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Slow granular flows: The dominant role of tiny fluctuations

Écoulements granulaires lents : Le rôle dominant des très petites fluctuations

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ABSTRACT

What mechanism governs slow flows of granular media? Microscopically, the grains experience enduring frictional contacts in these flows. However, a straightforward translation to a macroscopic frictional rheology, where the shear stresses are proportional to the normal stresses with a rate-independent friction coefficient, fails to capture important aspects of slow granular flows. There is now overwhelming evidence that agitations, tiny fluctuations of the grain positions, associated with large fluctuation of their contact forces, play a central role for slow granular flows. These agitations are generated in flowing regions, but travel deep inside the quiescent zones, and lead to a nonlocal rheology.

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RÉSUMÉ

Quel est le mécanisme qui gouverne les écoulements lents d'un milieu granulaire ? À l'échelle microscopique, les grains sont soumis à des contacts frottants. Cependant, une interprétation directe de ces écoulements par une rhéologie macroscopique, dans laquelle les contraintes de cisaillement seraient proportionnelles aux contraintes normales avec un coefficient de frottement indépendant de la vitesse, ne permet pas de reproduire certaines propriétés importantes des écoulements lents. Il est maintenant clair que, lorsqu'elles sont associées à de grandes fluctuations des forces de contact, de très petites fluctuations des positions des grains jouent un rôle capital dans ces écoulements. Bien que l'agitation des grains provienne de zones en mouvement, elle pénètre profondément dans les régions inactives, donnant ainsi naissance à une rhéologie non locale.

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1. Introduction & motivation

When the flow rate in a granular medium is so slow that the grains remain in contact most of the time, so that collisions play a minor role for the momentum transfer, we speak of slow or dense granular flows. Typical examples include the slow deformations of a granular pack under the motion of external boundaries, such as in Couette [1,2] and split-bottom [3]

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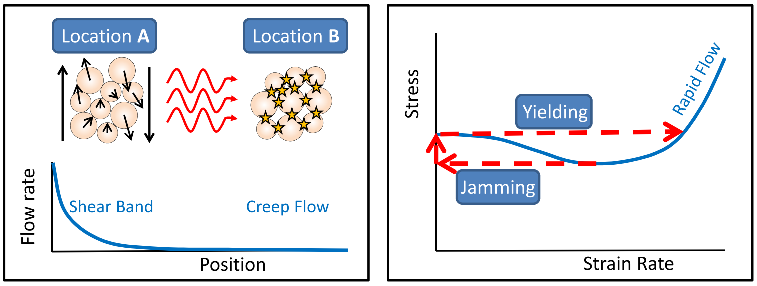

Fig. 1. (Color online.) (Left) Schematic depiction of the nonlocal "agitation" picture, where a localized flow in location A is associated with strong fluctuations in the microscopic motion. These lead to fluctuations in the contact forces that are carried through the medium (wavy lines) until they reach the nearly stationary zones (location B), where there is no appreciable grain motion, but the contact forces still fluctuate strongly (stars). (Right) Schematic depiction of a nonmonotonic flow curve. As a function of strain rate, the stresses initially decrease, before increasing sharply for rapid flows where collisions start to play a role. In controlled stress experiments, such a curve would lead to a hysteretic yielding-jamming behavior, as indicated.

geometries. We note that for faster flows, such as chute flows or avalanches, a successful description has been developed based on a rate-dependent friction coefficient [4–6]. In contrast, we still lack a predictive description for slow granular flows: given boundary conditions, what determines the flow rate and profile? How are stresses and strain rates related?

Slow granular flows differ in important ways from Newtonian flows, as friction plays a central role. This suggests that, first, shear stresses τ are proportional to the normal stresses P, and second, that the ratio of shear to compressive stress τ/P, which can be seen as an effective friction coefficient, μ_{eff}, does not vary strongly with rate and remains finite for strain rates going to zero [2,4–8].

These two assumptions form the core of a classical 'Mohr–Coulomb' picture of slow granular flows. First, they are consistent with rheological experiments such as in Couette, split-bottom or vane geometries, which find that the flow profiles do not vary with the driving rate, and the shear stresses are proportional to the confining pressure but rate independent [2,7–10]. Second, they give a simple picture for the yielding and jamming of granular flows: when τ/P is less than the effective friction coefficient of the medium, $\tau/P < \mu_{\mathrm{eff}}$, the granular material remains essentially solid and does not flow, whereas flow sets in when $\tau/P > \mu_{\mathrm{eff}}$—hence the frictional picture leads to a yielding criterion. This picture captures important aspects of the yielding of granular medium on an inclined plane: for small inclination angles θ, the sand remains fixed (here $\tau/P = \tan(\theta) < \mu_{\mathrm{eff}}$), whereas for sufficiently large angles (where $\tau/P = \tan(\theta) \geq \mu_{\mathrm{eff}}$), the sand starts to flow [4,6,8,11]. Third, this picture gives a simple explanation for the occurrence of sharp shear bands in granular flows, as small changes in the shear stress may lead to a drop from above to below the yielding criterion. For example, in Couette flows, the shear stress decays away from the inner cylinder with $1/r^2$ [12], leading to a narrow shear band near the inner wall, as observed experimentally [1].

However, a closer observation of these explanations, paired with a wealth of new experiments, illustrates important limitations of the Mohr–Coulomb picture. First, it ignores the crucial role played by fluctuations of the contact forces, which we will refer to as agitations, and second, such model cannot capture important aspects of the jamming/yielding transition. In the remainder of this paper, we will expand on these points (illustrated in Fig. 1), but before doing so, let us give the essential flavor of these.

Agitations: Let us consider the sharpness of the yielding criterion, which predicts the localization of flows in shear bands and a corresponding sharp separation between stationary, solid-like zones, and flowing zones [11,13–15]. There is by now overwhelming evidence against this picture: experiments clearly show that there is no sharp boundary between flowing and stationary zones, with (creep) flow occurring even far away from the main shear band. This makes the concept of shear band problematic; if one simply defines the shear band as all locations where the strain rate is finite, shear bands are very wide and often span the whole system. To distinguish the creep regions from the localized regions where most of the flow takes place, we will define shear bands as those regions where 99% of the strain is localized.

Moreover, several recent experiments [16–18] and theoretical works [19–22] indicate that for matter with granularity, the "fluidity" in location B, i.e. the local relation between stress and strain rate, is strongly influenced by the flow in location A. The physical picture is that localized flows lead to large fluctuations in the local contact forces, and that these fluctuations can propagate deep into the near-stationary zones. Even though the average shear stresses may be small there, the *fluctuations* in the contact forces lead to a finite probability that the local friction can be overcome. Consistent with this, the near-stationary or creep zones are observed to have a zero yield stress, and are thus not solid [14,17,18]. The resulting nonlocal behavior was first observed in the flow of emulsions [16], has also been observed in foams [23], and has been modeled by a diffusive model for the fluidity, leading to the introduction of a length scale that characterizes the nonlocality [16,20,21]. Kamrin and Koval have adapted these ideas for frictional rheologies, capturing granular Couette flows [19], and more recently, the full flow profiles and height dependence of split bottom granular flows [3,22,24,25].

Nonmonotonic friction: Let us now reconsider the assumption that μ_{eff} is constant at low rates (and eventually increases at rapid flow rates), which is motivated by the observed rate independence of the stresses for slow flows. Clearly, if the flow rate is not determined by the stresses, there is nothing that sets the flow rate for a given stress—this in itself is a major

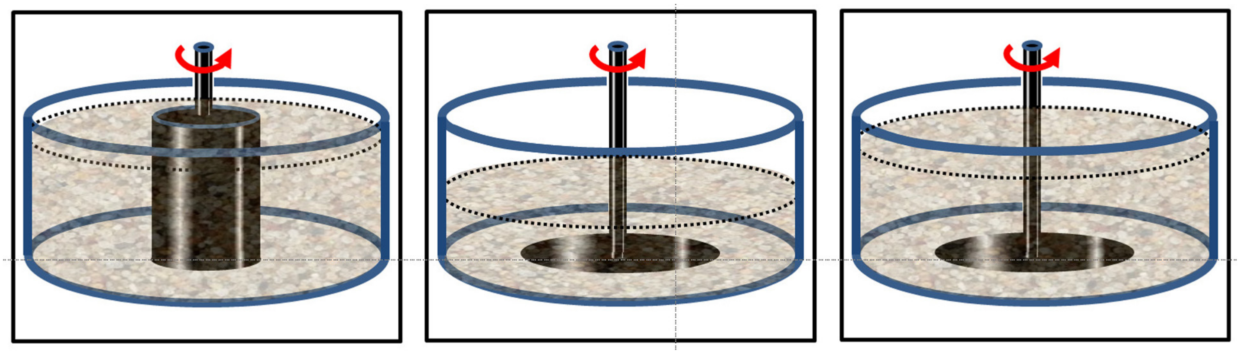

Fig. 2. (Color online.) Flow geometries and regions of main flow. (a) Couette geometry: a central rotating cylinder drives the flow, which remains localized near this cylinder. (b) Split-bottom cell, shallow filling height: a rotating disk drives the flow, which is localized in a cylindrical shape above the edge of the disk. (c) Split-bottom cell, deep filling height: the flow now localizes is a dome-like shape, pinned to the edge of the disk. For details, see [25].

problem. But even if we accept this limitation for the moment, and only consider that the frictional picture gives a yield criterion, i.e. determines the transition between flow and no flow, we run into problems.

Consider the inclined plane geometry, where we imagine to adiabatically increase the inclination angle θ. Once the granular material yields, observations suggest that the flow rate jumps to a finite value; it is not possible to obtain arbitrarily slow avalanches. This can in principle be captured by assuming that the static yield stress is larger than the dynamic yield stress, in analogy to static and dynamic friction [26]. This then leads to a difference between the static and dynamic angle of repose, consistent with observations.

However, this is not sufficient, as we can see by considering the case where we adiabatically lower the inclination angle θ from a value above the angle of repose: if the dynamic friction coefficient is monotonically increasing with the rate, this implies that the flow rate continuously decreases to zero when the stress (inclination angle) is lowered. In experiments, however, the flow is found to stop discontinuously: stress-controlled granular flows have a minimal flow rate [10,27,28]. As we will discuss, both the difference between dynamic and static angle and the finite jump in flow rate can be captured more naturally in a picture where the effective friction coefficient initially decreases with flow rate: the negative slope signals an instability that leads to a "forbidden" range of flow rates [12].

In this paper, we outline the contours of a more modern approach to slow granular flows, building on the observations of nonlocal rheology, and the suggestion that the friction is a nonmonotonic function of flow rate. We will not attempt to review the full literature, but hope that this paper serves to outline the main issues and open the relevant literature for the non-specialists. The outline of this paper is as follows. In Section 2 we describe experimental evidence for the crucial role of fluctuations in the flow of granular media. In Section 3 we present recent work on the so-called "secondary rheology", where one considers the situation where the driving of the main flow and the probing of the rheology are separated. In Section 4, we discuss some recent models to capture the nonlocal rheology. In Section 5 we finally present evidence for the nonmonotonic nature of the effective friction, and we close this paper with a brief outlook in Section 6.

2. Flow and agitations

The flow of granular media such as sand or rice is almost always inhomogeneous, with the flow often, but not always, localizing in shear bands near boundaries. A prime example is the localization observed in a Couette cell. In such a geometry, the granular medium fills the gap between an outer and inner cylinder, and their relative rotation generates a shear flow—often, the outer cylinder is stationary while the inner cylinder drives the flow. Apart from effects near the vertical boundaries, the shear stresses in such a system are purely set by geometry, and decay as $1/r^2$—for a narrow gap Couette system, this leads to a near-constant stress profile, useful for rheological measurements [12]. See Fig. 2.

For granular media, however, one observes a very strong localization on the particle scale of the flow near the inner cylinder, with the material that is 20 particle diameters away essentially appearing stationary: this is called shear banding. The flow profiles have been found to be rate independent over a wide range of driving rates, both in experiments and simulations [1,29–32].

This raises several questions: what causes this localization and what sets the width of the shear band? What is the flow profile connecting the shear band to the granular material far away? In the far-away zones, is the flow completely absent?

Localization: There are at least two mechanisms at play in shear band formation in Couette cells. First, the radial decay of the shear stresses leads to shear banding in a wide range of yield stress fluids—basically, for radii where the shear stress $\tau = \tau_0/r^2$ falls below the yield stress, the material is at rest. As the stresses in slow granular flows are essentially rate independent, one imagines that when the rotation rate of the inner cylinder is kept fixed, the stress in the material grow until just above the yielding point—a narrow region near the inner cylinder then flows, but the shear stress already drops below the yielding point nearby, leading to narrow shear bands.

In addition, it is clear that near the inner cylinder, the microscopic details of the granular packing are different. For monodisperse grains, smooth boundaries may cause crystallization, which locally increases the density and may lead to slip-planes, where ordered layers of grains easily slide past one another [1]. Conversely, rough boundaries or polydisperse beads may actually lead to more loosely packed grains near the boundary. Both mechanisms lead to a lowering of the yield stress near the boundary, leading to additional localization.

It will not come as a surprise then that the shear band localizes on the scale of a few grains—there is no other relevant length scale in the system. The details of the flow profile in the shear band are strongly dependent on particle shape, roughness of the cylinder, etc. [1,9,33]—there does not seem to be much hope for describing these profile quantitatively or at a continuum level.

Away from the flowing region: Observation with the eye suggests that far away from a shear band, the grains remain stationary. However, careful studies of these apparent stationary regions have revealed slow creep flows in these regions, and suggest that in non-cohesive granular media, there is no transition to a stationary zone away from a shear band—instead the strain rate decays continuously. A beautiful illustration of this was shown by Komatsu et al. [34] who made images of a stationary, localized flow with increasing exposure time. As a function of exposure time, the blurred region, i.e. the region where there had been flow increased without obvious bound, and a more qualitative measurement of the flow demonstrated an exponentially decaying flow profile over six decades in flow velocity. Earlier Couette experiments also hinted at exponential tails, although we stress again that the region near the wall does not appear to be universal [1]. By now, the exponential decay has been observed for a wide range of systems, and it has been established that the characteristic length of the exponential decay varies between one and several grain diameters (see [35], and references therein).

Split-bottom geometry: In order to study the slow flow of granular media away from side walls, we introduced the split-bottom geometry [3]. Here, the flow is driven by the slow rotation of a disk at the bottom of a container that is filled with granular media. This causes the pinning of a wide shear zone away from the lateral boundaries, as the shear zone emanates from the "split" in the bottom at the edge of the rotating disks [25]. If the container is large enough, and if the disk radius is larger then the filling height, the width of the flowing zones grow without any obvious bound: there is no localization in narrow shear bands [3,25]. On the other hand, when the filling height is larger than the disk radius, the flowing zone localizes near the bottom boundary, making this system also well-suited for measurements of the secondary rheology [17,36].

Surprisingly, the nature of the resulting granular flow is insensitive to the details of the grains used, and the location of the shear zone can be captured by a simple, universal scaling form [3], and can also be captured by a simple theoretical framework [37] that also predicts the filling height dependence of the driving stress [38]. This suggests that a continuum description should be possible for such flows [14]. Finally, the flow profile is particle independent, and can be described by an error function: there is thus no sharp boundary between flowing and non-flowing zones here. We note that in split-bottom geometries, the decay of the flow rate away from the main flowing zone is Gaussian, instead of exponential as in Couette flows [3,12].

Fluctuations: There have been vast numbers of studies that probe the nature of the fluctuations in granular flows. It is useful to distinguish these fluctuations into velocity/positional fluctuations and force fluctuations. Even though contact forces in granular media ultimately arise from the precise location of the grains, note that in practical situations, the Hertzian contacts are so stiff that they lead to an almost complete separation of scales. For example, the Hertzian deformation of a 1-mm glass bead under its own weight is of the order of a nanometer, and the deformation of the same glass bead at the bottom of a one meter deep pile is of the order of 100 nm—still 4 orders of magnitude below its diameter. This separation of scales allows us, for example, to study force statistics [39].

Clearly, even tiny grain motions thus lead to large force fluctuations, and once generated in a flowing zone, such fluctuations can propagate far into the granular medium, as evidenced by force measurements [29,30], and studies on fluctuating particle motions [9,31,33]. Indeed, it has long been accepted that such fluctuations play a crucial role in dense granular flow [40].

Concluding remarks: For plastic granular flows, a detailed study of the flow profiles has found that these never have sharp boundaries. Exponentially decaying flow profiles typically arise when shear bands localize near a boundary, whereas shear zones in the bulk of the material decay faster. Nevertheless, all evidence is that once there is flow anywhere in the system, all of the system is flowing, albeit slowly. This is in sharp contrast to the expected behavior of an ordinary yield stress fluid [12], and also is inconsistent with a simple frictional picture, where shear bands are expected to attain essentially zero width [14,37,41].

3. Secondary rheology

In most rheological studies, the motion of a single boundary, such as a sliding plate, rotating disk or rotating cylinder, drives the flow. The global rheological curve (relation between driving stress and flow rate) is then determined by the motion of this boundary. In such a case, nonlocal effects manifest themselves via subtle effects in the macroscopic rheology, or via details of the spatial flow profiles. However, the most direct manifestation of nonlocal rheology is that flows in location A influence the rheology in B. Hence, to probe this, one wishes to consider the case where the driving of the main flow and the probing of the rheology are separated (Fig. 3). Such (numerical) experiments are said to probe the *secondary rheology*.

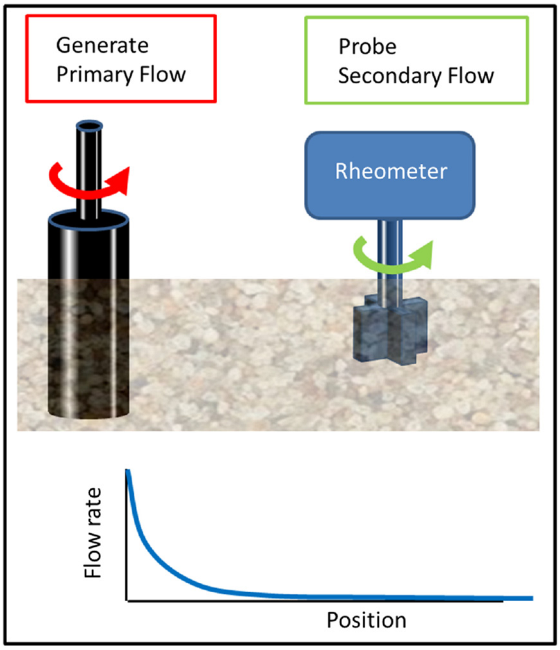

Fig. 3. (Color online.) Schematic picture, showing how a primary flow is generated (here by the rotation of a cylinder, left), which is mainly localized near the cylinder (profile below)—note that the primary flow rate remains finite but very small far away from the cylinder, and in fact, is not relevant for the secondary rheology [17]. The secondary flow is then probed, here by rheology employing a vane geometry, far away from the flowing zone (right).

In recent years, several examples of granular flow experiments have been designed to probe this secondary rheology, first by probing the sinking of a passive probe into a granular bed that is stirred far away from the probe [17], and second by doing explicit rheological measurements on objects rotating or dragged through a granular medium that is itself stirred [18,36]. In addition, numerical simulations probing the secondary rheology are also starting to appear [20]. In all cases, the far away zones appear to have a zero yield stress, and the secondary flow rates are proportional to the primary flow rate—but there are several differences and subtleties in the details of the observed secondary rheological laws.

3.1. Experiments

In the first example, the main flow is generated in a split-bottom cell, where the primary flow is driven by the rotation of a disk at the bottom of the container [17]. For large filling heights, the flow is localized near this disk, and (almost) no flow can be observed near the free surface [3,24], which is thus ideally suited for experiments on the secondary rheology. A probe particle, such as a steel ball, placed at the surface of the beads will get stuck at a small depth if the disk that drives the flow in a split-bottom cell is not rotating. However, when the disk begins rotating, such a probe particle immediately starts to sink into the grains. Several crucial properties of the secondary rheology can be determined from this sinking: *(i)* probes of large density continue to sink until they reach the bottom of the container, while lower-density probes reach a floating depth given by Archimedes' law. The latter is strong evidence for the absence of a finite yield stress. *(ii)* The overall rate of the secondary flows is directly proportional with the primary flow rate: hence, the sinking speed of a given probe particle does not only depend on its weight and size, but is also linearly proportional with the rotation rate of the disk that drives the primary flow. This is strong evidence for the direct influence that the primary flow has on the secondary rheology: the primary flow sets the relevant time scale for the secondary rheology, and no material time scale is at play. *(iii)* In the case of floating probes, the approach to this equilibrium depth is exponential in time, suggesting that the drag forces on the probe are linear in the velocity of the probe particle. *(iv)* Finally, the residual flow varies over many decades as a function of depth (in a similar way to the exponential flow profiles in Couette cells), whereas the effective viscosity in some cases is near depth independent. This is strong evidence for a nonlocal picture: it is the flow in A, rather than the residual flow in B, which sets the secondary rheology in B.

All of this strongly suggests that whenever there is flow anywhere in a granular system, all the material gets fluidized. Even though the absence of a yield stress and a linear relation between drag force and velocity suggests a viscous picture, we note here that this effective viscosity strongly varies with pressure, and can reach numerically large values—in some experiments at large filling heights, a steel ball of 25 mm diameter can take more than 24 hours to sink by 1 cm! Hence, in many situations, this fluidization is not easy to determine. A vivid illustration of this can be formulated as follows: suppose that you are standing still in the desert, and there is an avalanche on a remote sand dune. The fluidization picture then predicts that the sand you are standing on gets fluidized and that you would sink into the sand—however, very very slowly.

Shortly thereafter, Reddy et al. probed the secondary rheology in Couette geometries, and again found that a shear band in a granular medium dramatically changes the mechanical behavior of the material further away in the non-sheared region [18]. The experiments can be seen as microrheology experiments, where a constant force is applied to a small rod immersed outside the shear band. In the absence of a shear band, a yielding force is needed to move this rod, but once the system is flowing, the rod moves with respect to the residual flow for all finite forces—again there is no yielding criterion. Also here the secondary rates (i.e. the speed of the rod for low stresses in the presence of a finite primary rate) is proportional to the primary rate. In contrast to the work of Nichol, however, the relation between force and rod motion was found to be simply exponential, suggesting an Eyring-like activated process [18].

The melting of the yield stress, the proportionality between secondary and primary flows and the exponential relation between stress and secondary flow rate was also found for the secondary rheology in split-bottom cells, where a vane-shaped probe (instead of a sinking intruder) probes the rheology [36]. These experiments also addressed the following two questions: (i) what is the effect of pressure—are shear stresses still proportional to pressure? (ii) what is the role of anisotropy—granular flows lead to anisotropy of the granular fabric [42,43], which should influence the fluidity of the material. Indeed the local rheology was found to be anisotropic, with significant differences between co-flow, counter-flow, and perpendicular flow. Surprisingly, the anisotropy of the flow mainly manifests itself via the pressure dependence: for secondary flows counter to the main flow, the shear and normal stresses are proportional, whereas for co-flow the pressure dependence is more complex. A recent work by Andreotti seems to find a similar difference between co-flow and counter-flow situations in simulations [20]. Hence, local pressure and local anisotropy are crucial to describe the fluidity of granular media, thus stressing that anisotropy may be the next frontier needed to understand slow granular flows.

Concluding remarks: Despite subtle differences in the details of the secondary rheological laws, a coherent picture is emerging. The grain motion in the zones where the primary flow localizes causes strong force fluctuations in the local contact forces that propagate far away from the flowing zone, deep into the near-stationary zones. These fluctuations lead to a vanishing of the yield stress throughout the medium, and the fact that secondary flow rates are directly proportional to the primary flow rate suggests that there are no additional time scales to consider here—the agitations can be seen as quasi-static. We stress here that such agitations should be distinguished from those that are generated externally by, e.g., vibrations [8,38,44], as these generally have a much weaker effect.

4. Modeling

Clearly, several aspects of slow granular flows, such as the narrow and non-universal spatial profile of shear bands near boundaries, are not amenable to a continuum description. However, away from the boundaries, slow granular flows appear to behave more robustly and their features are smooth enough to give hope that there a continuum description may be found—this is why the split-bottom geometry, where the flow localizes away from the boundaries, plays a crucial role in probing the physics of slow granular flows.

Discrete modeling: Despite a lack of predictive continuum models that describe dense flows, various simulation techniques that model the individual particles appear to be able to capture many of the features of even complex flows [45]. Hence, important information on the form of continuum fields, such as strain rate and stress tensor, can be extracted to constrain possible continuum theories. For example, Depken et al. investigated the smooth granular flows in linear shear in the absence of gravity, slow inclined plane flows, and split-bottom geometries [14]. They assumed that the fluctuations in the flow imply that the material is fluidized, in the following sense: if there is a finite shear stress along a certain plane in the flow, then this should lead to a finite flow rate. In more technical terms, this means the stress and strain rate tensors should be colinear, i.e., have the same principle directions. This is by no means trivial, and many prior flow models violated this condition. Then, under the additional assumption that the shear stresses are proportional to pressure, but rate independent, they found that while these assumptions are consistent with the simple flow profiles in linear shear, such stresses are *inconsistent* with the observed flow profiles in the split-bottom cell—in these cases, the flow has to localize completely if stress balance is to be obeyed. To be consistent with the observed flow profiles, stress balance dictates that the effective friction coefficient has to vary throughout the shear band. Subsequent numerical work indeed found this subtle variation of the effective friction coefficient, and also provided strong numerical evidence for the assertion that the stress and strain rate tensors are colinear [15]. We note here that the issue of colinearity is far from settled, with other simulations finding small deviations from colinearity [46].

Nonlocal rheology: The next step in understanding the effects of fluidization of a particulate medium by its flow came in seminal work by Goyon *al.* on the flow of emulsions [16], which firmly established the existence of nonlocal rheology. First, from rheological measurements in a wide-gap Couette cell, these authors established that the bulk rheology of these emulsions was well fit by a Herschel–Bulkley law. Second, they tested that this global rheology also applies *locally* in wide-gap geometries. Third, they then went to narrow-gap geometries, where the stresses and flow rates have significant gradients on the bubble scale—the typical case in granular flows—and found that the observed flow profiles significantly deviate from those predicted by the global rheological law. To model this failure, Goyon et al. explored differences in local rheology near the wall and in the bulk and introduced a fluidity parameter—essentially the inverse viscosity—, which varies across the channel according to a simple diffusion equation. Fluidity is largest near the wall and extends toward the center of the channel over a characteristic flow-cooperativity length. Fitting their data to the resulting model, a wide range of profiles could be captured. Such nonlocal behavior was later observed in foams also [23], and has been captured in several models [16,20,21,47].

The physical picture underlying the nonlocal rheology is that strong shear induces strong fluctuations, which in turn induce plastic rearrangements further away [21]. Indeed, for the emulsion flows, one finds that away from the wall, where the stresses fall below the global yield stress, and where the local model predicts zero shear flow, the experiments still evidence shear.

Continuous modeling: The observations on emulsion flows are very reminiscent of the nonlocal effects observed in granular flows. On the basis of these observations, Kamrin and coworkers have developed continuum models for well-developed granular flow, combining nonlocal effects with the local theory that describes faster flows. In their model, a single material parameter captures the cooperativity, or nonlocality, of the flow. Technically, this model builds on the statistics of a kinetic elasto-plastic mechanism, which can be seen as similar to the famous SGR picture [48] where mesoscopic regions of material undergo plastic yielding and subsequent relaxation, but in contrast to SGR, there is no thermal but rather mechanical activation: flow induces flow.

This model works surprisingly well. It describes all 3D flow profiles observed in the split-bottom geometry—even though these profiles are functions of the filling height, and are qualitatively different for shallow and deep layers, the models captures all of this [22]. Moreover, this model also describes qualitatively different flows in Couette geometries [19]. Finally, this approach also captures secondary rheology [47]. This line of modeling is thus an incredibly important step forward in our understanding of slow granular flows.

Concluding remarks: Based on combining a frictional picture with the ideas on nonlocal rheology, models for slow flows have now emerged which, with the addition of a single material parameter, make accurate and correct predictions for the wide variety of flows observed in split-bottom and Couette geometries. It will be interesting to see if such models can be extended to describe anisotropic effects, as well as to investigate their application to recent observations on secondary rheology.

5. Nonmonotonic friction and yielding

As we have seen in the previous sections, we now have a good understanding of the phenomenology, mechanisms, and modeling of slow granular flows when they are in a steady state; but what about the jamming and yielding of these? As we argued in the introduction, the crucial issue here to consider is the (non)monotonicity of the effective rheology of granular media.

Nonmonotonicity of flow curves has a long history in rheology, in particular in connection with the formation of shear bands in spatially uniform stress fields [12]. In many cases, there is a mutual coupling between the organization and local flow rate which can cause instabilities and shear banding, as observed in polymers, worm-like micelles, or anisotropic colloidal particles, where shear induces alignment and orientational order. For spherical particles, migration and dilatancy may provide the ingredients for such a feedback. As a result, the flow curve that relates strain rate and stress may become nonmonotonic. The negatively sloped part of the flow curve is then associated with an instability: for an applied *global* shear rate in the unstable range, the system separates into bands of high and low shear rates that coexist at a common stress [49]. On the other hand, under a fixed shear stress, the flow rate becomes hysteretic and thus depends on history—moreover, the range of flow rates associated with the negatively sloped flow curve are inaccessible in constant-stress experiments.

It has been noted for a long time that the yielding and jamming of granular media has many similarities with this scenario [50]. Only recently this nonmonotonic behavior has been observed experimentally, both in split-bottom and in Couette flows, although we note here that the magnitude of the stress drop is so small that it may have been easily overlooked in earlier rheological experiments.

The nonmonotonic flow curves observed in granular media can best be expressed in terms of the effective friction coefficient as a function of flow rate. This coefficient was observed to start out at a finite value in the limit of vanishing flow rates, and then to gently decrease by perhaps 10% for increasing flow rates, before sharply increasing when kinetic effects become appreciable (here the inertial number becomes finite [4,8]). Similar nonmonotonicity was recently observed by Kuwano et al. [51]—at this moment of time, there is a need for more work, as nonmonotonicity is not widely accepted yet, and the number of experiments showing this is limited.

Let us now suppose that nonmonotonicity is real. The first consequence is that the plateau value, μ_0, and the minimum value of μ_m play a crucial role in the jamming and yielding of granular media. Basically, there are three regimes, depending on the value of the driving stress, which can be rewritten as a friction coefficient μ_d also. For $\mu_\mathrm{d} < \mu_\mathrm{m}$, there are microslip events inside the granular medium, but the system never yields completely. In the intermediate range where $\mu_\mathrm{m} < \mu_\mathrm{d} < \mu_0$, global and microslip yielding coexist, where as for $\mu_\mathrm{d} > \mu_0$, there are no microslips anymore—any microscopic slip event grows out to a full-blown yielding event and a steady flow [8].

The second consequence is seen by sweeping the stress up and down, where one observes a concomitant hysteretic transition between flow and no flow. By using weak vibrations, the rheological curves can be tuned, and it could be established that hysteresis and nonmonotonicity of the flow curves are directly linked, also for finite vibration strength [8].

6. Outlook

We have outlined main developments in our understanding and modeling of slow granular flows. Despite all progress made, several open issues remain. First and foremost, it is an open question if the weak negative slopes in the rheological curves are due to a similar negative slope in the local friction coefficient, or in fact are due to "self-agitation" of the medium—in the latter case, the fluidization and rheological instability would be naturally related. Second, the nature of the secondary rheology is not well understood, in particular as most data seem to suggest an exponential relation, but some do not. Third, evidence is emerging that anisotropy plays a crucial role for slow granular flows—more experiments are needed, and no clear theoretical picture has emerged. Finally, other local characteristics of the medium could also play a role, in particular local packing density.

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