \equiv \vec{v}_B/c$. $\vec{w}$ being a unit vector, we get the following equation:

$$\left(\frac{\tau}{\delta t}\right)^2 + 2\left(\frac{\tau}{\delta t}\right)\vec{n} \cdot \vec{\beta}_B - \left(1 - \beta_B^2\right) = 0$$

the solution of which is:

$$\delta t = \frac{\tau}{\sqrt{1 - [\vec{n} \times (\vec{n} \times \vec{\beta}_B)]^2} - \vec{n} \cdot \vec{\beta}_B} = \frac{L}{c}\frac{\sqrt{1 - [\vec{n} \times (\vec{n} \times \vec{\beta}_B)]^2} + \vec{n} \cdot \vec{\beta}_B}{1 - \beta_B^2}$$

it is thus clear that at the first order in β, we have:

$$\delta t_{AB} = \frac{L}{c}(1 + \vec{n} \cdot \vec{\beta}_B) \tag{13}$$

For the reciprocal delay, we get obviously:

$$\delta t_{BA} = \frac{L}{c}(1 - \vec{n} \cdot \vec{\beta}_A) \tag{14}$$

so that the difference is:

$$\Delta t = \delta t_{AB} - \delta t_{BA} = \frac{L}{c}\vec{n} \cdot (\vec{\beta}_A + \vec{\beta}_B)$$

The unit vector $\vec{n}$ and the vectors $\vec{\beta}_{A,B}$ can be calculated from the orbital evolution of LISA derived in the preceding section (see Eqs. (11), (12)). We have for instance, for link 2–3:

$$\Delta t_{2,3} = \frac{L}{c}\vec{n}_1 \cdot (\vec{\beta}_2 + \vec{\beta}_3) = \frac{2\Omega RL}{c^2}\cos\Phi - \frac{e\Omega RL^2}{c^2} + \mathcal{O}(e^2) \tag{15}$$

$$\simeq \frac{2\Omega RL}{c^2}\cos\Phi - \frac{\sqrt{3}}{6}\frac{\Omega L^2}{c} = \frac{2\Omega RL}{c^2}\cos\Phi - \frac{2\Omega S}{c^2} \tag{16}$$

where $\Omega \equiv 2\pi/1$ year, $\Phi \equiv \Omega t$, and where $S \equiv L^2\sqrt{3}/12$ is nothing but the area swept by a line joining the photon to LISA's center of mass during the propagation between spacecraft #2 and #3 (namely 1/3 of the area of the triangle). Other links give a similar result, up to time shifts of four and eight months, respectively. We see that the global non-reciprocal time-delay results from a modulated part (aberration) with a 1-year period, and a static part that can be attributed to the Sagnac effect. The Sagnac effect for a rotating triangle at angular velocity $-\Omega$ would give $\delta t = -4\Omega S/c^2$. Here, seen from a co-moving inertial frame, the triangle seems to combine its orbital motion (Ω) with a counter-rotation ($-\Omega$) with a relative angle of $\pi/3$ between the two angular momenta, so that the efficient Sagnac delay in Eq. (16) is reduced by a factor $1 - \cos\pi/3 = 1/2$ with respect to the general formula. The orders of magnitude of the two effects are very different. The periodic aberration term has an amplitude of 3.3 ms (or ∼1000 km equivalent path), whereas the constant Sagnac term is about 16 μs (or ∼5 km equivalent path).

4. Detection scheme

4.1. One-way detection

Let us call h_+, $h_\times$ the two polarization components of a gravitational wave propagating along a direction determined by the unit vector $\vec{w}$ in the barycentric frame. In other words, $-\vec{w}$ is the direction of the source seen by an observer at rest in the Solar System. Seen from LISA, the same source has a direction:

$$\vec{w}_L = \vec{w} - \vec{w} \times (\vec{w} \times \vec{\beta}) + \mathcal{O}\left(\beta^2\right)$$

due to aberration ($\vec{\beta} \equiv \vec{v}/c$). The velocity $\vec{v}$ of each station has the same order of magnitude as LISA's one as a whole (and as the Earth's one), so that $\beta \sim 10^{-4}$, which allows us to neglect this correction, and replace $\vec{w}_{\rm L}$ with $\vec{w}$ in formulas involving $h_{+,\times}$. If we denote θ the latitude and ϕ the longitude, we have:

$$\vec{w} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \tag{17}$$

We obtain two new unit vectors $\vec{\theta}$ and $\vec{\phi}$ by:

$$\vec{\theta} = -\frac{\partial\vec{w}}{\partial\theta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}, \qquad \vec{\phi} = \frac{\partial\vec{w}}{\cos\theta\,\partial\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \tag{18}$$

the three vectors $(\vec{w}, \vec{\theta}, \vec{\phi})$ form a direct orthonormal frame. In particular, we assume h_+, $h_\times$ to be the polarization components in that frame. This means that we can represent the gravitational wave amplitude by the transverse-traceless expression:

$$h_{0i} = 0, \qquad h_{ij} = h_+(t)(\theta_i\theta_j - \phi_i\phi_j) + h_\times(t)(\theta_i\phi_j + \phi_i\theta_j) \quad (\forall i, j = 1, 2, 3)$$

Now, in a flat space–time (of Minkowskian metric $\eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$ perturbed by our gravitational wave of amplitude $h_{\mu\nu}$, the line element is given by:

$$\mathrm{d}s^2 = (\eta_{\mu\nu} + h_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

so that a null space–time element corresponding to a light path has thus the following form:

$$0 = \mathrm{d}s^2 = c^2\,\mathrm{d}t^2 + h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

For a laser link, or a light path joining two points A, B of coordinates $\vec{x}_{\rm A}$ and $\vec{x}_{\rm B}$, with

$$L = \|\vec{x}_{\rm B} - \vec{x}_{\rm A}\|, \qquad \vec{n} = \frac{\vec{x}_{\rm B} - \vec{x}_{\rm A}}{L}, \quad \text{and} \quad \vec{\mathrm{d}x} = \vec{n}\,\mathrm{d}l$$

we have:

$$\mathrm{d}l = c\,\mathrm{d}t\left(1 - \frac{1}{2}h_{ij}n^i n^j\right) \tag{19}$$

let us denote:

$$H(t) \equiv h_{ij}n^i n^j = h_+\xi_+ + h_\times\xi_\times$$

with

$$\begin{cases} \xi_+(\Phi, \theta, \phi) = (\vec{n}\cdot\vec{\theta})^2 - (\vec{n}\cdot\vec{\phi})^2 \\ \xi_\times(\Phi, \theta, \phi) = 2(\vec{n}\cdot\vec{\theta})(\vec{n}\cdot\vec{\phi}) \end{cases} \tag{20}$$

We assume t the current time coordinate, i.e. the time at which we receive a given photon at $\vec{x}_{\rm B}$, and $t_{\rm R}$ the retarded time, i.e. the time at which the photon was emitted at $\vec{x}_{\rm A}$. We can integrate Eq. (19) from $\vec{x}_{\rm A}$ to $\vec{x}_{\rm B}$, giving:

$$\frac{L}{c} = t - t_{\rm R} - \frac{1}{2}\int_{t_{\rm R}}^{t} H\left[t' - \frac{1}{c}\vec{w}\cdot\vec{x}(t')\right]\mathrm{d}t' \tag{21}$$

with

$$\vec{x}(t) = \vec{x}_A + c(t - t_{\rm R})\vec{n}$$

or as well, replacing $t_{\rm R}$ by $t - L/c$ in the integral (we develop a first-order theory in H):

$$t_{\rm R} = t - \frac{L}{c} - \frac{1}{2}\int_{t-L/c}^{t} H\left[(1 - \vec{n}\cdot\vec{w})t' - \frac{1}{c}\vec{w}\cdot\vec{x}_{\rm A} + t_{\rm R}\vec{n}\cdot\vec{w}\right]\mathrm{d}t'$$

It is convenient to introduce the Fourier Transform $\tilde{H}(f)$ of H, and the final result is:

$$t_{\rm R} = t - \frac{L}{c} - \frac{1}{2}\int_{\mathbb{R}} \tilde{H}(f)\mathrm{e}^{-2\mathrm{i}\pi f t}\frac{\mathrm{e}^{2\mathrm{i}\pi f t\vec{w}\cdot\vec{x}_{\rm B}/c} - \mathrm{e}^{2\mathrm{i}\pi f t(\vec{w}\cdot\vec{x}_{\rm A} + L)/c}}{-2\mathrm{i}\pi f(1 - \vec{w}\cdot\vec{n})}\,\mathrm{d}f \tag{22}$$

We see that the travel delay from A to B is modulated by the gravitational wave. In other words, the gravitational wave causes a phase modulation in the received light. We can find the frequency modulation by:

$$\delta\nu(t) = \nu_0 \frac{\mathrm{d}}{\mathrm{d}t} t_\mathrm{R}(t)$$

where ν_0 is the frequency of the laser emitter at A. The relative frequency shift, analogous to a modulated Doppler effect, is thus, using Eq. (22):

$$\frac{\delta\nu(t)}{\nu_0} = -\frac{1}{2}\frac{1}{1-\vec{w}\cdot\vec{n}}\left[H\left(t-\frac{\vec{w}\cdot\vec{x}_\mathrm{B}}{c}\right) - H\left(t-\frac{\vec{w}\cdot\vec{x}_\mathrm{A}+L}{c}\right)\right] \tag{23}$$

We have

$$\vec{x}_\mathrm{A} = \vec{r}_0 + \frac{L}{\sqrt{3}}\vec{u}_\mathrm{A}, \qquad \vec{x}_\mathrm{B} = \vec{r}_0 + \frac{L}{\sqrt{3}}\vec{u}_\mathrm{B},$$

so that, in the very low frequency approximation, we have simply:

$$\frac{\delta\nu(t)}{\nu_0} \simeq -\frac{1}{2}\frac{1}{c}\dot{H}(t-\vec{w}\cdot\vec{r}_0/c)\frac{L-L(\vec{u}_\mathrm{B}-\vec{u}_\mathrm{A})\cdot\vec{w}/\sqrt{3}}{1-\vec{w}\cdot\vec{n}} = -\frac{1}{2}\frac{L}{c}\dot{H}(t-\vec{w}\cdot\vec{r}_0/c) \tag{24}$$

4.2. TDI

4.2.1. Basic TDI

As seen from Eq. (24), the signal delivered by one optical link is a modulated Doppler shift of amplitude h, which means less than 10^{-22}. The best scheme for laser frequency stabilization gives presently a spectral density as low as 10 $\mathrm{Hz}/\sqrt{\mathrm{Hz}}$ in the LISA frequency range. This is in relative value about 10^{-13} $\mathrm{Hz}^{-1/2}$, thus much higher than the signal. In a real Michelson interferometer, the arms being almost symmetrical, the frequency fluctuations are in a common mode on the splitter and are rejected from the output dark fringe. This is why the demand on stability is determined only through the small departures from symmetry and greatly relaxed. Here, the history of the two beating waves (the local oscillator and the light coming from 5 Mkm) are essentially different, and the laser frequency fluctuations are in direct competition with the signal. The idea developed since years (firstly by Tinto et al. [4]), under the name of "Time-Delay Interferometry" (TDI), is to use the redundancy of the six laser links to synthesize interferometric configurations by restoring some symmetry in the light paths. A true interferometric configuration being impossible, one constructs virtual digital interferometers from the sampled data. We present firstly a very simplified model in which we neglect the non-reciprocal time delays on long optical paths, and only one laser on each spacecraft. Reality is somewhat different, but this will give basic ideas on the method. Following [5], let us denote by U_k, V_k ($k = 1, 2, 3$) the six data flows. The laser instantaneous frequencies are $\delta_k(t) \equiv \delta\nu(t)/\nu_0$. If we consider the data resulting only from these fluctuations, we have, for instance, for the beat note at spacecraft #1 coming from spacecraft #3:

$$U_1(t) = \delta_3(t - L_2/c) - \delta_1(t) \tag{25}$$

U_2, U_3 are obtained by cyclic permutations of indices. The same way for the beat note at spacecraft #1 coming from spacecraft #2:

$$V_1 = \delta_1(t) - \delta_2(t - L_3/c) \tag{26}$$

V_2, V_3 are obtained by cyclic permutations of indices. Still following [5], we introduce the delay operators D_k ($k = 1, 2, 3$) defined by their effect on any function of time $f(t)$:

$$(D_k f)(t) = f(t - L_k/c)$$

with that notation, we can write, under vector form:

$$\vec{U} = \begin{pmatrix} U_1 = D_2\delta_3 - \delta_1 \\ U_2 = D_3\delta_1 - \delta_2 \\ U_3 = D_1\delta_2 - \delta_3 \end{pmatrix}, \qquad \vec{V} = \begin{pmatrix} V_1 = \delta_1 - D_3\delta_2 \\ V_2 = \delta_2 - D_1\delta_3 \\ V_3 = \delta_3 - D_2\delta_1 \end{pmatrix}$$

We see that the sum $\vec{U} + \vec{V}$ has the algebraic signature of a curl operator:

$$\vec{U} + \vec{V} = \vec{D} \times \vec{\delta}$$

It is therefore possible to immediately exhibit an identically null combination by taking the divergence:

$$\sum_{k=1}^{3} D_k(U_k + V_k) = 0$$

This is the first example of a “silent combination”, or of a “TDI observable”. The question of the determination of all possible silent combinations has been discussed in [5]. We represent any combination C by a linear construction:

$$C = \sum_{k=1}^{3} p_k V_k + q_k U_k = \langle \mathcal{C} | \mathcal{U} \rangle \quad \text{with } \mathcal{C} \equiv (\vec{p}, \vec{q}) \text{ and } \mathcal{U} \equiv (\vec{V}, \vec{U}) \tag{27}$$

so that the problem amounts to find all 6-uples of formal polynomials:

$$(p_1, p_2, p_3, q_1, q_2, q_3)$$

(in the D_1, D_2, D_3 formal variables), giving a null result in Eq. (27). It is a classical problem of algebraic geometry, and the conclusion is that the set of all possible TDI observables has the structure of a module (the so-called first module of syzygies) on the ring of formal polynomials in D_k. Generating sets can be found by standard procedures. For instance, in our simple model, all TDI observables may be found by combinations of the following generators:

$$\begin{cases} \alpha = (1, D_3, D_1 D_3, 1, D_1 D_2, D_2) \\ \beta = (D_1 D_2, 1, D_1, D_3, 1, D_2 D_3) \\ \gamma = (D_2, D_2 D_3, 1, D_1 D_3, D_1, 1) \\ \zeta = (D_1, D_2, D_3, D_1, D_2, D_3) \quad \text{already found above} \end{cases} \tag{28}$$

For example, the “Michelson” X TDI observable is obtained by:

$$X = \alpha - D_3 \beta - D_2 \gamma + D_2 D_3 \zeta = \big(1 - D_2^2, 0, -D_2\big(1 - D_3^2, 1 - D_3^2, -D_3\big(1 - D_2^2\big), 0\big) \tag{29}$$

showing that only arms 2 and 3 are involved. Two more “Michelson” observables may be built, involving other arms. We see, on the expressions (28), that β is obtained from α by a transform that we call $\mathcal{C}$, which consists in cyclically permuting the ranks within each sub 3-uple separately, and simultaneously cyclically permuting the indices. γ is obtained in turn the same way from β. ζ is invariant under $\mathcal{C}$. The real situation is more complex in practice because due to the orbital motion of LISA, as seen above, the delays are non-reciprocal, so that we have not three but six delay operators. Moreover, the periodic deformation of the triangle (at second order in e) makes the propagation time delays time dependent, so that the delay operators do not commute. Noise cancelling generators can however be found, at the price of a greater complexity [6].

4.2.2. Extended TDI

Not only the frequency laser noise compete with the gravitational signal, but also contributions having eventually the same effect, but due to mechanical causes. The motions of the optical benches and of the test masses add a specific phase noise (let us recall that the gravitational signal modulates the light distance between the test masses, not their relative velocities). Moreover, we have not one laser per spacecraft, but two, so that the relative frequency between those must be taken into account. Extending the preceding analysis, we denote by C_k^* the relative phase noise of the laser of node k facing node $k+1$ [mod 3], and by C_k the same for the laser of node k facing spacecraft $k-1$ [mod 3]. In the same spirit, we denote by $\vec{V}_k^*$ the velocity of the optical bench of node k towards spacecraft $k+1$ [mod 3], and $\vec{V}_k$ the velocity of the twin bench of spacecraft k towards spacecraft $k-1$ [mod 3]. The same way, we adopt the notation $\vec{v}_k^*$, $\vec{v}_k$ for the residual velocities of the test masses when servoed. We must also take into account the fact that the phase noises caused by the test masses result from specular reflection of the incoming beam (a factor of 2 in the coming formula). All this put together, we get an extended formula for the noise (we keep the notation of [7]):

$$U_1(t) = C_3(t - L_2) - C_1^*(t) - \vec{n}_2 \cdot \vec{V}_3(t - L_2) - \vec{n}_2 \cdot \vec{V}_1^*(t) + 2\vec{n}_2 \cdot \vec{v}_1^*(t) + y_1^* \tag{30}$$

$$V_1(t) = C_1(t) - C_2^*(t - L_3) - \vec{n}_3 \cdot \vec{V}_2^*(t - L_3) - \vec{n}_3 \cdot \vec{V}_1(t) + 2\vec{n}_3 \cdot \vec{v}_1(t) + y_1 \tag{31}$$

other being obtained by cyclic permutations of indices. y_k, y_k^* account for the shot noises. To close the information, relative measurements are made on board each spacecraft between the two benches and lasers by means of an optical fiber. We denote by z_k, z_k^* the two data flows on board spacecraft k. We have:

$$z_1(t) = C_1(t) - C_1^*(t) + 2\vec{n}_3 \cdot \big[\vec{v}_1(t) - \vec{V}_1(t)\big] + \sigma_1(t) \tag{32}$$

and for the reciprocal measurement:

$$z_1^*(t) = C_1^*(t) - C_1(t) - 2\vec{n}_2 \cdot \big[\vec{v}_1^*(t) - \vec{V}_1^*(t)\big] + \sigma_1(t) \tag{33}$$

where the σ_k account for the noise introduced by propagation in the fiber (assumed reciprocal). Despite the increased complexity, the preceding basic study helps finding a new module of silent combinations. If we temporarily forget the test masses motions and the shot noise, we can write:

$$U_1 = D_2[C_3 - \vec{n}_2 \cdot \vec{V}_3] - \big[C_1^* + \vec{n}_2 \cdot \vec{V}_1^*\big]$$
$$V_1 = [C_1 - \vec{n}_3 \cdot \vec{V}_1] - D_3\big[C_2^* + \vec{n}_3 \cdot \vec{V}_2^*\big]$$

and one more set of observables can be constructed, eliminating the fiber noises:

$$Z_1 = \frac{1}{2}(z_1 - z_1^*) = C_1 - C_1^* - \vec{n}_3 \cdot \vec{V}_1 - \vec{n}_2 \cdot \vec{V}_1^*$$

If now we introduce the new observables:

$$\tilde{C}_1 \equiv C_1 - \vec{n}_3 \cdot \vec{V}_1, \qquad \tilde{C}_1^* \equiv C_1^* + \vec{n}_2 \cdot \vec{V}_1^*$$

we see that Eqs. (30), (31) become simply:

$$U_1 = D_2\tilde{C}_3 - \tilde{C}_1^*, \qquad V_1 = \tilde{C}_1 - D_3\tilde{C}_2^* \tag{34}$$

plus three new variables:

$$Z_k \equiv \tilde{C}_k - \tilde{C}_k^*$$

The data flow can thus be represented by a set of 9-uples $\mathbf{U} = (\mathbf{V}_k, \mathbf{U}_k, \mathbf{Z}_k)$ so that silent combinations (or TDI observables) will be generated by a family of 9-uples of polynomials $\mathbf{X} = (\mathbf{p}_k, \mathbf{q}_k, \mathbf{r}_k)$, a TDI observable being of the generic form:

$$x = \langle \mathbf{X} | \mathbf{U} \rangle = \sum_{k=1}^{3} [p_k V_k + q_k U_k + r_k Z_k]$$

According to Eq. (34), the structure of the subset (V, U) is identical to the set presented in the "basic" model: this is a way to express the fact that the phase noise coming from the optical benches cannot be distinguished from the intrinsic laser noise. The TDI-generating module will therefore be an extension of the former. As in the preceding subsection, we can find a generating set $(\alpha, \beta, \gamma, \zeta)$. For instance, we have:

$$\alpha = \big[1, D_3, D_1D_3, 1, D_1D_2, D_2, -1 - D_1D_2D_3, -(D_1D_2 + D_3), -(D_1D_3 + D_2)\big]$$

(β and γ can be obtained as usual by $\mathcal{C}$). We have:

$$\zeta = \big[D_1, D_2, D_3, D_1, D_2, D_3, -(D_1 + D_2D_3), -(D_2 + D_3D_1), -(D_3 + D_1D_2)\big]$$

which is invariant under $\mathcal{C}$. For the Michelson observable, the first 6-uple is unchanged (see Eq. (29)) and the last 3-uple is:

$$\big[D_2^2 + D_3^2 - D_2^2D_3^2, -1, 0, 0\big]$$

two other Michelson observables can be found by $\mathcal{C}$.

4.2.3. Second-generation TDI

The reader has certainly remarked that the phase noise suppression by the TDI assumes that the values of the inter-spacecraft are perfectly known. In practice, these values not only have some inherent inaccuracy, but are changing in time. It is therefore necessary to assess the needed precision. Owing to the best frequency stabilization schemes, an acceptable residual phase noise is acceptable if the distances L_i are known with an accuracy better than 200 m [7], i.e. much less than the differences in flight time computed in 3.2.2. As seen in 3.2.2, the delays of light propagation from one spacecraft to another are not reciprocal, so that the delay operators introduced above are only approximate and are used only in analytical calculations of the GW signal. For actually eliminating the phase noises, more accurate delays must be employed, and consequently, the delay operators are six (two per link). It is nevertheless possible to find a module of generators generalizing TDI observables [6,8]. Moreover, the propagation delays are time dependent, since the distances between spacecraft are constant only at the first order with respect of the eccentricity. The result is that the delay operators do not commute in general. It is again possible to find observables having a negligible residual phase noise. This is called "second-generation TDI" [6,9].

4.3. Signal-to-noise ratio, sensitivity

4.3.1. Spectral density of noise

If we assume a perfect cancellation of the phase noise by use of TDI observables, we are left with the residual data flows (see Eqs. (30), (31)):

$$\begin{cases} U_1 = 2\vec{n}_2 \cdot \vec{v}_1^* + y_1^* \\ V_1 = 2\vec{n}_3 \cdot \vec{v}_1 + y_1 \\ Z_1 = \vec{n}_3 \cdot \vec{v}_1 + \vec{n}_2 \cdot \vec{v}_1^* \end{cases}$$

A generic TDI combination $\mathbf{A} = (\mathbf{p}_i, \mathbf{q}_i, \mathbf{r}_i)$ when applied to the 9-uple $\mathbf{U} = (\mathbf{V}_k, \mathbf{U}_k, \mathbf{Z}_k)$ gives:

$$\langle \mathbf{A}|\mathbf{U}\rangle = (2p_1 + r_1)\big(\vec{n}_3 \cdot \vec{v}_1 + \vec{n}_2 \cdot \vec{v}_1^*\big) + (2p_2 + r_2)\big(\vec{n}_1 \cdot \vec{v}_2 + \vec{n}_3 \cdot \vec{v}_2^*\big) + (2p_3 + r_3)\big(\vec{n}_2 \cdot \vec{v}_3 + \vec{n}_1 \cdot \vec{v}_3^*\big) + \sum_{k=1}^{3}\big[p_k y_k + q_k y_k^*\big]$$

In the Fourier space, assuming a common spectral density $S_{\mathrm{acc}}(f)$ for the motions of the various test masses and $S_{\mathrm{qn}}(f)$ for the shot noise at all photodiodes, we get the overall spectral density of noise as:

$$S_A(f) = \sum_{k=1}^{3}\big[|2p_k + r_k|^2 + |2q_k + r_k|^2\big] S_{\mathrm{acc}}(f) + \sum_{k=1}^{3}\big[|p_k|^2 + |q_k|^2\big] S_{\mathrm{qn}}(f) \tag{35}$$

At this point, we can assume equal length arms ($L_k = L$), and we get for instance, for the Michelson TDI observables (see [11] for details):

$$S_{X,\mathrm{noise}}(f) = \big[8\sin^2(4\pi f L) + 32\sin^2(2\pi f L)\big] S_{\mathrm{acc}}(f) + 16\sin^2(2\pi f L) S_{\mathrm{qn}}(f) \tag{36}$$

The estimates reported in LISA documents are:

$$S_{\mathrm{acc}}(f) = s_1 \left[\frac{1\ \mathrm{Hz}}{f}\right]^2, \qquad S_{\mathrm{qn}}(f) = s_2 \left[\frac{f}{1\ \mathrm{Hz}}\right]^2$$

with:

$$s_1 = 2.5 \times 10^{-48}\ \mathrm{Hz}^{-1}, \qquad s_2 = 5.3 \times 10^{-38}\ \mathrm{Hz}^{-1}$$

4.3.2. Spectral density of signal

It results from Eq. (23) that the contributions of the GW signal to the six data flows are given by:

$$V_1(t) = \frac{1}{2(1 - \vec{w} \cdot \vec{n}_3)}\big[H_3(t - \mu_1) - H_3(t - \mu_2 - L_3)\big] \tag{37}$$

$$U_1(t) = -\frac{1}{2(1 + \vec{w} \cdot \vec{n}_2)}\big[H_2(t - \mu_1) - H_2(t - \mu_3 - L_2)\big] \tag{38}$$

and others under $\mathcal{C}$. We have set $\mu_i \equiv \vec{w}.\vec{r}_i$, $H_i(t) \equiv h_+(t)\xi_{i,+} + h_\times(t)\xi_{i,\times}$ and finally (see Eq. 20):

$$\xi_{i,+} = (\vec{\theta} \cdot \vec{n}_i)^2 - (\vec{\phi} \cdot \vec{n}_i)^2, \qquad \xi_{i,\times} = 2(\vec{\theta} \cdot \vec{n}_i)(\vec{\phi} \cdot \vec{n}_i) \tag{39}$$

In the frequency domain we get ($\omega \equiv 2\pi f$):

$$V_1(f) = F_{V_1,+}(f)h_+(f) + F_{V_1,\times}(f)h_\times(f)$$

$$U_1(f) = F_{U_1,+}(f)h_+(f) + F_{U_1,\times}(f)h_\times(f)$$

where we have introduced the transfer functions:

$$F_{V_1,+,\times}(f) = \frac{1}{2(1 - \vec{w} \cdot \vec{n}_3)}\big[\mathrm{e}^{\mathrm{i}\omega\mu_1} - \mathrm{e}^{\mathrm{i}\omega(\mu_2 + L_3)}\big]\xi_{3,+,\times}$$

$$F_{U_1,+,\times}(f) = -\frac{1}{2(1 + \vec{w} \cdot \vec{n}_2)}\big[\mathrm{e}^{\mathrm{i}\omega\mu_1} - \mathrm{e}^{\mathrm{i}\omega(\mu_3 + L_2)}\big]\xi_{2,+,\times}$$

When dealing with gravitational amplitudes, we can neglect the length differences in the arms ($L_i = L$) and then:

$$F_{V_1,+,\times}(f) = -i\pi f L \mathrm{e}^{\mathrm{i}\pi f(L - \mu_3)} \mathrm{sinc}\big(\pi f L(1 - \vec{w} \cdot \vec{n}_3)\big)\xi_{3,+,\times}$$

$$F_{U_1,+,\times}(f) = i\pi f L \mathrm{e}^{\mathrm{i}\pi f(L - \mu_2)} \mathrm{sinc}\big(\pi f L(1 + \vec{w} \cdot \vec{n}_2)\big)\xi_{2,+,\times}$$

For a generic combination $X = (\vec{p}, \vec{q})$, the transfer function is now:

$$F_{X,+,\times} = \sum_{k=1}^{3}(\tilde{p}F_{V_1,+,\times} + \tilde{q}F_{U_1,+,\times})$$

where the $\tilde{p}$,$\tilde{q}$ are true polynomials in the unique variable $\exp(2\mathrm{i}\pi f L)$. The spectral density of signal is now given, in terms of the spectral densities of the two polarization states ($S_{h,+}, S_{h,\times}$) by:

$$S_{X,\mathrm{signal}} = \left|\sum_{k=1}^{3}(\tilde{p}_k F_{V_k,+} + \tilde{q}_k F_{U_k,+})\right|^2 S_{h,+} + \left|\sum_{k=1}^{3}(\tilde{p}_k F_{V_k,\times} + \tilde{q}_k F_{U_k,\times})\right|^2 S_{h,\times} \tag{40}$$

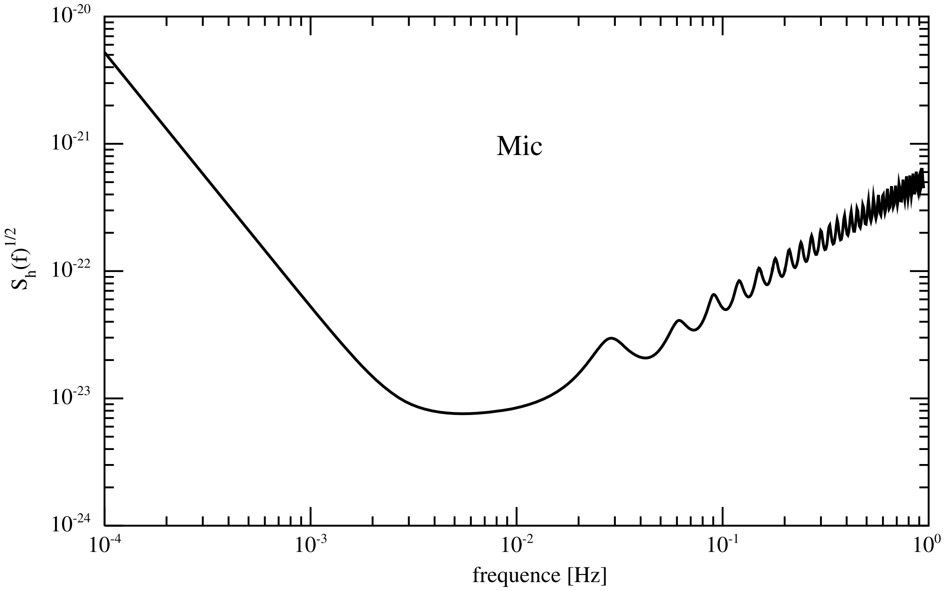

Fig. 3. Spectral sensitivity of the Michelson TDI observable.

The signal to noise ratio is obviously obtained by dividing for example (40) by (36) if the Michelson combination is relevant. After an average on the source direction on the full sky, and requiring a SNR of 5 for integration over one year of the signal, assumed as periodic, we get a sensitivity curve of the form displayed in Fig. 3.

5. Very low frequency approximation

At very low frequencies, the gravitational responder of LISA can be expanded in a Taylor series with respect to time-delays. We discuss the result for the TDI observables.

5.1. Expansion of the gravitational response

Following Eq. (29), and neglecting the asymmetries of the triangle ($L_a \sim L \Rightarrow D_a \sim D\ \forall a = 1, 2, 3$), we can express the Michelson X observable as:

$$X = \left(1 - D^2\right)[V_1, 0, -DV_3, U_1, -DU_2]$$

Now, using Eq. (24), we have:

$$\begin{cases} V_1 = \dfrac{L}{2c}\dot{H}_3(t - \vec{w}\cdot\vec{r}_0/c) \\ V_3 = \dfrac{L}{2c}\dot{H}_2(t - \vec{w}\cdot\vec{r}_0/c) \\ U_1 = -\dfrac{L}{2c}\dot{H}_2(t - \vec{w}\cdot\vec{r}_0/c) \\ U_2 = -\dfrac{L}{2c}\dot{H}_3(t - \vec{w}\cdot\vec{r}_0/c) \end{cases}$$

We introduce the directional functions $\xi_{a,+,\times}$ by:

$$\xi_{a,+} = (\vec{n}_a\cdot\vec{\theta})^2 - (\vec{n}_a\cdot\vec{\phi})^2, \qquad \xi_{a,\times} = 2(\vec{n}_a\cdot\vec{\theta})(\vec{n}_a\cdot\vec{\phi}) \quad (a = 1, 2, 3) \tag{41}$$

so that we get after some algebra, neglecting the third time derivative:

$$X \simeq \frac{L}{c}\left[\dot{H}_3(t_R) - \dot{H}_2(t_R) - \frac{L}{c}\left(3 - \frac{\vec{w}\cdot\vec{u}_3}{2\sqrt{3}}\right)\ddot{H}_3(t_R) + \frac{L}{c}\left(3 - \frac{\vec{w}\cdot\vec{u}_2}{2\sqrt{3}}\right)\ddot{H}_2(t_R)\right] \tag{42}$$

with $t_R \equiv t - \vec{w}\cdot\vec{r}_0/c$, $H_a \equiv h_+\xi_{a,+} + h_\times\xi_{a,\times}$.

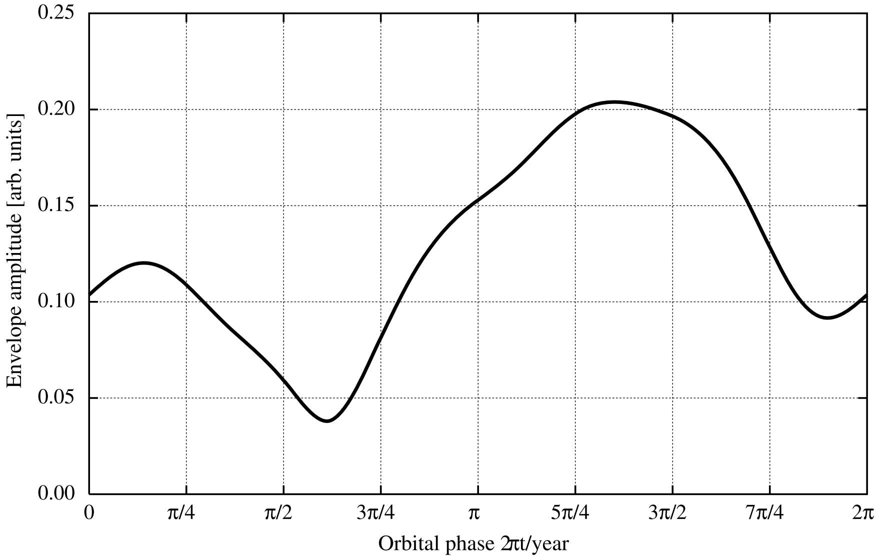

Fig. 4. Example of an amplitude modulation of a monochromatic signal (arbitrary parameters).

5.2. *Amplitude modulation for a monochromatic wave*

We consider a monochromatic source as a galactic binary, so that the polarization components may be defined as follows: if the z axis is supposed to be along the sight line, and if the angular momentum, assumed in the x, z plane, makes an angle ι with the z axis, we have:

$$h'_+(t) = h\big(1+\cos^2\iota\big)\cos\omega t, \qquad h'_\times(t) = 2h\cos\iota\sin\omega t \tag{43}$$

The unit vectors along the x and y axes make an angle ψ with our transverse vectors θ and ϕ, so that in our frame, the components are rather, due to the spin 2 nature of the gravitational tensor field:

$$\begin{cases} h_+(t) = \cos 2\psi\, h'_+(t) - \sin 2\psi\, h'_\times(t) \\ h_\times(t) = \sin 2\psi\, h'_+(t) + \cos 2\psi\, h'_\times(t) \end{cases}$$

After substituting Eq. (42) and using Eq. (43), we obtain:

$$X(t) = \frac{\omega L}{c}\big(e_1(t_R)\cos\omega t + e_2(t_R)\sin\omega t\big) \tag{44}$$

with:

$$\begin{cases} e_1(t) = 2\cos\iota(-\sin 2\psi\, F_+ \cos 2\psi\, F_\times) + \dfrac{\omega L}{c}\big(1+\cos^2\iota\big)(\cos 2\psi\, G_+ \sin 2\psi\, G_\times) \\ e_2(t) = -\big(1+\cos^2\iota\big)(\cos 2\psi\, F_+ + \sin 2\psi\, F_\times) - 2\dfrac{\omega L}{c}\cos\iota(\sin 2\psi\, G_+ - \cos 2\psi\, G_\times) \end{cases} \tag{45}$$

and with the following directional functions that have been used:

$$F_{+,\times}(\theta,\phi,t) = \xi_{3,+,\times} - \xi_{2,+,\times}, \qquad G_{+,\times}(\theta,\phi,t) = F_{+,\times} \times \left(3 - \frac{\vec{w}\cdot\vec{u}_3}{2\sqrt{3}}\right)$$

The envelope $E(t)$ of the signal is:

$$E(t) = \sqrt{e_1^2(t) + e_2^2(t)}$$

and is clearly modulated with a one-year period, due to the motion of the LISA through the directional functions F, G (see in Fig. 4 an example of such a modulation). The direction of the source in encoded in this modulation, so that by fitting the envelope, it is possible to retrieve the sky location.

6. Some technological challenges

The preceding discussion assumes some technological issues solved. An impressive amount of R&D work has indeed been cumulated since twenty years. Let us briefly summarize the most important issues encountered in the development of the LISA concept, and which could be a significant advantage for a similar future project.

6.1. Drag-free operation

The propagation of light, which contains the information on GW, is assumed to link two free falling test masses. Any orbiting object undergoes various perturbing effects like radiation pressure, it is therefore necessary to protect these test massed from spurious forces by a shield (the external shell of the spacecraft). This shell must in turn to be servoed on the test mass position despite of these spurious forces. It is thus necessary to have a readout system delivering error signals in case of relative motion, and a system producing restoring forces. In the LISA mission, the readout was capacitive, between the test mass (a cubic gold-platinum block of 2 kg) and a surrounding cage. The feedback forces were applied by micronewton thrusters using accelerated ions. The residual acceleration noise was foreseen at the level of $3 \times 10^{-15}\ \mathrm{m\,s^{-2}\,Hz^{-1/2}}$ (see [12] for more technical details).

6.2. Stabilized lasers

A key point is the frequency stability of the onboard lasers. TDI observables are free of laser noise, up to a compromise between the accuracy of the spacecraft ranging (knowledge of the actual light distances) and the frequency stability of the oscillators. A stability of the order of $10\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$ is presently achieved by servoeing the lasers on an ultra-stable reference cavity. Proposals have been made to use a molecular spectral line (e.g., iodine) as a frequency etalon.

6.3. LISA Pathfinder

A technological demonstrator “LISA Pathfinder” will test the possibility of placing two test masses in free fall inside the same spacecraft. An interferometric system will measure the relative accelerations of the two test masses [1]: this will be a crucial test. Pathfinder is expected to be launched in 2014.

7. Conclusion

Besides its fascinating science case, a LISA mission raises some interesting fundamental questions related to the geometry and light propagation, which are, after all, the basis of General Relativity. We have tried to present here a central problem: the cancellation of the laser phase noises. For this we had to study the motion of the three spacecraft along a year, to compute accurately the flight time of a photons between two moving masses, and to derive silent (TDI) combinations of data. We also considered how GW modulate the signal, and gave an idea of how it is possible to extract extrinsic parameters from TDI observables.

References

[1] http://lisa.nasa.gov, http://www.esa.int/esaSC/120376_index_0_m.html.
[2] http://lisa.nasa.gov/Documentation/LISA-LIST-RP-436_v1.2.pdf.
[3] W.H. Clohessy, R.S. Wiltshire, J. Aerospace Sci. 27 (1960) 653–658.
[4] M. Tinto, J. Armstrong, Phys. Rev. D 59 (1999) 102003.
[5] S. Dhurandhar, R. Nayak, J.-Y. Vinet, Phys. Rev. D 65 (2002) 102002.
[6] http://www.livingreviews.org/lrr-2005-4, Albert Einstein Institute, Potsdam.
[7] LISA System and Technology Report (ESA-SCI (2000) 11).
[8] R. Nayak, J.-Y. Vinet, Phys. Rev. D 70 (2004) 102003.
[9] D.A. Shaddock, Phys. Rev. D 68 (2003) 061303.
[10] B. Chauvineau, T. Regimbau, J.-Y. Vinet, S. Pireaux, Phys. Rev. D 72 (2005) 122003.
[11] S. Dhurandhar, R. Nayak, R. Koshti, J.-Y. Vinet, Class. Quant. Grav. 22 (3) (2005) 481–487.
[12] The LISA Yellow Book ESA/SRE (2011), 3 Feb. 2011.