C. R. Physique 14 (2013) 685–699

Contents lists available at ScienceDirect

Comptes Rendus Physique

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Disordered systems / Systèmes désordonnés

Out-of-equilibrium dynamics of classical and quantum complex systems

Dynamique hors équilibre de systèmes complexes classiques et quantiques

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ARTICLE INFO

Article history:
Available online 14 October 2013

Keywords:
Out-of-equilibrium dynamics
Disordered systems
Driven dynamics

Mots-clés :
Dynamique hors équilibre
Systèmes désordonnés
Dynamique sous sollicitation

ABSTRACT

Equilibrium is a rather ideal situation, the exception rather than the rule in Nature. Whenever the external or internal parameters of a physical system are varied, its subsequent relaxation to equilibrium may be either impossible or take very long times. From the point of view of fundamental physics, no generic principle such as the ones of thermodynamics allows us to fully understand its behaviour. The alternative is to treat each case separately. It is illusionary to attempt to give, at least at this stage, a complete description of all non-equilibrium situations. Still, one can try to identify and characterise some concrete, but still general features of a class of out-of-equilibrium problems – yet to be identified – and search for a unified description of these. In this report, I briefly describe the behaviour and theory of a set of non-equilibrium systems and I try to highlight common features and some general laws that have emerged in recent years.

RÉSUMÉ

L'équilibre est une situation plutôt idéale, l'exception plutôt que la règle dans la Nature. Chaque fois que les paramètres externes ou internes d'un système physique subissent une modification, sa relaxation subséquente vers l'équilibre peut, soit être impossible, soit prendre très longtemps. Du point de vue de la physique fondamentale, aucun principe générique tel que ceux de la thermodynamique ne permet de comprendre complètement son comportement. L'alternative consiste à traiter chaque cas séparément. Il est illusoire de tenter de donner, au moins à ce stade, une description complète de toutes les situations hors équilibre. Mais on peut essayer d'identifier et de caractériser quelques traits concrets, mais toujours généraux, d'une classe de problèmes hors équilibre – restant à identifier – et de rechercher une description unifée de ceux-ci. Dans cette contribution, je décris brièvement le comportement et la théorie d'un jeu de systèmes hors équilibre et je tente de mettre en lumière des traits communs et quelques lois générales qui ont vu le jour au cours des dernières années.

1. Introduction

This text is a stroll in the non-equilibrium dynamics of classical and quantum physical systems with a few or many degrees of freedom, closed or in contact with an environment. While in classical mechanics the theory of dynamical systems has dealt with very rich out-of-equilibrium phenomena for more than a century, in condensed matter or statistical

1631-0705/$ – see front matter
http://dx.doi.org/10.1016/j.crhy.2013.09.004

physics, focus was set on equilibrium properties until relatively recently. Micro-canonical, canonical or grand canonical ensembles were used depending on the conditions of preference or relevance. The relaxation of a tiny perturbation away from equilibrium is also sometimes described in textbooks and undergraduate courses. However, important problems in physics and other sciences have obliged us to move away from the equilibrium hypothesis and face the difficulty of grasping and explaining out-of-equilibrium dynamics in these areas as well.

The scope of this report is a rather wide one and, in consequence, the presentation is not intended to be technical. I will shortly describe various out-of-equilibrium situations in different fields of science ranging from physics to biology, and including present-day themes of study in computer science. Beyond understanding each of these and other problems on their own, the ultimate aim of the analysis should be to identify generic features (dynamic scaling, thermodynamic-like properties, *etc.*) and methods of study that could be of wide applicability. Success in this direction has been achieved in recent years and I will shortly describe some of these remarkable findings.

As departure from equilibrium can be found at all scales, some of the examples are formulated in terms of variables that behave classically, but others, drawn from the microscopic world, are set in terms of quantum mechanics. In the instances that I have chosen to treat, relativistic effects are not important and will be neglected throughout. I organise the body of the text in three main sections, dealing with background, classical dynamics, and quantum problems. For lack of space, and also because many of the questions posed apply to both realms albeit in a slightly modified way, I devote a more extended presentation to classical problems. In the concluding section, I list a number of open problem that I personally find interesting and at analytic reach.

2. Background

I start by recalling the basics of classical mechanics, the construction of equilibrium statistical mechanics, the emergence of collective phenomena and phase transitions, and some peculiar effects induced by quenched disorder, competing interactions, and constraints. I next explain why these systems give rise to yet not well understood, nor satisfactorily described, evolutions.

2.1. Dynamical systems

Dynamical systems [1–3] consist of classical variables that, given an initial condition, evolve in time following some deterministic rule that univocally yields the trajectory in phase space. The evolution laws can be discrete in time, and be called maps, or they can be continuous. Maps, and in particular cellular automata [4] in which the dynamic variables take only a discrete set of values, are useful to model non-physical systems. Population dynamics in which each time step represents a generation are an example. In the microscopic modelling of physical systems, time is a continuous variable. The classification and understanding of all possible dynamic trajectories for different choices of the rule, embedding box and initial conditions is the goal of this branch of theoretical physics. A *conservative* dynamical system conserves the volume in phase space, while a *dissipative* one does not. A specially interesting question is whether qualitative changes affect the trajectories when some parameter is varied at, *e.g.*, *bifurcation points*. The theory of *deterministic chaos* deals with non-linear systems with a few degrees of freedom that, in spite of following a deterministic dynamic rule, show a sustained erratic temporal behaviour. Chaotic systems are extremely sensitive to small perturbations on the initial conditions. Coherent collective behaviour in *coupled map lattices* [5], such as global synchronisation and spatio-temporal patterns, occur for strong spatial coupling, while the propagation of a perturbation in both space and time when a large number of individual chaotic maps are set in interaction has been observed in the weak coupling regime. Such persistent disorder in space and time called *spatio-temporal chaos* is caused by instabilities in the deterministic dynamics and not by external noise. A traditional example with these phenomena is the atmosphere–ocean evolution. Unless otherwise stated, these problems feel no influence from the environment.

In *Hamiltonian dynamical systems*, the evolution is dictated by Hamilton's equations of motion that do conserve the volume in phase space. Whenever the Hamiltonian does not depend explicitly on time, the mechanical energy is a constant of the motion. The simplest example is the one of a single classical particle embedded in a d-dimensional space with some volume, possibly taken to be infinite. Hamiltonian dynamical systems are said to be *integrable* when there exist as many (linearly independent) constants of the motion as, say, coordinate type degrees of freedom.

The theory of dynamical systems has reached maturity many years ago and, after having recalled some definitions and introduced some terminology, I will not discuss it further here. Excellent books describe it from the very mathematical to the rather practical viewpoints [1–3].

2.2. Equilibrium

The *ergodic hypothesis* states that the diverging time average along the trajectory of every observable expressed as an integrable function, and for almost all initial conditions, equals its average over a measure on phase space that is left invariant by the dynamics. This hypothesis is the usual starting point for the development of *equilibrium statistical mechanics*, the central postulate of which states that, after all transient effects have died out, an isolated Hamiltonian system can be

found in each one of its accessible states with equal probability. This principle introduces time-independent probabilistic concepts in the description of dynamical systems and establishes the *micro-canonic* distribution.

Questions on ergodicity and equilibration of closed systems have recently regained interest, boosted by the development of quantum systems in almost perfect isolation. I will come back to this problem in Section 4. However, these issues are highly non-trivial in the classical limit as well. Integrable systems are not ergodic. Non-integrability, as introduced by non-linear terms, does not necessarily imply the rapid establishment of ergodicity, as demonstrated by the almost exact finite-period recurrence of the celebrated Fermi–Pasta–Ulam model of a large number of one-dimensional linear harmonic oscillators anharmonically coupled [6]. What initially appeared as a paradox was later shown to be a fact, with the Kolmogorov–Arnold–Moser theorem [7] that asserts that the evolution of a large set of initial conditions of an integrable Hamiltonian system non-linearly perturbed can remain quasi-periodic if the strength of the perturbation is sufficiently small (how small is small is a hard problem-dependent question).

Accepting that ergodicity established, one rarely uses the micro-canonic ensemble. The *canonical* (or grand-canonical) equilibrium distribution is built by partitioning the very large system into a subsystem of interest and a much larger part that is effectively 'integrated out'. Through some mathematical manipulations (that need the additivity of the energy of the two subparts and the large size limit of the 'bath'-part to ensure the absence of temperature fluctuations) one derives the equilibrium distributions of energy (and particle number) fluctuations of the selected part. The connection with macroscopic observables and thermodynamics is next established.

The same procedure can be formulated dynamically, starting from the Hamiltonian evolution of the full coupled system [8]. With convenient choices for the bath model and the coupling between bath and system, the calculations can be carried out to derive effective dynamic equations for the 'reduced system'. Under further assumptions on the initial conditions of the bath's degrees of freedom that typically introduce stochasticity into the description, one deduces *generalised Langevin equations* for the selected part. This calculation gives a formal foundation to one of the most commonly used tools to tackle the dynamics of open classical systems.

2.3. Macroscopic systems: phase transitions

One of the most striking features of large equilibrium systems is their collective behaviour and the possibility of undergoing *phase transitions*, that is to say, sharp changes in the macroscopic behaviour at special values of the parameters [9,10]. Phase transitions can usually be detected with the measurement of an order parameter, generically defined as the average of a simple observable (*e.g.,* the magnetisation density in a magnet) that vanishes in one phase and is different from zero in another. They can only arise in the limit of a diverging number of degrees of freedom as it is associated with the non-analyticity of the free energy. In the Ginzburg–Landau approach, one postulates the order-parameter-dependent free-energy, the extrema of which characterise the equilibrium and metastable states in the different phases. This idea is at the basis of the *free-energy landscape* widely used in the description of disordered systems that I will sketch below and in Section 2.4.2.

In *first-order phase transitions*, the order parameter jumps at the critical point to a finite value right on the ordered side of the transition. This is accompanied by discontinuities in various thermodynamic quantities and the divergence of a first derivative of the free-energy density. The high and low temperature phases coexist at the transition. These transitions often exhibit hysteresis and memory effects, since the system can remain in the metastable phase when the external parameters go beyond the critical point. Examples are the melting of 3d solids and the condensation of a gas into a liquid.

In a *second-order phase transition*, there is no phase coexistence since the phases on either side of the transition are identical at the critical point. The order parameter is continuous, but the linear susceptibility, a second derivative of the free energy, diverges. By virtue of the fluctuation–dissipation theorem, this implies the divergence of the correlation length of the order parameter fluctuations and the scale-free character of the critical state. These transitions can be classified in *universality classes*, depending on space dimensionality, the dimension of the order parameter, and the symmetries of the problem. They are usually accompanied by *spontaneous symmetry breaking*, *i.e.* the fact that the system has several equivalent equilibrium states related by symmetry in the ordered phase. *Spontaneous broken ergodicity* is another common feature of these transitions. Once the system is ordered in one of the components of the equilibrium distribution, the long-time average (in which the system remains in one the disjoint ergodic components) differs from the statistical one (performed over all components). One can reconcile the two results by, in the statistical average, summing over the configurations in one of the ergodic components only or, in the dynamic approach, taking the diverging time limit before the infinite size one.

The Berezinskii–Kosterlitz–Thouless transition is special in that it separates a disordered phase from a critical one. Topological defects proliferate in the disordered phase and they bind in pairs in the one with quasi long-range order and correlation functions that decay algebraically. Physical realisations are 2d planar ferromagnets, superconducting films, Josephson-junction arrays, especially tailored nematic liquid crystals, and toy models for two-dimensional turbulence. The 2d *xy* model is the paradigm in this class of *topological phase transitions*.

All these features are, of course, very well established. The general picture is captured by mean-field theory, while the critical properties are obtained with perturbative renormalisation group (RG) techniques [9,10].

2.4. Some interesting problems

I introduce now a number of problems in which the picture sketched above gets more complicated, already at a static level.

2.4.1. Disorder

No material is perfectly homogeneous: impurities of different kinds are distributed randomly throughout the samples. Such *disorder* can be of two types: annealed or quenched. In the former case, the relaxation times of the impurities and host variables are of the same order, and the equilibrium properties of the full system are given by the partition sum over all configurations of host and impurities. In the latter, there is a sharp separation of time scales: the one for the impurities is much longer than the one of the host variables. It is the case of some magnetic systems in which the diffusion time of the impurities is so long that these remain frozen.

A natural effect of disorder on many body systems is to favour disordered configurations (*e.g.*, to lower the critical temperature). However, as quenched randomness naturally generates a complex potential energy landscape with many wells and barriers that scale and are organised in a highly non-trivial way, novel phenomena are also encountered.

As a particular sample is not expected to be special in any sense, the full ensemble of statistically equivalent samples is studied at once, assuming that for most interesting observables, the typical behaviour is equal to the mean. A certain number of specific prescriptions to carry out the average exist: the replica theory, the supersymmetric method, and the dynamical approach [11]. Effective interactions are introduced by the averaging procedure, but no spatial heterogeneities are left over. Scaling or RG methods to be mentioned below allow one to make concrete predictions on the effect of spatial fluctuations induced by disorder.

Much attention has been payed to the effect of *weak quenched disorder* (inducing no frustration) on systems in which the nature of the phases are not modified by the impurities but the critical phenomenon is (see, *e.g.*, [12] and references therein). On the one hand, the critical exponents of second-order phase transitions might be modified by disorder; on the other hand, disorder may smooth out the discontinuities of first-order phase transitions rendering them continuous or even pushing them to zero temperature, as it was rigorously proven to occur in the 2d random field Ising model. The Harris criterion estimates the fluctuations of a second-order critical temperature induced by the disorder spatial fluctuations. Finite regions of the system can order due to fluctuations in the quenched randomness below the critical point of the pure case and above the one of the disordered case. These properties manifest in non-analyticities of the free energy, the so-called Griffiths singularities. For instance, deviations from the Curie–Weiss ($\chi = 1/T$) behaviour appear below the Néel temperature of dilute anti-ferromagnets in a uniform field.

Spin-glass materials [13] are the archetype of systems with *strong disorder*. They are very imbalanced mixtures of a majority of non-magnetic (sometimes metallic) elements with a minority of magnetic elements. The compound is prepared in the liquid phase and it is very rapidly cooled down and let solidify. The location of the magnetic impurities is random and the ensuing ferromagnetic and anti-ferromagnetic interactions induce frustration. These systems show a second-order phase transition between a paramagnetic and a spin-glass phase in which the spins take local and different preferred directions. In the SK model, defined on a complete graph with two-spin interaction strengths drawn from a probability distribution function, the spin-glass phase is very peculiar, with a very complex free-energy landscape as a function of local magnetisations. The order parameter is no longer a simple observable, but a relatively complex functional. These results, found with the replica theory [14], were later confirmed by the analysis of metastable states and, more recently, pure probability theory [15,16].

The study of the ordered static phases in systems with strong quenched randomness beyond mean-field is very difficult. The nature of the spin-glass phase has been a matter of debate for more than 30 years, as the competing droplet scaling picture [17] assumes that it is just a ferromagnet in disguise with two equilibrium states related by spin reversal symmetry. Which one of the two pictures prevails in finite dimensions remains an open question.

Very few analytic techniques have been successfully applied to finite-dimensional strongly disordered models in equilibrium. Four kinds of RG methods are currently being explored. The first one is a real-space variation of the Migdal–Kadanoff block renormalisation on regular lattices. Differently from this, the Dasgupta–Ma or strong disorder RG consists in the progressive elimination of the degrees of freedom with higher energy, to obtain an effective low-energy theory. It assumes that disorder-induced fluctuations dominate with respect to any other source of fluctuations. It becomes asymptotically exact if the distribution of disorder broadens without limit at large scales [18]. The method has been successfully applied to random quantum spin chains, the random field Ising model, low-dimensional diffusion in random media, notably, the Sinai problem. The field theoretical functional RG takes into account the evolution of the statistic properties of the full random potential under re-scaling. This method has been mainly used to study interfaces in random media, but connections with the SK model have also been established [19]. Finally, extensions of the so-called exact RG to deal with the effects of disorder are currently being investigated [20].

2.4.2. Structural glasses

In *structural glassy systems* [21], the dynamic variables are not quenched, but still interact in a competing way. Their low-temperature configurations may look disordered, but still have macroscopic properties similar to those of a crystalline state. Structurally, a liquid and a glass look similar, but while the former cannot support stress and flows, the glass has

solid-like properties as crystals, it supports stress, and does not easily flow in reasonable time-scales. A dynamic crossover between these two 'states' is only observed.

The importance of the glassy problem is stressed by the ubiquitous existence of materials with the above-mentioned properties. Glasses occur over an astounding range of scales. Macroscopic examples include granular media like sand and powders. Unless fluidized by shaking or during flow, these quickly settle into jammed, amorphous configurations. Jamming can also be caused by applying stress, in response to which the material may effectively convert from a fluid to a solid, refusing further flow. Colloidal suspensions contain smaller (typically micrometre-sized) particles suspended in a liquid and form the basis of many paints and coatings and at high density tend to become glassy unless crystallisation is specifically encouraged. On smaller scales, there are atomic and molecular glasses such as window glass formed by quick cooling of a silica melt or the polymer plastics in drink bottles. Finally, on the nanoscale, vortex lines in type-II superconductors can also form intrinsic glasses due to entanglement even in the absence of external disorder. Lennard–Jones binary mixtures undergo a glassy crossover when cooled across a temperature $T_{\rm g}$ or when compressed across a density $n_{\rm g}$.

The glassy systems discussed in the previous paragraph are essentially dynamical, in the sense that no important components in them can be thought of being quenched and disordered, and exerting random forces on other variables in the ensemble. In mean-field *disordered* systems such as, for instance, a directed elastic manifold embedded in an infinite-dimensional space under a random potential with short-range correlations, or a spin model on a fully connected graph with multi-spin random interactions, a mixed kind of phase transition occurs, in which the order parameter is discontinuous, but all first derivatives of the free-energy density remain finite. A more careful analysis of the free-energy density as a function of local order parameters indicates the following organisation [22–24]. A high-temperature disordered phase is associated with a single absolute minimum of it. At a temperature commonly called $T_{\rm d}$, the Gibbs–Boltzmann measure is broken into an exponentially large (in the number of degrees of freedom) number of pure states (local minima of the order-parameter-dependent free-energy). Averaged observables are the result of the weighted average over these states and, surprisingly enough, a smooth continuation of the high-temperature behaviour is found, and the thermodynamic free energy remains analytic. The entropy of these states (defined as the logarithm of its number) is called *complexity*, and it vanishes (*entropy crisis*) at a static transition temperature, $T_{\rm s} < T_{\rm d}$. For $T < T_{\rm s}$, the states that yield the dominant contribution to the statics are the equilibrium glassy phase. This is called a *random first-order transition*. Although these models have quenched random variables, differently from structural glasses that do not, their behaviour resembles in so many ways the one of *fragile glasses* that they are accepted (at least by a part of the community) as their mean-field model. In finite dimensions, transitions should be replaced by crossovers and infinite life-times by very long ones.

2.4.3. Constraints and frustration

Frustration is the name given to the impossibility of satisfying all competing interactions simultaneously. In certain systems, the local minimisation of the interaction energy on a frustrated unit gives rise to a macroscopic degeneracy of the ground state for some sets of parameters, unconventional phase transitions and the emergence of critical phases among other interesting features. Traditional examples are *constrained classical magnets*, with anti-ferromagnets on a triangular lattice as the most prominent one.

Similar features are observed in systems with hard local constraints that are the result of complex microscopic interactions. Spin-ice samples are materials of this kind [26]. These systems exist naturally in 3d and have been engineered in the laboratory in 2d. On the theoretical side, the celebrated *vertex models* are a quite incredible playground that realises many of these special features.

2.4.4. Physics and computer science

Combinatorial optimisation can be formulated as the minimisation of a cost or energy function defined over elementary degrees of freedom, that can be recast as one instance of a spin model on a random graph and with disordered many-body interactions [27]. The randomness in the ensemble of instances is the exact parallel of the disorder in physical samples. The difference arises in the questions asked in one and the other context. While in the physical context ones determines averaged macroscopic quantities with the usual stat–mech approach, in the optimisation context one seeks the microscopic configurations that are the exact energy minima of, moreover, the *worst instance* of a problem. Indeed, physics focuses on averaged properties and usually disregards worst-case events. With this *proviso* in mind, physical tools have still been useful to better understand optimisation problems.

Analytic tools borrowed or inspired from disordered spin models such as the cavity approach and belief or survey propagation methods [27] gave a rather complete understanding of the statistical properties of these problems. A phase transition between a phase with one or many zero energy states and one in which the absolute energy minimum is strictly positive has been identified. Interesting optimisation problems, for instance, constrained satisfaction ones, turn out to have a cost function landscape similar to the one of disordered spin models of the random first-order transition class, with a partition of the solution space into clusters with a finite configurational entropy and a condensation transition associated with this complexity tending to zero. This complex structure influences the performance of physical algorithms.

Algorithms that use the physically gained knowledge of the free-energy landscape in phase space have proven useful to tackle *typical* randomly generated instances of hard optimisation problems. The hope is that knowledge on the averaged and typical behaviour might be helpful to design algorithms to attack *hard* single instances as well. For instance, it was proposed to tune a simulated annealing procedure to slowly follow thermal configurations towards a ground (zero-energy)

state until the condensation transition, but not beyond. The threshold of algorithms that do not respect local physical laws may be better, but it will depend on the stochastic rules used to wander in the configuration space.

3. Classical dynamics

I here discuss, in some detail, the dynamics of classical complex problems.

3.1. Dynamics close to equilibrium

Onsager theory (reciprocity relations) characterises the behaviour of observables that are conserved in the full isolated systems (*e.g.*, energy or matter), when these are perturbed not very far from equilibrium with a variation of their associated intensive parameter (*e.g.*, temperature or pressure). A linear Taylor expansion relates the current of the extensive quantities and the variation of their associated intensive parameter *via* kinetic or transport coefficients (*e.g.*, the familiar heat equation). This approach naturally leads to an exponential decay of the perturbation towards equilibrium. (Exponential relaxation near equilibrium is not necessarily enforced when one adopts a more microscopic approach.)

The stochastic dynamics of systems perturbed away from equilibrium by an external perturbation or an internal fluctuation are also well documented in the literature. Equilibration has consequences upon (many) time-dependent correlation functions: they are stationary, meaning that they just depend upon the dynamics during the time delays involved in their definition but not upon the starting time of the measurement, and they are linked to linear responses (susceptibilities) by fluctuation–dissipation theorems [24].

I will not expand upon the dynamics close to equilibrium here as our interest is in far-from-equilibrium situations. I just wanted to mention these two well-known facts to go beyond them below.

3.2. Dynamics far from equilibrium

The main reasons for being far from equilibrium are the following.

– I have already mentioned the possibility that an isolated classical Hamiltonian system of integrable or even non-integrable kind may not reach equilibrium (KAM theorem). One example of this kind is given by systems with long-range interactions, such as plasmas or self-gravitating ensembles, that, in the thermodynamic limit and with no dissipation nor external source of fluctuations, get trapped in out-of-equilibrium stationary states [25]. The fate of a quantum isolated system under similar conditions, named a *quantum quench*, receives great attention nowadays.

- At the most extreme microscopic level, a closed physical classical system should be described by an Hermitian Hamiltonian and its dynamics be invariant under time-reversal. In some cases (arguing some coarse-graining procedure) one departs from this natural rule and ascribes *ad hoc* updates to the degrees of freedom (in continuous or discrete time) that may not lead to equilibrium.
- The initial condition of an open system can be far away from the equilibrium states at the working conditions and the *relaxation dynamics be too slow* to take the system from one to the other. In many cases of interest, the time needed to equilibrate is a growing function of the size of the confining box or the number of variables that may diverge for all practical purposes. This situation can be realised classically and quantum mechanically as well.
- Open systems can be maintained out of equilibrium by external baths at different temperatures, particle reservoirs at different chemical potentials, external fields, *etc.* In such cases, there is transport of heat from the hot bath to the cold one, mass from one reservoir to another, *etc.* Quite generally, a stationary state, in which instantaneous observables do not depend upon time, establishes, and it is called a *non-equilibrium steady state* (NESS). This setting also appears classically and quantum mechanically.
- At a coarse-grained level external drive on a classical stochastic system can be mimicked by transition rules that do not satisfy detailed balance. Such dynamics may also lead to a NESS.

The above-listed situations are realised in a number of problems that have, or still are, intensively studied. I discuss some of them and I simultaneously sketch some methods used to deal with them below.

3.2.1. Dynamical systems

I just mention here an idea developed in this field that has been recently borrowed to study the non-equilibrium dynamics of disordered systems. In the 1970s, Ruelle proposed to construct a dynamic partition function to count trajectories rather than microscopic states. This approach, closely linked to *dynamic large deviation theory*, was worked out in some simple problems such as the Lorentz lattice gas. A renewed interest in this approach was gained in recent years with the derivation of *fluctuation theorems* [29–31], see Section 3.4, as a consequence of symmetry properties of such a dynamic large deviation function. The subsequent derivation of what is called *stochastic thermodynamics* [32,33] is also attracting much attention. The method is also used in the context of glassy systems (see, *e.g.*, [34]).

3.2.2. Open dissipative single-particle systems

Possibly, the simplest open system with out-of-equilibrium dynamics – though this is not always appreciated – is *Brownian motion*, i.e. the erratic rattling of a colloidal particle immersed in a fluid and under no external force.

Both experimentally and with, *e.g.*, the Langevin approach, one observes that, for any set of initial conditions, after some relaxation time, the velocity of the particle becomes Maxwell-distributed. This was the requirement used by Langevin to postulate the friction and noise term as well as its statistics. The velocity relaxation time depends upon the ratio between the particle mass and the strength of the friction and, in most cases of interest, it is extremely short. In these cases, the inertia term can be dropped, and the Langevin equation becomes an over-damped first-order stochastic differential equation. Among these instances are colloidal suspensions, a 'soft condensed matter' example; spins in ferromagnets coupled with lattice phonons, a 'hard condensed matter' case; and proteins in the cell, a 'biophysical' instance. In contrast, the position does not reach an equilibrium measure. Indeed, the mean-square displacement increases with time, *i.e.* the particle undergoes *diffusion*, unless the motion is confined by a finite box or an external potential. Moreover, position correlation functions *are not stationary*, and the *fluctuation–dissipation relations* between its linear response and the correlation are not the ones dictated by equilibration [24].

In this very simple example, one reckons that different dynamic stochastic variables can behave very differently in the same time regime, with the velocity being equilibrated and the position undergoing out-of-equilibrium dynamics.

This is also the simplest problem in which quenched disorder plays a dominant role. The diffusive properties of the particle can be heavily influenced by a quenched random potential in ways that depend upon its statistical properties (range of correlations) and the dimensionality of space [28]. The problem has applications to the hopping conductivity of disordered materials and the diffusion of a test particle in a porous medium among many others. It has also been used as a caricature of the dynamics of many-body complex systems with glassy nature, in which the particle represents the system's configuration and its motion occurs in a configurational space decorated with a disordered landscape. A weak effect of quenched randomness is the modification of the transport coefficient, *e.g.*, the diffusion constant, while a much stronger one is the appearance of *anomalous diffusion*, *i.e.*, a different temporal dependence of the mean-square displacement, even for white noise. Perturbative [28] and Dasgupta–Ma [18] RG techniques have been successfully applied to this problem. Anomalous diffusion has also been found in clean, but geometrically constrained systems as, *e.g.*, on fractal structures, or in particles moving in flat regular spaces, but in contact with a bath with coloured noise. The latter problem is of biological interest, as one uses Brownian particles as tracers to characterise the milieu in which they move.

3.2.3. Elastic manifolds

More interesting features are expected to be found in the relaxation dynamics of many-body or extended systems. The dynamics of manifolds with d internal dimensions embedded in $(N+d)$-dimensional spaces with N the dimension of the transverse space [35–40] is often encountered in physics. A directed line ($d=1$) mimics vortex lines in $(N+d=3)$-dimensional high-T_c superconductors or stretched polymers. An interface is a frontier separating two regions covered by two phases. It could be the border between water and oil in a liquid mixture, or between positive and negative magnetisation in a magnet. Growth phenomena such as the burning front in a forest, the advance of a crack in a rock, fluid invasion in porous media, the growth of a surface on a substrate due to material deposition combined (or not) with material diffusion, or even the growth of a bacterial colony, define interfaces that can be mimicked as elastic manifolds. One distinguishes cases in which the interfaces are closed and therefore not directed, from those in which they are single-valued with respect to some reference plane and therefore directed. A manifold in a random medium is sometimes considered to be a 'baby' spin-glass problem due to the frustration induced by the competition between the elastic energy that tends to reduce the deformations and quenched disorder that tends to distort the structure.

Interface models can be defined by restricted solid-on-solid models or with continuous equations. The former are discrete and advantageous to numerical simulations. Different rules for the particle deposition (random, ballistic, random with surface relaxation) are used. Continuous models often describe the surfaces at larger length scales: a coarse-graining process is employed to describe the surface with a continuous function. Manifolds under the effect (or not) of quenched random potentials, with (Kardar–Parisi–Zhang) or without (Edwards–Wilkinson, Mullins–Herring) non-linear interactions, and with short- or long-range contributions to the elastic energy are modelled with variations of a functional Langevin equation.

The morphology of an interface is usually characterised by its width. Under certain circumstances interfaces *roughen*, that is to say, the space-averaged dispersion of its position about its mean increases with time until reaching saturation at a value that grows with the linear size of the manifold. In the presence of quenched disorder, the asymptotic roughness undergoes a crossover at a temperature and disorder strength dependent scale. For interfaces with smaller linear length the statics and dynamics are controlled by thermal fluctuations. For interfaces with longer linear length disorder takes control and changes the scaling laws. These results were suggested by scaling arguments and they were later set into firmer grounds with the functional RG [37–40].

The relaxation dynamics of clean and disordered elastic manifolds, before the linear length-dependent crossover to saturation, occurs out of equilibrium and present *ageing effects* and non-trivial linear responses, with *violations of the equilibrium fluctuation–dissipation theorem*. Their behaviour has many points in common with the one observed in glassy systems [41]. A growing length can be identified in these problems and its properties studied carefully.

The driven dynamics of an elastic directed manifold in a random medium present two hallmarks intensively studied in the last two decades. At zero temperature, a finite force is needed to set the manifold into motion. The associated *de-pinning*

transition is a dynamic critical phenomenon with similarities with a second-order phase transition, where the velocity plays the role of the order parameter and the force the one of the control parameter. At finite temperature, the manifold moves for arbitrarily small forces, but the scaling of its mean velocity with force strength and temperature is highly non-trivial, characterising what is called *creep motion* [37–40].

A ferromagnetic material, such as iron, under a slow and smooth increase of a magnetic field, gets magnetised through a random sequence of steps associated with changes in the size and orientation of microscopic clusters of aligned atomic magnets (spins). The magnetic changes are often detected *via* acoustic noise. The amount of such Barkhausen noise increases in systems with impurities, crystal dislocations, *etc.*, and it is therefore used to detect the presence of unwanted internal material stresses and to test the sample's mechanical properties [42]. In general, the response of an elastic manifold to external driving through a disordered medium is not smooth, but exhibits discontinuous and collective jumps called *avalanches*, which extend over a broad range of space and time scales. The distribution functions of the bursts' magnitude and duration, as well as other characteristics, have been characterised. A self-similar pattern emerges with scaling laws [43].

The pulled averaged and fluctuating dynamics of manifolds in random media have been studied with the functional RG, but the non-stationary relaxation in finite-dimensional transverse space still remains out of the reach of this technique. A *dynamic Gaussian variational approach* in the limit of an infinite transverse space dimensionality is the sole analytic tool we can count upon to study this question at present.

3.2.4. Critical dissipative dynamics

In *quenches to a critical point* [44], as realised by, *e.g.*, the instantaneous change of external temperature from, say, infinity to T_c, patches with equilibrium critical fluctuations grow in time, but their linear extent never reaches the equilibrium correlation length that diverges. Clusters within clusters of neighbouring spins pointing in the same direction grow in time with fractal interfaces between them. The system builds correlated critical Fortuin–Kasteleyn clusters with fractal dimension $d_{\rm FK} = (d+2-\eta)/2$, with η the anomalous exponent, in regions growing algebraically as $R_c(t) \simeq t^{1/z_{eq}}$, with z_{eq} the dynamic critical exponent. The relaxation time diverges close to the phase transition as a power law of the distance to criticality $\tau \simeq (g-g_c)^{-\nu z_{eq}}$, with ν the exponent that controls the divergence of the correlation length. Such a slow relaxation is named *critical slowing down*. Scaling arguments and perturbative RG calculations [45–47] give explicit expressions for many of the quantities of interest. Numerical simulations probe the exponents and scaling functions beyond the available perturbative orders. In the asymptotic time regime, the space–time correlation functions have a *multiplicative* scaling form as, for instance, $C(r,t) \simeq C_{\rm st}(r) C_{\rm ag}(r/R_c(t))$ with $C_{\rm st}(r) \simeq r^{-2(d-d_{\rm FK})} = r^{2-d-\eta}$, taking into account the fact that the equilibrium structures have a fractal nature (hence their density decreases as their size grows), and the fact that the order parameter vanishes at the second-order critical point. The dependence on $r/R_c(t)$ in the ageing factor expresses the similarity of configurations at different times once lengths are measured in units of $R_c(t)$. At distances and times such that $r/R_c(t) \ll 1$, the equilibrium power-law decay, $C_{\rm st}(r)$, is recovered, thus $C_{\rm ag}(x) = 1$ at $x \to 0$. $C_{\rm ag}(x)$ falls off rapidly for $x \gg 1$ to ensure that spins be uncorrelated at distances larger than $R_c(t)$. Similar factorisations apply to higher-order correlation functions and linear responses. The equilibrium fluctuation–dissipation theorem holds at short time and length scales, where the stationary relaxation is relevant, but violations appear at long time and length scales [48,49].

3.2.5. Dynamics across a phase transition.

When a system with a *first-order phase transition* is taken to a region in the phase diagram in which it is still locally stable, but metastable with respect to the new absolute minimum of the free energy, its evolution towards the new equilibrium state occurs by *nucleation* of the stable phase. The theory of simple nucleation [50,51] establishes that the time needed for one bubble of the stable state to conquer the sample grows as an exponential of the free-energy barrier over the thermal energy available, $k_B T$. Once the bubble has reached a critical size that also depends on this free-energy barrier; it very rapidly conquers the full sample. The textbook example is the magnetic magnetisation reversal in, *e.g.*, an Ising model in equilibrium under a magnetic field that is suddenly reversed. As multiple nucleation and competition between different states intervenes, the problem gets harder to quantify and very relevant to the mean-field theory of fragile structural glasses as realised by the random first-order phase transition scenario.

Take a system with a well-understood *second-order phase transition* across the critical point by tuning a control parameter from the disordered and symmetric to its ordered and symmetry-broken phase. The system will tend to order locally in one of the, possibly many, new stable states that have the same free-energy density. *Phase-ordering* [52] is characterised by a patchwork of large domains, the interior of which is basically thermalised in one of the equilibrium phases, while their boundaries slowly move and tend to become smoother due to their elastic energy. The patterned structure is not quiescent, ordered regions tend to grow on average with a linear length $R(t)$, but the time needed to fully order the sample diverges with the system size. This picture suggests the splitting of the degrees of freedom (spins) into two categories, providing statistically independent contributions to correlation and linear response functions. Bulk spins contribute to a quasi-equilibrium stationary contribution, while interfacial spins account for the non-equilibrium part. At late times and in the scaling limit $r \gg \xi$, $R(t) \gg \xi$ and $r/R(t)$ arbitrary, with ξ the equilibrium correlation length, the system enters a *dynamic scale-invariant regime*, in which there exists a single characteristic length, $R(t)$, such that the domain structure is, in statistical sense, independent of time when lengths are scaled by $R(t)$. In practice, this means that all time and space dependence in correlation functions appear as ratios between distances and $R(t)$. For instance, the space–time correlation decomposes as $C(r,t) \simeq C_{\rm st}(r) + C_{\rm ag}(r/R(t))$, with $C_{\rm st}(0) = 1 - \langle\phi\rangle^2$, $C_{\rm st}(r) \to 0$ for $r \to \infty$, $C_{\rm ag}(0) = \langle\phi\rangle^2$ and $C_{\rm ag}(x) \to 0$ for

$x \to \infty$. In clean systems, the characteristic length grows algebraically in time, $R(t) \simeq t^{1/z}$ with z a dynamic exponent that defines the dynamic universality class. For systems with competing interactions, this law can be notably slowed down, and logarithmic growth has been obtained in these cases.

Phase ordering kinetics are rather well understood qualitatively, although a full quantitative description is hard to develop as the problem is set into the form of a non-linear functional Langevin equation for, say, a scalar ($N = 1$) field ϕ with a double-well Ginzburg–Landau free-energy and no small parameter. This problem is ubiquitous and hence very important. Coarsening has been identified in relatively complex systems such as frustrated magnets or kinetically constrained models (see Section 3.2.6). It is not clear yet whether all glassy dynamics can fall into some kind of coarsening, albeit of yet unknown type.

Weak quenched disorder renders the dynamics of macroscopic systems even slower than in clean cases. Take, for instance, the 3d random field Ising model, with paramagnetic and ferromagnetic equilibrium phases, as an example. For probabilistic reasons, the fields can be very strong and positive in some region of the sample and favour positive magnetisation, and very strong and negative in a neighbouring region and favour negative magnetisation. It will then be very hard to displace the phase boundary and let one of the two states conquer the local volume. The probability of finding such rare regions can be quantified and the time needed to displace the domain wall can be estimated with the activated Arrhenius argument. They influence very strongly the equilibrium and out-of-equilibrium dynamics on both sides of the critical point. The relaxation of a perturbation away from equilibrium becomes slower than exponential, the dynamic counterpart of the Griffiths essential singularities of the free energy. The assumption of a power-law dependence of free-energy barriers with size combined with an Arrhenius argument suggests $R(t) \simeq \ln^{1/\psi} t$ for the growing length.

The dynamics of systems with *strong quenched randomness* is very complex. After a quench from high to low temperatures, they not only show very slow out-of-equilibrium relaxation, never reaching thermal equilibrium, but they also display very intriguing memory effects under complicated paths in the parameter space [53]. Although it is not clear whether the dynamics occur via the growth of domains, scaling of dynamic correlation functions describe numerical data quite precisely and, somehow surprisingly, with a power law $R(t) \simeq t^{1/z}$. This fact is not compatible with mean-field predictions of a much complex time dependence that could perhaps only establish at much longer time scales. But the power-law growth of R is not compatible with the droplet picture predictions either. The proposal in this model is that droplet-like low-energy excitations of various sizes on top of the ground state should render the dynamics strongly heterogeneous, both in space and time. At low enough temperatures, the evolution should be dominated by thermal activation. The typical free-energy gap of a droplet with respect to the ground state and the free-energy barrier to nucleate a droplet are assumed to scale as L^θ and L^ψ, respectively, with θ and ψ two non-trivial exponents. Static order is assumed to grow as in standard coarsening systems with two equilibrium states related by symmetry. Dynamical observables such as the two-time auto-correlation function should then follow universal scaling laws in terms of a growing length $R(t) \simeq \ln^{1/\psi} t$. Before drawing conclusions, one must keep in mind that the analysis of experimental and numerical data is difficult given the limited range of time scales available in both cases. In [44], an efficient strategy for data analysis with the goal of finding the best $R(t)$ for dynamic scaling is discussed and might be of help in future analysis of this question.

3.2.6. Glassy dynamics

The *random first-order scenario* of the glassy arrest developed from the exact solution to fully-connected disordered spin models with three (or more) body interactions (see Section 2.4.2). These models present a very peculiar static and dynamic behaviour that is supposed to be the mean-field description of the glass phenomenology [22]. The dynamic approach detects the change in free-energy landscape at T_d since the relaxation time diverges at this temperature and the system cannot reach equilibrium for any temperature below it [24]. For the relaxation dynamics from a disordered initial configuration T_s does not play any role. The out-of-equilibrium dynamics are dominated by threshold states that are high in free energy and have flat directions in the free-energy landscape. The separation of time scales observed in the relaxation of two-time correlation functions can then be rationalised as the sum of two processes: a transverse and a longitudinal motion in the phase space, with the former being similar to a confined relaxation, while the latter is associated with diffusion along flat channels. The latter shows ageing effects and scaling properties, although the putative growing length is not easy to visualise [44].

The equilibrium diverging relaxation time (of a local dynamics respecting detailed balance) close and above T_d can be set in correspondence to the equilibrium diverging correlation length of a *point-to-set spatial correlation* [54]. This is shown with a formal upper bound between relaxation time and correlation length that applies to finite-dimensional systems as well.

Another approach to glassy dynamics is given by *kinetically constrained models* [55]. These are similar to lattice gases in that one and only one particle can occupy each vertex of a lattice, but only if a number of local conditions are met, typically, that there is at least a number of empty sites in the immediate neighbourhood. Single particle jumps between nearest neighbour sites under the same constraint provide the dynamics. The rates satisfy detailed balance with respect to a Boltzmann measure factorized over different sites and energetically trivial as the particles do not interact. These rules are supposed to be the result of the coarse-graining of a dense molecular system, although there is no formal proof of this fact. These models are trivial thermodynamically, but they may have a rich dynamics with ergodicity breaking, slow relaxation, ageing, and many other interesting features for many choices of the constraints.

3.2.7. Externally driven systems

The problems mentioned above can be subjected to external perturbations in the form of forces that do not derive from a potential, baths at different temperatures connected on the borders, *etc.* In this section, I mention a number of models that are held out of equilibrium by persistent driving forces that are receiving much attention at present.

Driven lattice gases [56] of asymmetric exclusion type (ASEP) [57–59] are used to mimic mass transport in various contexts. In 1d, they model molecular motors motility along filaments in the cytoskeleton, vehicle circulation along highway lanes or generic queueing problems. These models consist of a linear lattice with open or periodic boundary conditions. At each time step, one particle is chosen at random and it moves to the next site in one direction with probability p or backwards with probability $1-p$, only if that site is empty. Jumps over particles are forbidden. In open cases, particles can enter the lattice with a particular probability and leave with another probability, typically at the ends of the line.

The 1d ASEP [57–59] is quite special in that it is a genuine out-of-equilibrium model for which an exact solution for the stationary regime, with a current of particles, is known. There is a huge mathematical and theoretical physics activity around these problems due to their connection with powerful techniques (matrix product Ansatz, the Bethe Ansatz, combinatorics, random matrix theory) and their mapping to other physical problems such as interface growth and, in the particular, the KPZ equation for long length and time scales, or the XXZ spin-$\frac{1}{2}$ quantum Heisenberg chain with appropriate boundary conditions.

Reaction–diffusion processes [60] involve (possibly different kinds of) particles that freely diffuse in space and undergo reactions (annihilation, coagulation or transformations from one type to another) when they are within some prescribed range. Particle propagation can be modelled as a continuous or discrete time random walk, either on a lattice or in the continuum. Such models have been used to model population dynamics, epidemic spreading, chemical reactions, *etc.* Breaking detailed balance allows dynamic phase transitions in 1d or a non-vanishing dynamic-order parameter in 2d models with continuous symmetry, thus circumventing the Mermin–Wagner theorem.

These systems are quite special in that fluctuations govern their behaviour in a much more dramatic way than at equilibrium. For instance, in cases in which spontaneous particle decay competes with a production or branching process, fluctuations may generate a phase transition and not only alter its properties. A mean-field treatment in which correlations and spatial fluctuations are neglected (kinetic rate equations) predicts a unique active phase, while the careful treatment of spatial fluctuations (with numerical simulations or approximate approaches) demonstrates the existence of phase transitions between different asymptotic states of active (with a non-vanishing particle density) and inactive (just free of particles) kind. The continuous transition found is analogous to a second-order equilibrium phase transition, and requires the tuning of appropriate reaction rates as control parameters.

The analytic treatment of these problems is very rich [61]. The master equation for a reaction–diffusion process can be re-expressed as a Schrödinger-like equation with a non-Hermitian Hamiltonian by using second-quantized bosonic creation–annihilation operators (Doi–Peliti formalism). With the coherent state representation and an appropriate continuum limit, one constructs a path-integral for a statistical field theory. At this point, one can apply all the field-theoretical machinery and, in particular, develop RGs. In some cases, the field theory can be transformed into a form that imposes an effective Langevin equation for a (complex) field with multiplicative non-equilibrium (imaginary) noise that does not respect the Einstein relation between noise and dissipation. However, in some cases, evidencing which are the effective field theories remains a hotly debated question, and there is no complete phase transition classification for their non-equilibrium phase transitions. Experiments on reaction–diffusion physical systems are still quite rare.

Pattern formation is the spontaneous formation of macroscopic spatial structures in open systems constantly driven far from equilibrium. Patterns can be stationary in time and periodic in space, periodic in time and homogeneous in space or periodic in both space and time. The key words 'dissipative structures' or 'self-organisation' are also attached to this phenomenon that was initially studied in fluid dynamics and chemical reactions but also appears in solid-state physics, soft condensed matter and non-linear optics, and is now central to biology with, for instance, morphogenesis and the dynamics of active matter (see below) playing a predominant role. The traditional example is Rayleigh–Bénard convection: a fluid placed between two infinite horizontal plates that are perfect heat conductors at different temperature. At a threshold value of the temperature difference, the uniform state with a linear temperature profile becomes unstable towards a state with convective flow. In the context of chemistry the paradigm are systems with competition between temporal growth rates and diffusivity of the different species.

The theory of pattern formation is mature and reviewed in great detail in [62]. These systems are usually described with over-damped dissipative deterministic partial differential equations. The exact form of the equations depends on the problem at hand, ranging from Navier–Stokes equations for fluid dynamics to reaction–diffusion equations for chemical systems. A well-studied case is the non-linear Schrödinger equation.

The spatio-temporal structures are found from the growth and saturation of modes that are unstable when a control parameter is increased beyond threshold. A parallel between the kind of bifurcation of fixed-point solutions and the order of a phase transition can be established. Concretely, a linear stability analysis of the uniform state reveals the mechanism leading to the pattern formation. The analysis of non-linear effects is often realised numerically. Weak Gaussian noise can select a particular pattern among many equivalent ones.

The constituents of *active matter* [63], be them particles, lines or other, absorb energy from their environment or internal fuel tanks and use it to carry out motion. In this new type of soft condensed matter, energy is partially transformed into mechanical work and partially dissipated in the form of heat. The units interact directly or through disturbances propagated

in the medium. Some realisations are bacterial suspensions, the cytoskeleton in living cells, or even swarms of different animals.

In other driven systems, such as sheared fluids, vibrated granular matter and driven vortex lattices, energy input occurs on the boundaries of the sample. Instead, in active matter, the energy input is located on the internal units and is homogeneously distributed in the sample. Moreover, the effect of the motors can be dictated by the state of the particle and/or its immediate neighbourhood and it is not necessarily fixed by an external field.

Active matter is typically kept in a non-equilibrium steady state and displays out-of-equilibrium phase transitions that may be absent in their passive counterparts. It often exhibits unusual mechanical properties, very large responses to small perturbations, and large fluctuations not consistent with the central limit theorem. Much theoretical effort has been recently devoted to the description of different aspects, such as the self-organisation of living microorganisms, the identification and analysis of states with spatio-temporal structure, such as bundles, vortices and asters, and the study of the rheological properties of active particle suspensions (to identify the mechanical consequences of biological activity). A rather surprisingly result was obtained with a variational solution to the many-body master equation of the motorised version of the hard sphere fluid often used to model colloids: instead of stirring and thus destabilising ordered structures, the motors do, in some circumstances, enlarge the range of stability of crystalline and amorphous structures relative to the ones with purely thermal motion. The relation of this fact with the emergence of a lower than ambient effective temperature (see Section 3.3) was suggested.

3.3. Effective thermodynamics?

The quest for an approximate thermodynamic description of non-equilibrium systems or, to start with, the identification of effective parameters acting as the equilibrium ones, has a long history that I will not review here. I will simply focus on one notion, the effective temperature, $T_{\rm eff}$, that has proven to be a successful concept in classical glassy physics. The answer to the simple-looking question 'how does a glassy model respond to an infinitesimal perturbation?' had a perturbingly simple answer: the linear response is linked to the strength of the ambient noise and the spontaneous fluctuations through fluctuation–dissipation relations (FDRs) in a way that resembles strongly the equilibrium fluctuation–dissipation theorem (FDT) [24]. These relations were later shown to hold, at least within the time scales that a simulation can access, in a host of more realistic models including coarsening, driven powders, colloidal suspensions, models for high-$T_{\rm c}$ superconductors, active matter and so on and so forth. The recognition that the FDRs could be grasped in terms of $T_{\rm eff}$ paved the way to a more intuitive interpretation of the dynamics of out-of-equilibrium systems with slow dynamics, and led to the extension of other thermodynamic concepts [64]. Within the RFOT description of fragile glasses $T_{\rm eff}$ is observable-independent, it can be deduced from the variation of the complexity (or configurational entropy) around the energy reached asymptotically (as in a micro-canonical definition), and it satisfies many other welcome properties. The $T_{\rm eff}$ idea found further support from the fact that it replaces the bath temperature in Fluctuation Theorems, see below, when the system that is driven far from equilibrium is itself unable to equilibrate with its environment. Some of these properties were also checked in a variety of glass models, although in some of them, such as some kinetically constrained systems or trap models, they do not hold. Despite an enormous effort, we still lack a satisfactory understanding of its microscopic origin and real-space interpretation as well as the precise limits of validity of its thermodynamic interpretation.

3.4. Fluctuation theorems

In the last decade a number of exact results for externally driven systems were proven. The fluctuation theorem characterises the fluctuations of the entropy production over long time-intervals in certain driven steady states. It was first observed numerically [65], and later proven for reversible hyperbolic dynamical systems [66] and open systems with stochastic dynamics. This remarkable result suggested the search of other relations of similar kind. Constraints on the probability distributions for work, heat, and entropy production, depending on the nature of the system and the choice of non-equilibrium conditions, were found (Jarzinsky equation, Crooks relation, *etc.*) [33]. One especially appealing consequence of these relations is that one could use them to obtain free-energy differences of equilibrium states by performing out-of-equilibrium measurements.

4. Quantum physics

The study of canonical (or macro canonical) equilibrium properties of quantum strongly interacting many-body systems has been the main focus of condensed matter. These properties can be described with a combination of mean-field techniques, RG, and scaling arguments to the same level of satisfaction as their classical counterparts.

Having said this, an important difference with classical statistical physics is that the reduction of a coupled system into a subpart by the integration of a large piece that is taken to be the bath gives rise to long-range interactions (in imaginary time) that have non-trivial effects on the behaviour of the selected part. Equilibrium phase transitions can be modified by the bath both in location as in order. Quite generally, one observes that the order phase is enlarged by the coupling to the bath. In order to observe these effects, it is necessary to use an exact treatment of the bath (and not a Markov approximation).

Quantum non-equilibrium phenomena are continuously growing in importance, but their theoretical understanding is still at a very early stage. Significant advances in the field of ultra-cold atoms have allowed one to engineer quantum many-body systems in almost perfect isolation from the environment [67]. This ensures their coherent unitary evolution for sufficiently long times to be studied in detail. Moreover, thanks to the ability to rapidly tune different parameters, *e.g.*, the interaction strength between the atoms or the creation of controlled excitations, the non-equilibrium dynamics have been accessed.

Besides, the physical behaviour of traditional out-of-equilibrium dissipative quantum systems [8] is also a topic of great interest. In condensed-matter systems, one usually selects the relevant variables, say the spins, and treats the coupling to all other variables as the coupling to the bath, *e.g.*, phonons in a solid. In mesoscopic cases, one is mostly interested in local couplings to current-carrying leads, *i.e.* fermionic baths. In cold atomic gases, the light fields confining the gas can also lead to dissipative processes.

Therefore, the problems discussed in Section 2 that were set in microscopic terms naturally admit quantum extensions and physical realisations. In the following, I briefly mention some problems and methods that are being used in this area at present.

The thermalisation (or not) of isolated quantum systems, evolving with unitary dynamics after an abrupt or slow change in a parameter (a quantum quench), is actively being investigated [68,69]. The analytic solution of some models, especially integrable systems in 1d, has allowed one to discuss this problem in great detail and suggested that generalised Gibbs ensembles could be the relevant measure to describe them [70]. The possible connection with many-body localisation has also been evoked [71]. Beyond integrability, numerical methods in the form of variations of the density matrix renormalisation group technique (t-DMRG) using matrix product states [72] allow an almost exact simulation of large strongly interacting quantum systems in 1d. The large N quantum $O(N)$ model provides a natural playground for analytic calculations and $1/N$ expansions that suggest the existence of non-thermal long-lived quasi-stationary states in the symmetric phase [73] and other non-trivial effects. Approximate methods of analytic kind, as time-dependent variational approaches for strongly interacting fermionic systems (*e.g.,* the Hubbard model) [74,75] or numerical type, as diagrammatic Monte Carlo techniques and others [76] are currently being used in this context. Tools developed by the glassy systems community, as the use of fluctuation–dissipation relations (see Section 3.3) of quantum nature in this case, may be useful to investigate whether usual thermal equilibration establishes (or not) in this context [77].

I have already mentioned the non-trivial effect of a quantum bath on the phase transitions of a large quantum system. A sufficiently strong quantum bath may also affect the evolution of a single quantum degree of freedom as shown by the highly non-trivial localisation transition of a two-level system under the effect of a bath made of quantum oscillators at zero temperature [78]. A complete treatment of the memory induced by the bath is necessary to capture this phenomenon as well.

Impurity motion in low-d quantum liquids [79] has been a major field of research in the last decade, now realised in cold atom experiments. The 'artificial' experimental design of these systems has now become possible by confining cold atoms in optical nanotubes. Using these techniques, the diffusion of impurity atoms in contact with a Luttinger liquid (LL) with tuneable impurity-LL interaction was studied. Due to the external trapping potential, the minority atoms undergo damped oscillations indicating that dissipation takes place in this system. This problem is one realisation of *quantum Brownian motion* in a confining potential [8,80]. It can be dealt with analytic methods, *e.g.* quantum generating functionals, but also with the t-DMRG, where the bath is taken to be constituted by a finite number of atoms confined in the same harmonic trap.

A *quantum phase transition* occurs at zero temperature and it is driven uniquely by quantum fluctuations. It corresponds to a change in the symmetry of the ground state when the strength of the quantum fluctuations is appropriately tuned at zero temperature. Experiments on atoms trapped in an optical lattice that undergo a Mott insulator transition have prompted the study of equilibrium behaviour and non-equilibrium dynamics close to and across quantum phase transitions [67].

Quantum coarsening phenomena and *quantum critical dynamics* have not been studied in sufficient depth yet. One case that has been studied in detail is the mean-field representation as realised by, *e.g.*, the large N quantum $O(N)$ model coupled with an ensemble of quantum harmonic oscillators or with electron baths [81]. A separation of time-scales, as in the classical limit, and dynamic scaling with the classical growing length were found. The short-time and short-length dynamics are controlled by quantum and thermal fluctuations in equilibrium, while the long scales behave as classically with ageing and breakdown of the quantum fluctuation–dissipation theorem that takes, in this regime, a classical form. The non-trivial effect of the quantum bath was also reckoned. However, other relatively simple problems such as dissipative quantum spin chains set out of equilibrium remain to be analysed at the same level of completeness. (Some attempts to extend t-DMRG methods to include the coupling with a bath may be of help in this context, see, *e.g.*, [82].)

It is well known that quenched disorder can have fantastic effects on quantum systems with Anderson localisation as the most prominent example. I will not dwell upon this or other well-understood problems such as transport in mesoscopic systems here. Concerning the physical systems discussed in Section 2, one of the hallmarks of finite-dimensional quantum spin models with weak disorder are Griffiths–McCoy singularities of their free-energy, that lead to a highly non-trivial paramagnetic phase and critical behaviour [83]. Spin-glass phases have been identified in many condensed matter systems at very low temperature. *Quantum glassy phases* exist also in electronic systems and structural glasses. Wigner glasses are other examples of relevance modelled as elastic systems under the effect of quenched randomness.

The mean-field analysis of quantum dissipative systems with quenched randomness can be performed with the same level of detail as the one for the classical counterparts. A definition and study of metastable states (in phase space) in

mean-field quantum glasses has been done with an extension of the Thouless–Anderson–Palmer approach. The results are compatible with the analysis of the free-energy density with the replica Matsubara approach. The out-of-equilibrium relaxation of these systems coupled with a quantum environment, for a generic initial density matrix not correlated with the quenched randomness hence mimicking a quench from the disordered phase, can be investigated with the Schwinger–Keldysh closed-time formalism [84]. For details on these calculations, see [85] and references therein. Even more so than in the classical case, going beyond mean-field is extremely hard.

In spin models on hyper-random graphs with finite or infinite connectivity and multi-spin interactions, the random first-order phase transition becomes a genuine first-order one at low temperatures [85]. This fact has formidable consequences to the failure of quantum annealing methods to solve hard optimisation problems that can be mapped, as already mentioned in Section 2.4.4, onto these models. The quantum annealing or adiabatic quantum algorithm is the quantum counterpart of thermal simulated annealing. The idea is to change the parameters in a quantum Hamiltonian to take the system across a convenient path in the phase diagram. The progressive vanishing of quantum fluctuations should then allow it to go from a simple ground state to the one of the hard classical problem one is actually interested in. This strategy will obviously fail when going across a first-order transition line [86].

A problem of great interest is the characterisation of the (*e.g.*, mass or energy) current flowing through a system coupled with external reservoirs at different temperatures or chemical potentials on its borders. Again, the literature on this topic is huge. I just want to mention here that the *full-counting statistics*, that is to say, all cumulants of the current, have recently been computed with conformal field theory methods in 1d quantum problems at criticality [87] (see this reference for other methods applied to these same problems).

By the end of our discussion on classical systems, I mentioned two generic ideas that have attracted considerable attention in recent years: effective temperatures in systems in slow relaxation or weakly driven by external forces (see Section 3.3) and fluctuation theorems (see Section 3.4). Deviations from the equilibrium fluctuation–dissipation theorem in quantum (mean-field) systems were found [64] and more recently some driven quantum problems were analysed from this perspective as well (see, *e.g.* [88]). However, the thermodynamic sense of this notion in the quantum context has not been explored yet. In parallel, quantum fluctuation theorems have been proposed and are currently being explored. I cite here only one very recent proposal to analyse quantum work statistics with an experimental set-up [89] from which the rich literature on this subject can be re-constructed.

5. Conclusions

In the body of these notes I have tried to summarise our current understanding of out-of-equilibrium complex systems by focusing on a certain number of problems. Non-equilibrium physics is a very wide area of research and it is also rapidly evolving. My summary is necessarily partial. In particular, I have tried to cite review articles that could give a global description of the topics touched upon in the text. Consequently, I have not made justice to original articles in the field.

In this section I will simply list a number of problems that, in my view, would be interesting, and possible, to address theoretically in the near future.

- Conformal field theory combined with Coulomb gas methods yielded a myriad of results on a large diversity of equilibrium geometric objects at 2d critical points (critical percolation, self-avoiding walks, loop-erased random walks, q-states Potts models, *etc.*). More recently, conformally invariant stochastic growth, the so-called SLE [90,91], allowed one to re-derive these results in more mathematical terms as well as to extend them in various directions. Could these be of help in out-of-equilibrium cases as well?
- Phase ordering kinetics or coarsening is a qualitatively well-understood phenomenon [52]. However, there is no efficient analytic tool allowing one to prove dynamic scaling, derive the growing length systematically, obtain the scaling functions, *etc.* This remains an open problem since long ago.
- I gave a short description of the statics and dynamics of models with quenched disorder that is based on the exact solution to mean-field models [22–24]. Whether these results hold true in finite dimensions is still an open question. The improvement of various RG techniques may be of help to decide upon this point.
- The same *incognita* concerns the behaviour of glassy systems with no quenched randomness [22].

As already said, most of these questions can be posed including quantum fluctuations.

Acknowledgements

I wish to thank all my collaborators in this field for many years of very interesting exchanges and ANR-BLAN-0346 (FAMOUS) for financial support.

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