C. R. Physique 15 (2014) 539–546

Contents lists available at ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

Grand Prix Mergier–Bourdeix 2013 de l'Académie des sciences

Ginzburg–Landau vortices, Coulomb gases, and Abrikosov lattices

Vortex de Ginzburg–Landau, gaz de Coulomb et réseaux d'Abrikosov

Sylvia Serfaty [a,b]

[a] *UPMC (Université Paris-6) & CNRS, UMR 7598 "Laboratoire Jacques-Louis-Lions", 75005 Paris, France*
[b] *Courant Institute, New York University, 251 Mercer st, New York, NY 10012, USA*

ARTICLE INFO

Article history:
Available online 25 June 2014

Keywords:
Ginzburg–Landau equations
Superconductivity
Vortices
Abrikosov lattice
Coulomb gas
One-component plasma

Mots-clés :
Équations de Ginzburg–Landau
Supraconductivité
Vortex
Réseau d'Abrikosov
Gaz de Coulomb
Plasma à un constituant

ABSTRACT

This is a review of a mathematical analysis of vortices in the Ginzburg–Landau model: phase transitions and effective energies that govern optimal patterns formed by the vortices, in particular the Abrikosov lattice, are discussed. Analogies with Coulomb gases are also mentioned.

RÉSUMÉ

Cet article présente une analyse des vortex dans le modèle de Ginzburg–Landau : les transitions de phase ainsi que les énergies effectives qui gouvernent les structures optimales formées par les vortex, en particulier le réseau d'Abrikosov, sont discutées. Des analogies avec les gaz de Coulomb sont aussi évoquées.

1. The Ginzburg–Landau model of superconductivity

As discovered by Kamerlingh Onnes in 1911, certain metals, when cooled down below a critical temperature, lose their resistivity, and permanent currents can flow without energy dissipation. This phenomenon is called superconductivity, and (together with its brother, superfluidity) is one of the most striking macroscopic manifestations of a quantum phenomenon, see [1–3] for a presentation.

A characteristic feature of superconductors is that they exhibit the Meissner effect: they expel an applied magnetic field (by creation of an opposite magnetic field generated by a superconducting current)—this is responsible for the classic photograph of a magnet levitating above a superconductor. If the magnetic field is too large, however, then it destroys the superconductivity and penetrates the sample. Superconducting materials are often classified as type-I or type-II according to their response: type-I superconductors have one critical field at which the material undergoes a transition from superconducting to normal, while type-II ones have two critical fields, between which—as first discovered by Shubnikov—they

E-mail address: serfaty@ann.jussieu.fr.

http://dx.doi.org/10.1016/j.crhy.2014.06.001
1631-0705/

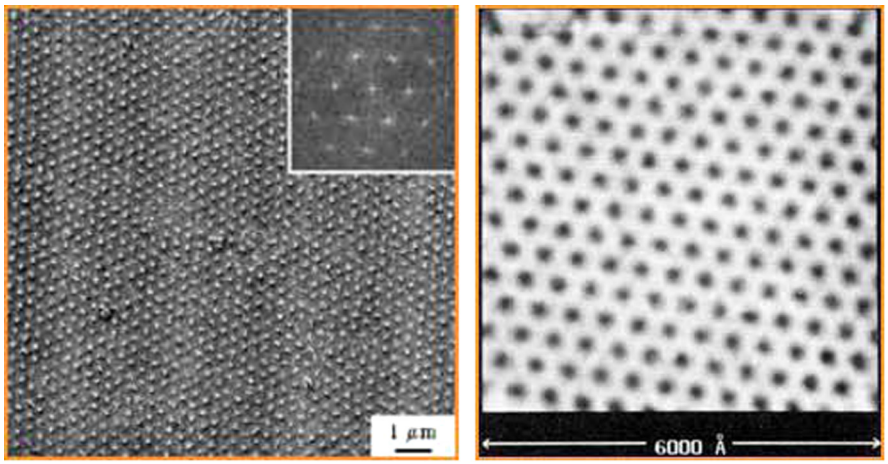

Fig. 1. Abrikosov lattices.
Source: H.F. Hess et al. *Phys. Rev. Lett.* 62, 214 (1989).

allow for a *mixed phase* with partial penetration of the magnetic field via *vortices*, which are small regions of normal phase surrounded by a loop of superconducting current. When the field is large enough, the experiments (dating from the 1960's) show that vortices arrange themselves in perfect triangular lattices, cf. http://www.fys.uio.no/super/vortex/ or Fig. 1. These are named *Abrikosov lattices* after the physicist Abrikosov who had predicted, from the Ginzburg–Landau model, that periodic arrays of vortices should appear [4]. These vortices repel each other like Coulomb charges would, while being confined inside the sample by the applied magnetic field. Their triangular lattice arrangement is the result of these two opposing effects.

1.1. The model

In the 1950's Landau and Ginzburg introduced their celebrated model on phenomenological grounds [5]. Gorkov later showed that it coincides with the quantum Bardeen–Cooper–Schrieffer (BCS) theory [6] near T_c, see also [7], and for a rigorous derivation of Ginzburg–Landau from BCS, see [8]. In the Ginzburg–Landau model, the energy of a superconductor occupying Ω, in the presence of a constant applied field H_{ex}, when the exterior region is insulating, is:

$$G(\psi, A) = G_0 + \int_{\mathbb{R}^3} \frac{|\operatorname{curl} A - H_{ex}|^2}{8\pi} + \int_{\Omega} \frac{1}{2m^*} \left| \left(\hbar \nabla - \frac{ie^*}{c} A \right) \psi \right|^2 + \alpha |\psi|^2 + \beta |\psi|^4 \tag{1}$$

Besides the physical constants $\hbar$ and c, additional constants m^* and e^* are present (see [1] for an explanation of these constants) as well as two quantities α and β that depend on the temperature T and on the superconducting material. Near the so-called critical temperature T_c, it is assumed that β is a positive constant and α is proportional to $T - T_c$ and has the same sign. The quantity G_0 represents the energy of the normal state and does not depend on ψ or A. From then on, we consider that we are below the critical temperature T_c. After some nondimensionalizing procedure (described for example in [9, Chap. 2]) and reduction to a two-dimensional domain, the energy functional can be reduced to:

$$G_\varepsilon(\psi, A) = \frac{1}{2} \int_{\Omega} \left| (\nabla - iA)\psi \right|^2 + |\operatorname{curl} A - h_{ex}|^2 + \frac{1}{2\varepsilon^2} \left(1 - |\psi|^2\right)^2 \tag{2}$$

This is an idealized situation where the sample is assumed to be a three-dimensional infinitely long cylinder with cross-section Ω, submitted to an external field parallel to the axis of the cylinder and of intensity h_{ex}. It can also be used to describe thin films, such as high-T_c superconducting compounds for which the Ginzburg–Landau theory has turned out to work quite well [10].

In the above energy functional, the parameter ε is a material constant, it is the inverse of the "Ginzburg–Landau parameter" usually denoted κ. It is also the ratio between the "coherence length", usually denoted ξ (roughly the vortex-core size), and the "penetration length" of the magnetic field, usually denoted λ. We are interested in the regime of small ε, corresponding to high-κ, or extreme type-II superconductors, also called the London limit.

The energy depends on two unknown functions ψ and A. ψ is a complex-valued function, called *order parameter* and indicating the local state of the sample: $|\psi|^2$ is the density of *Cooper pairs* of superconducting electrons. With our normalization $|\psi| \leq 1$, and where $|\psi| \simeq 1$ the material is in the superconducting phase, while where $|\psi| = 0$, it is in the normal phase (i.e. behaves like a normal conductor), the two phases being able to coexist in the sample. A vortex is an object

centered at an isolated zero of ψ, around which the phase of ψ has a nonzero winding number, called the *degree of the vortex*. A typical vortex centered at a point x_0 is the ansatz $\psi = \rho e^{i\varphi}$ with $\rho(x_0) = 0$ and $\rho(x) = f(\frac{|x-x_0|}{\varepsilon})$ where $f(0) = 0$ and f tends to 1 as $r \to +\infty$, i.e. its characteristic core size is ε, and

$$\frac{1}{2\pi} \int_{\partial B(x_0, R\varepsilon)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

is its degree.

The vector field A is the gauge field or vector potential of the magnetic field. The induced magnetic field in the sample is deduced by $h(x) = \nabla \times A = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$, it is thus a real-valued function in Ω.

Note that the Ginzburg–Landau model is also the simplest gauge theory with Abelian gauge $\mathbb{U}(1)$. For further details on the model, we refer to [5,11,12,1–3] on the physics side, [13,9] on the mathematics side.

The Ginzburg–Landau model has led to a large amount of theoretical physics literature, probably most relevant to us is the book by De Gennes [11]. However, a precise mathematical proof of the phase transition at the first critical field, and of the emergence of the Abrikosov lattice as the ground state for the arrangement of the vortices was still missing. A series of mathematical works in the 1990's opened the way cf. e.g. [14–16].

1.2. Rigorous results on critical fields and vortices

We summarize here the results of joint work with Étienne Sandier [9], on the vortices in ground states of the energy G_ε, in other words minimizers of G_ε and describe via mathematical proofs, the values of the critical fields for which vortices appear, the vortex patterns for energy minimizers, and the limiting energies that govern their interaction, in the limit $\kappa \to \infty$, or equivalently $\varepsilon \to 0$.

Recall that a complex-valued map ψ can be written in polar coordinates $\psi = \rho e^{i\varphi}$ with a phase φ which can be multi-valued. Given a configuration (ψ, A), we define its *vorticity* by

$$\mu(\psi, A) = \operatorname{curl} j + \operatorname{curl} A = \operatorname{curl} j + h \tag{3}$$

where $j = \langle i\psi, \nabla_A \psi \rangle$ is the superconducting current, with the notation $\langle a, b \rangle = \frac{1}{2}(a\bar{b} + \bar{a}b)$.

When ε is small, we have the approximate (formal) relation:

$$\mu(\psi, A) \approx 2\pi \sum_i d_i \delta_{a_i} \tag{4}$$

where a_i's are the vortices of ψ and d_i's their degrees, and δ_p the Dirac mass centered at a point p. Thus the quantity $\mu(\psi, A)$ is appropriate as a proxy for the vortices of ψ (it is formally like the vorticity for fluids). We also have the following relation between the induced magnetic field h and the vorticity

$$\begin{cases} -\Delta h + h = \mu(\psi, A) \approx 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega \end{cases} \tag{5}$$

which is known as the *London equation*. Thus the induced magnetic field behaves like a potential generated by point charges at the vortices, in an electrostatic analogy. The London equation indicates how the magnetic field penetrates in the sample through the vortices.

In fact, the relation (5) is only approximately true; what gives a more correct picture is to write

$$-\Delta h + h = 2\pi \sum_i d_i \delta_{a_i}^{(\varepsilon)}$$

where $\delta_x^{(\varepsilon)}$ denotes the Dirac mass at x smeared out at the scale ε (or in other words a smooth function of integral 1, with support in a disc of radius ε), characteristic length scale of the vortices.

1.3. Formal correspondence with a Coulomb gas

It turns out that it is more convenient to express the energy of a configuration (ψ, A) in terms of the induced magnetic field h, via the London equation (5). Some computations (with the help of all the mathematical machinery developed to describe vortices), cf. [17–19], lead eventually to the conclusion that everything happens as if the Ginzburg–Landau energy G_ε of a configuration were equal to

$$\begin{aligned} G_\varepsilon(\psi, A) &\simeq \frac{1}{2} \int_\Omega |\nabla h|^2 + |h - h_{\text{ex}}|^2 \\ &= \frac{1}{2} \iint_{\Omega \times \Omega} G_\Omega(x, y) \left(2\pi \sum_i d_i \delta_{a_i}^{(\varepsilon)} - h_{\text{ex}}\right)(x) \left(2\pi \sum_i d_i \delta_{a_i}^{(\varepsilon)} - h_{\text{ex}}\right)(y) \end{aligned} \tag{6}$$

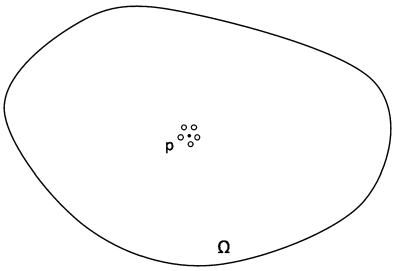

Fig. 2. Minimizers with small number of vortices.

where G_Ω is a type of Green (or Yukawa) kernel, solution to

$$\begin{cases} -\Delta G_\Omega + G_\Omega = \delta_y & \text{in } \Omega \\ G_\Omega = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

and h solves (5). With this way of writing, and in view of the logarithmic nature of G_Ω, one recognizes essentially a pairwise (screened) Coulomb interaction of positive charges in a constant negative background ($-h_{\text{ex}}$), supplemented with the (large but not infinite) self-interaction of all the charges. This leads to viewing the vortex system essentially as a Coulomb gas (we will discuss the Coulomb gas subsequently).

1.4. First critical field

One may guess that at the first critical field H_{c_1}, i.e. the smallest h_{ex} for which vortices appear in energy-minimizers, the induced magnetic field is close to that of configurations without vortices, which in view of (5) is the solution h_0 to

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega \end{cases} \tag{8}$$

In fact, the value of H_{c_1} can be guessed by perturbing h off of h_0 and, starting from (6), expanding G_ε in a suitable manner. Letting

$$\lambda_\Omega = \left(2\left|\min(h_0 - 1)\right|\right)^{-1} \tag{9}$$

and $\Lambda = \{x \in \Omega / h_0(x) = \min h_0\}$, we find (see [17] for a simple derivation) that vortices first become favorable at the points of Λ, and when $h_{\text{ex}} \geq \lambda_\Omega |\log \varepsilon|$. More precisely, we prove (see [20], [9, Chap. 11]) that

$$H_{c_1} = \lambda_\Omega |\log \varepsilon| + C_\Omega + o(1) \quad \text{as } \varepsilon \to 0$$

and that below H_{c_1} there is no vortex, while for $h_{\text{ex}} = H_{c_1}$ isolated vortices appear near points of Λ (with at most one vortex near each isolated point of Λ). This improves on the formal expansion of [11], which gave $H_{c_1} \approx \frac{1}{2}|\log \varepsilon|$; the two expansions actually agree when the domain Ω becomes very large, because then $\lambda_\Omega \to \frac{1}{2}$.

We assume then on for simplicity that Λ is reduced to only one point, denoted $\bar{p}$ (this is the case for example when Ω is convex), and denote $Q(x) = \langle D^2 h_0(\bar{p})x, x\rangle$ its second-order differential, assumed to be definite positive. We are then able to characterize further transitions: we prove [20,9] that there exists an increasing sequence of additional "critical" fields $H_2, H_3 \ldots$ with

$$H_n = \lambda_\Omega |\log \varepsilon| + (n-1)\lambda_\Omega \log \frac{|\log \varepsilon|}{n} + \text{constant-order terms}$$

separated by increments of log $|\log \varepsilon|$, for which a second, third, ..., vortex becomes favorable. Each time the optimal vortices are located close to $\bar{p}$ as $\varepsilon \to 0$ (cf. Fig. 2) and after blowing-up at the scale $\sqrt{\frac{h_{\text{ex}}}{n}}$ around $\bar{p}$, they converge as $\varepsilon \to 0$ to configurations that minimize an *effective interaction energy* given by

$$H_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^{n} Q(x_i) \tag{10}$$

Observe that H_n contains a repulsion and a confinement term; it is in fact exactly the Hamiltonian of a two-dimensional Coulomb gas of n particles in a confining potential Q. When Q has rotational symmetry, numerical minimization (see Gueron–Shafrir [21]) yields very regular shapes (regular polygons for $n \leq 6$, regular stars) which look very much like the birth of a triangular lattice as$n$ becomes large (their density tends to be uniform, supported in a fixed disc of $\mathbb{R}^n$ as $n \to \infty$), see Fig. 3.

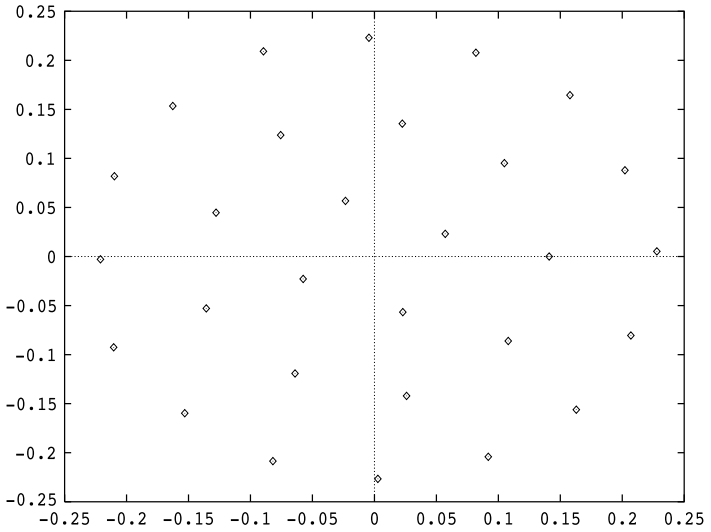

Fig. 3. Numerical minimization of H_n by Gueron–Shafrir [21], $n = 29$.

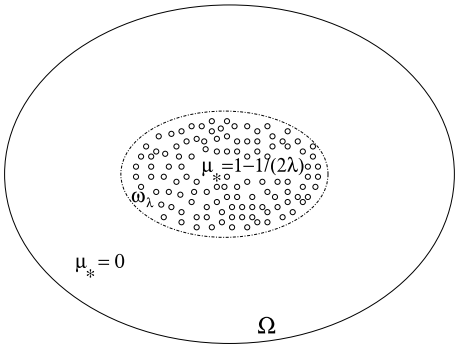

Fig. 4. Optimal density of vortices according to the obstacle problem.

In [22] and [9, Chap. 12], we prove the existence of branches of local minimizers of (2) (i.e. metastable solutions) of similar type that have arbitrary bounded numbers of vortices, all of degree $+1$, and the locations of the vortices in these solutions also minimize (15). These solutions exist for wide ranges of the parameter h_{ex}. All these results are in very good agreement with experimental observations and theoretical findings [11,1,23].

When the applied field h_{ex} becomes such that $h_{\text{ex}} - H_{c_1} \gg \log\ |\log \varepsilon|$, then we show that the number of vortices in any minimizer diverges as $\varepsilon \to 0$. The distribution of the vortices can then be described in averaged form, by characterizing the limit of $\mu(\psi, A)$ normalized by the expected number of vortices. We show [9, Chap. 9] that for $h_{\text{ex}} - H_{c_1} \gg \log\ |\log \varepsilon|$, vortices of minimizers still concentrate around the point $\bar{p}$, but when zooming at the appropriate scale their average distribution is uniform in a subregion near $\bar{p}$, given as the unique minimizer of

$$\mathcal{F}(\mu) = -\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_{\mathbb{R}^2} Q(x)\,\mathrm{d}\mu(x) \tag{11}$$

This is a standard minimization problem in potential theory, known as the capacitor problem (with external field), first studied by Gauss and solved by Frostman (cf. [24] for details). When $h_{\text{ex}} - H_{c_1}$ is of the same order as $|\log \varepsilon|$ then as long as $h_{\text{ex}} \ll \frac{1}{\varepsilon^2}$, then we show [9, Chap. 7] that the optimal distribution of vortices is uniform and proportional to h_{ex} in a subregion of $\omega_{h_{\text{ex}}}$ depending on h_{ex} and characterized via an *obstacle problem* (see Fig. 4). In other words, the normalized vorticity $\frac{1}{h_{\text{ex}}}\mu(\psi, A)$ behaves like a multiple of the characteristic function of $\omega_{h_{\text{ex}}}$, as $\varepsilon \to 0$. This subregion $\omega_{h_{\text{ex}}}$ is reduced to the point $\bar{p}$ when $h_{\text{ex}} = H_{c_1}$ and then grows as h_{ex} increases. When $h_{\text{ex}} \gg |\log \varepsilon|$, then $\omega_{h_{\text{ex}}}$ tends to cover the whole domain Ω.

1.5. Next-order results

The results mentioned above show that above H_{c_1}, for $\lambda > \lambda_\Omega$, the number of vortices is proportional to $h_{\rm ex}$ and they are uniformly distributed in a subregion of the domain, but it is still far from explaining the optimality of the Abrikosov lattice. To (begin to) explain it, one needs to look at the next order in the energy asymptotics, and at the blown-up of (5) at the inverse of the intervortex distance scale, which here is simply $\sqrt{h_{\rm ex}}$. For simplicity, let us reduce to the case $\lambda = 1$ (or $h_{\rm ex} \gg |\log \varepsilon|$) where the limiting optimal measure is $\mu_* = \mathbf{1}_\Omega$ and the limiting $h \equiv 1$.

Once the blow-up by $\sqrt{h_{\rm ex}}$ is performed and the limit $\varepsilon \to 0$ is taken, (5) becomes

$$-\Delta h + 1 = 2\pi \sum_a \delta_a \quad \text{in } \mathbb{R}^2 \tag{12}$$

where the limiting blown-up points a form an infinite configuration in the plane, and these are now true Dirac masses (one may in fact reduce to the case where all degrees are equal to $+1$, other situations being energetically too costly).

One may recognize here essentially a *jellium* of infinite size, and $E = \nabla h$ the electric field generated by the points (its rotated vector field $j = -E^\perp$ corresponds to the superconducting current in superconductivity). The jellium model was first introduced by Wigner [25], and it means an infinite set of point charges with identical charges with Coulomb interaction, screened by a uniform neutralizing background, here the density -1. It is also called a *one-component plasma*. It then remains first to identify and define a limiting interaction energy for this jellium, and second to derive it from G_ε. Of course defining the total Coulomb interaction of such a system is delicate because several difficulties arise: first, the infinite number of charges and the lack of local charge neutrality, which lead us to defining the energy as a thermodynamic limit; second the need to remove the infinite self-interaction created by each point charge, now that we are dealing with true Dirac masses. Note that h satisfying (12) has a logarithmic singularity near each a, and thus $|\nabla h|^2$ is not integrable; however, when removing small balls of radius η around each a, adding back $\pi \log \eta$, and letting $\eta \to 0$, this singular energy can be subtracted or "renormalized", roughly like

$$W(E) = \lim_{R\to+\infty} \frac{1}{|B_R|}\left(\lim_{\eta\to 0} \int_{B(0,R)\setminus \cup_a B(a,\eta)} |E|^2 + \pi(\log\eta)\sum_i 1\right) \tag{13}$$

In the particular case where the configuration of points Λ has some periodicity, i.e. if it can be seen as n points $a_1, \cdots, a_n$ living on a torus $\mathbb{T}$ of appropriate size, then W can be expressed much more simply as a function of the points only:

$$W(a_1, \cdots, a_n) = \frac{\pi}{|\mathbb{T}|}\sum_{j\neq k} G(a_j - a_k) + \pi \lim_{x\to 0}\big(G(x) + \log|x|\big), \tag{14}$$

where G is the Green's function of the torus (i.e. solving $-\Delta G = \delta_0 - 1/|\mathbb{T}|$). The Green function of the torus can itself be expressed explicitly in terms of some Eisenstein series and the Dedekind Eta function. The definition (13) thus allows us to generalize such a formula to any infinite system, without any periodicity assumption. We show that ground states, once zoomed in at the scale $\sqrt{h_{\rm ex}}$, form patterns that tend to minimize W, and we obtain an expansion of the ground-state energy:

$$\min G_\varepsilon = C_2 h_{\rm ex}^2 + C_1 h_{\rm ex} + o(h_{\rm ex})$$

where C_2 is a constant identified via the mean-field limit and C_1 is a constant that depends in a simple way of $\min W$.

The question of central interest to us thus becomes that of understanding the minimum and minimizers of W. We have the partial minimization result:

Theorem 1. *(See [19].) The minimum of W over perfect lattice configurations (of density 1) is achieved uniquely, modulo rotations, by the triangular lattice.*

By triangular lattice, we mean the lattice $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$, properly scaled. The proof relies on results from number theory about the minimization of the Epstein zeta function of a lattice.

In view of the experiments showing Abrikosov lattices in superconductors, it is then natural to formulate the

Conjecture 1. *The "Abrikosov" triangular lattice is a global minimizer of W.*

This question belongs to the more general family of crystallization problems. If this is true, then it would justify the emergence of the Abrikosov lattice in the regime of applied fields considered here $h_{\rm ex} \ll h_{\rm ex} \ll H_{c_2}$ and in the asymptotics of small ε.

2. The 2D Coulomb gas

The connection with the jellium is what prompted us to examine in [26] the consequences that our study could have for the 2D classical Coulomb gas. More precisely, we are thinking of a 2D Coulomb gas of n particles $x_i \in \mathbb{R}^2$ in a confining potential Q (growing sufficiently fast at infinity) with Hamiltonian

$$H_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log|x_i - x_j| + n \sum_{i=1}^{n} Q(x_i), \tag{15}$$

which we already encountered in (15).

The Gibbs measure for the same Coulomb gas at temperature $1/\beta$ is

$$\mathrm{d}\mathbb{P}_n^\beta(x_1, \cdots, x_n) = \frac{1}{Z_n^\beta} \mathrm{e}^{-\frac{\beta}{2} H_n(x_1, \cdots, x_n)} \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{16}$$

where Z_n^β is the associated partition function, i.e. a normalization factor that makes $\mathrm{d}\mathbb{P}_n^\beta$ a probability measure. The study of this Gibbs measure also finds motivation in the *Ginibre ensemble* of random matrices [27], the connection between Coulomb gases and random matrices was first pointed out by Wigner [28] and Dyson [29]. For general background and references, we refer to [30].

The Hamiltonian H_n is easily seen to be connected to (11) in the limit $n \to \infty$: in fact one can prove [24] that under suitable assumptions on Q, $\min H_n \approx n^2 \min \mathcal{F}$ as $n \to \infty$ and that for sequences of minimizers $(x_1, \ldots, x_n)$ of H_n, we have convergence of the empirical measures $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ to the equilibrium measure μ_0 (minimizer of (11)). This can be called the mean field limit, and is analogous to the obstacle problem distribution found for Ginzburg–Landau. Deriving this limit is significantly easier than for Ginzburg–Landau, due to the discrete nature of the starting energy, and the fact that all charges are $+1$. As in the Ginzburg–Landau situation, one can go to the next order in the expansion of $\min H_n$. The connection with the Ginzburg–Landau situation is made by defining analogously the potential generated by the charge configuration using the mean-field density μ_0 as a neutralizing background, this yields the following equation playing the role of the analogue to (5):

$$h_n = -2\pi \Delta^{-1}\left(\sum_{i=1}^{n} \delta_{x_i} - n\mu_0\right) \quad \text{in } \mathbb{R}^2.$$

The next step is again to express this in the blown-up coordinates at scale $\sqrt{n}$ (analogous to the $\sqrt{h_{\mathrm{ex}}}$ scale previously) around x_0, $x' = \sqrt{n}(x - x_0)$, via h_n' the solution to

$$h_n'(x') = -2\pi \Delta^{-1}\left(\sum_{i=1}^{n} \delta_{x_i'} - \mu_0\left(x_0 + \frac{x'}{\sqrt{n}}\right)\right). \tag{17}$$

When taking $n \to \infty$, the limit equation to (17) is

$$-\Delta h = 2\pi\left(\sum_a \delta_a - \mu_0(x_0)\right) \quad \text{in } \mathbb{R}^2 \tag{18}$$

analogue of (12), corresponding to another infinite jellium with uniform neutralizing background $\mu_0(x_0)$.

Expanding the energy to next order is done via a suitable splitting, by analogy with Ginzburg–Landau. In fact, in this setting the splitting procedure is quite simple: it suffices to write $\nu_n := \sum_{i=1}^n \delta_{x_i}$ as $n\mu_0 + (\nu_n - n\mu_0)$. Noting that

$$H_n(x_1, \cdots, x_n) = \iint_{\triangle^c} -\log|x - y| \, \mathrm{d}\nu_n(x) \, \mathrm{d}\nu_n(y) + \int Q(x) \, \mathrm{d}\nu_n(x)$$

where $\triangle$ denotes the diagonal, inserting the indicated splitting of ν_n, we eventually find an exact decomposition, and this leads us to a next-order expansion of the ground-state energy:

$$\min H_n = n^2 \mathcal{F}(\mu_0) - \frac{n}{2}\log n + n\left(\frac{1}{2\pi}\min W - \frac{1}{2}\int \mu_0 \log \mu_0\right) + o(n) \tag{19}$$

where W is the same renormalized energy as in (13). Again if Conjecture 1 were established, this would indicate that points in zero-temperature Coulomb gases should form a crystal in the shape of an Abrikosov triangular lattice (or perturbations of one with equal average energy), in agreement with predictions in the literature (cf. [31] and references therein).

When (15) is considered for $x_i \in \mathbb{R}$ instead of $\mathbb{R}^2$, then it is the Hamiltonian of what is called a *log gas*. The corresponding results are proven in [32], together with the definition of an appropriate one-dimensional version of W, for which the

minimum is this time shown to be achieved by the lattice configuration (or "clock distribution") $\mathbb{Z}$. We also treat the case of higher dimensional Coulomb gases in [33].

The expansion to next order of H_n is valid for arbitrary configurations, not only ground states. They can thus be inserted into the Gibbs measure, yielding new results on the next-order asymptotic expansion of the partition function [26,33] (in contrast to the one-dimensional log gas case where Z_n^β is known, at least for Q quadratic, for all β by Selberg integrals), and information about thermal states and how they should crystallize as the temperature tends to 0.

Also, the result relates the computation of Z_n^β to that of the unknown constant $\min W$, so to prove Conjecture 1 it would suffice in principle to know how to compute Z_n^β for a 2D Coulomb gas.

Minimizers of H_n are also viewed as *weighted Fekete sets* in approximation theory [24], and our conjecture is equivalent to a conjecture of [34] on the order n term in the expansion of the minimal logarithmic interaction on the 2-sphere, as shown by [35].

For other self-contained presentations of these topics, one can refer to [18,17].

References

[1] M. Tinkham, Introduction to Superconductivity, second edition, McGraw-Hill, New York, 1996.
[2] J. Tilley, D. Tilley, Superfluidity and Superconductivity, second edition, Adam Hilger Ltd., Bristol, 1986.
[3] J.F. Annett, Superconductivity, Superfluids and Condensates, Oxf. Ser. Condens. Matter Phys., 2004.
[4] A. Abrikosov, On the magnetic properties of superconductors of the second type, Sov. Phys. JETP 5 (1957) 1174–1182.
[5] V.L. Ginzburg, L.D. Landau, in: D. Ter Haar (Ed.), Collected Papers of L.D. Landau, Pergamon Press, Oxford, 1965.
[6] J. Bardeen, L. Cooper, J.R. Schrieffer, Theory of superconductivity, Phys. Rev. (2) 108 (1957) 1175–1204.
[7] A. Larkin, A. Varlamov, Theory of Fluctuations in Superconductors, Oxford Science, Oxford, UK, 2005.
[8] R. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, Microscopic derivation of Ginzburg–Landau theory, J. Am. Math. Soc. 25 (3) (2012) 667–713.
[9] E. Sandier, S. Serfaty, Vortices in the Magnetic Ginzburg–Landau Model, Birkhäuser, 2007.
[10] G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, V.M. Vinokur, Vortices in high-temperature superconductors, Rev. Mod. Phys. 66 (4) (1994) 1125–1388.
[11] P.-G. De Gennes, Superconductivity of Metal and Alloys, Benjamin, New York and Amsterdam, 1966.
[12] D. Saint-James, E.J. Thomas, Type-II Superconductivity, Pergamon Press, Oxford, 1969.
[13] S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Prog. Nonlinear Differ. Equ. Appl., vol. 77, Birkhäuser, 2010.
[14] S.J. Chapman, A hierarchy of models for type-II superconductors, SIAM Rev. 42 (4) (2000) 555–598.
[15] Q. Du, M. Gunzburger, J.S. Peterson, Analysis and approximation of the Ginzburg–Landau model of superconductivity, SIAM Rev. 34 (1) (1992) 54–81.
[16] F. Bethuel, H. Brézis, F. Hélein, Ginzburg–Landau Vortices, Prog. Nonlinear Partial Differ. Equations Their Appl., Birkhäuser, 1994.
[17] S. Serfaty, Coulomb Gases and Ginzburg–Landau Vortices, Zurich Lecture Notes in Mathematics, Eur. Math. Soc., forthcoming, arXiv:1403.6860.
[18] S. Serfaty, Ginzburg–Landau vortices, Coulomb gases, and renormalized energies, J. Stat. Phys. 154 (3) (2014) 660–680.
[19] E. Sandier, S. Serfaty, From the Ginzburg–Landau model to vortex lattice problems, Commun. Math. Phys. 313 (3) (2012) 635–743.
[20] S. Serfaty, Local minimizers for the Ginzburg–Landau energy near critical magnetic field, part I, Commun. Contemp. Math. 1 (2) (1999) 213–254;
S. Serfaty, Local minimizers for the Ginzburg–Landau energy near critical magnetic field, part II, Commun. Contemp. Math. 1 (2) (1999) 295–333.
[21] S. Gueron, I. Shafrir, On a discrete variational problem involving interacting particles, SIAM J. Appl. Math. 60 (1) (2000) 1–17.
[22] S. Serfaty, Stable configurations in superconductivity: uniqueness, multiplicity and vortex-nucleation, Arch. Ration. Mech. Anal. 149 (1999) 329–365.
[23] V.A. Schweigert, F.M. Peeters, P. Singha Deo, Vortex phase diagram for mesoscopic superconducting disks, Phys. Rev. Lett. 81 (13) (1998) 2783–2786.
[24] E. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, 1997.
[25] E. Wigner, On the interaction of electrons in metals, Phys. Rev. 46 (1934) 1002.
[26] E. Sandier, S. Serfaty, 2D Coulomb gases and the renormalized energy, arXiv:1201.3503v2, 2014, Ann. Probab., in press.
[27] J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. 6 (1965) 440–449.
[28] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, Ann. Math. (2) 62 (3) (1955) 548–564.
[29] F. Dyson, Statistical theory of the energy levels of a complex system. Part I, J. Math. Phys. 3 (1962) 140–156;
F. Dyson, Statistical theory of the energy levels of a complex system. Part II, J. Math. Phys. 3 (1962) 157–165;
F. Dyson, Statistical theory of the energy levels of a complex system. Part III, J. Math. Phys. 3 (1962) 166–175.
[30] P.J. Forrester, Log-gases and Random Matrices, Lond. Math. Soc. Monogr. Ser., vol. 34, Princeton University Press, Princeton, NJ, USA, 2010.
[31] A. Alastuey, B. Jancovici, On the classical two-dimensional one-component Coulomb plasma, J. Phys. 42 (1) (1981) 1–12.
[32] E. Sandier, S. Serfaty, 1D log gases and the renormalized energy: crystallization at vanishing temperature, arXiv:1303.2968.
[33] N. Rougerie, S. Serfaty, Higher dimensional Coulomb gases and renormalized energy functionals, arXiv:1307.2805.
[34] J.S. Brauchart, D.P. Hardin, E.B. Saff, The Next-Order Term for Optimal Riesz and Logarithmic Energy Asymptotics on the Sphere, Contemp. Math., vol. 578, 2012, pp. 31–61.
[35] L. Bétermin, Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere, arXiv:1404.4485.