C. R. Physique 16 (2015) 257–266

Contents lists available at ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

Coarsening dynamics / Dynamique de *coarsening*

Coarsening phenomena

Phénomènes de coarsening

Leticia F. Cugliandolo

Sorbonne Universités, Université Pierre-et-Marie-Curie (Paris-6), Laboratoire de physique théorique et hautes énergies, 4, place Jussieu, 75252 Paris cedex 05, France

ARTICLE INFO

Article history:
Available online 28 April 2015

Keywords:
Coarsening
Phase ordering kinetics
Domain growth

Mots-clés :
Coarsening
Cinétique de croissance d'ordre
Croissance de domaines

ABSTRACT

This article gives a short description of pattern formation and coarsening phenomena and focuses on recent experimental and theoretical advances in these fields. It serves as an introduction to phase ordering kinetics as part of the dossier 'Coarsening dynamics' in *Comptes rendus Physique*, edited by Federico Corberi and Paolo Politi.

RÉSUMÉ

Le présent article donne une brève description du processus de formation de motifs et du phénomène de *coarsening* ; il met l'accent sur les développements expérimentaux et théoriques récents dans ces domaines. Il tient lieu d'introduction à la cinétique de croissance d'ordre dans le dossier « Dynamique de *coarsening* » des *Comptes rendus Physique*, coordonné par Federico Corberi et Paolo Politi.

1. Introduction

When a parameter in a macroscopic homogeneous system is changed, the homogeneous state often becomes unstable and the system evolves into a time-dependent inhomogeneous configuration with growth of order. There are two major alternatives. One is the asymptotic selection of a persistent length associated with a given *pattern*. The other one is *coarsening*, with an increase in the length scale with time only arrested by the full homogenisation of the sample.

Problems of this kind are found in, essentially, all branches of science. Phase ordering kinetics occur at very different scales. Clustering dynamics occurred in the early universe. *Coarsening* and *pattern formation* are observed in solids as well as in fluids. In social sciences, models for, *e.g.*, opinion dynamics involve coarsening phenomena. In developmental biology, pattern formation describes the mechanism whereby initially equivalent cells in a developing tissue in the embryo assume complex forms and functions. Bacterial colonies grow into fancy spatial patterns depending on the surrounding conditions. Many other examples could be used to illustrate these phenomena.

Our understanding of coarsening and pattern formation has progressed following two main routes. One is theoretical and the main actors have been mathematicians, mostly specialised in probability methods, theoretical physicists and theoretical

E-mail address: leticia@lpthe.jussieu.fr.

http://dx.doi.org/10.1016/j.crhy.2015.02.005
1631-0705/

chemists. The other one is experimental, with important contributions made by the applied physics community. Indeed, in the latter field of research, the interest in phase-ordering kinetics is due to the fact that material properties usually depend upon the morphology of the phase separating sample. In metallurgy, a rather old domain of material science, the motivation for blending immiscible components is to create materials with interpenetrating domains that can have better properties than their components. In more recent material engineering, creating micro-patterns and nano-patterns of polymeric materials is of great interest for microdevice fabrication.

Phase-ordering kinetics occur in systems relaxing to equilibrium, but it also exists under external drive. The dynamics could follow microscopic physical rules and respect a detailed balance allowing the system to reach thermal equilibrium asymptotically, or they could go beyond this limitation, in other physical situations or in models for social sciences, economy or other non-physical branches of science.

Coarsening often occurs after crossing a phase transition. A well-known case is the temperature quench through a critical point that separates the disordered high temperature phase from an ordered low-temperature phase. The dynamics across such (standard) phase transitions are rather well understood qualitatively, although a full quantitative description of specific cases is hard to develop. Different mechanisms for pattern formation and coarsening exist and have been studied in great detail; let us mention a few of them here.

We focus first on the freely relaxing case with dynamics allowing for equilibration. In the late stages of coarsening, the configurations within domains of, say, two equilibrium phases have already reached equilibrium and the excess free energy of the system, which is now localised at the interface between the two phases, decreases with time.

In systems with *non-conserved scalar order parameter* (see the main text for the precise definition of these and other technical terms employed in the introduction) as realised, for instance, in magnetic systems, our analytical understanding of phase-ordering kinetics is based on the dynamics of these interfaces, and the driving force behind coarsening is just surface tension. This kind of dynamics is often referred to as *curvature driven*.

The simplest example of dynamics with *conserved order parameter* is Ostwald ripening, the phenomena whereby smaller particles in solution dissolve and deposit on larger particles in order to reach a more thermodynamically stable state in which the surface to volume ratio is minimised. As the larger particles grow, the area around them is depleted of smaller particles. This phenomenon was originally observed in solid solutions, but is also common in emulsions, such as the oil-in-water one, and many others.

In phase separating mixtures, another realisation of conserved order parameter dynamics, two possible mechanisms for the initial domain formation exist:

- if the quench ends at a point outside the spinodal curve, the system is stable against small fluctuations. Rare large fluctuations of thermally activated origin create “critical droplets” (nucleation) that subsequently coarsen (growth);
- if the quench ends at a point inside the spinodal curve, the system is unstable against small fluctuations, leading to phase separation by unstable growth.

In the late stages of growth, both mechanisms give rise to qualitatively similar coarsening motifs. Depending upon the relative density of the two components, one can find isolated droplets of the minority phase immersed in the majority one, with their further evolution following Ostwald ripening, or a bi-continuous structure with domains of the two kinds percolating across the sample. In these systems, the curvature sets up a gradient in the chemical potential that causes molecules to diffuse from regions of high positive curvature to regions of low or negative curvature. This is an evaporation–condensation mechanism. This mechanism operates in binary alloys, while segregation in binary fluids is drastically modified by flow fields.

The excess free energy in a quenched system with scalar order parameter is stored in sharp domain walls. In cases in which the order parameter is a vector, other kinds of topological defects exist. During the ordering process, the system has to eliminate some of them to reach configurations typical of equilibrium over larger and larger scales. This occurs, for instance, in the 2d *xy* model quenched below its Kosterlitz–Thouless temperature, with the progressive annihilation of vortex–antivortex pairs in the course of time.

The mechanisms mentioned above are not the only ones leading to coarsening. In problems in which microscopic dynamics do not necessarily respect a detailed balance, the evolution may lead to absorbing states *via* a coarsening process in which the interfaces are of totally different natures and do not have surface tension. This is the case of, *e.g.*, the voter model.

I do not intend to write here a complete overview of the theoretical description of pattern formation and coarsening phenomena, as several excellent review articles, *e.g.*, [1–6], and textbooks [7–10] treating these topics already exist, and many technical aspects will be covered in the following articles. I will, instead, start by giving a very short survey of the main features of phase-ordering kinetics referring the reader to already published references and the relevant chapters in this volume for further details. I will then describe some open questions and recent methods that are currently being explored and that I personally find interesting and promising.

2. Experiments

Traditionally, coarsening systems are studied in the lab with scattering methods. These give access to the time-dependent structure factor, $S(k,t)$, and the dynamic scaling assumption was formulated to describe the time and wave-vector dependence of this observable. More subtle features, such as the Porod tails characterising the long wave-vector dependence of S, were also uncovered from this kind of experimental measurement [3,5]. In spite of the success of Fourier-space methods, real-space techniques are better suited to give direct access to the microscopic mechanisms for coarsening and, therefore, a more detailed understanding of these phenomena. Indeed, a variety of visualisation methods that are currently being developed should allow one to observe different aspects of coarsening systems in much more detail than what has been done so far.

Direct imaging of the domain structure is easier to achieve in two-dimensional systems. For instance, temperature-controlled polarising microscopy was used in [11,12] to study the chiral domain structure in electric-field-driven deracemization of an achiral liquid crystal. Optical microscopy is commonly used to study domain growth in polymer thin films [13]. Magnetic force microscopy, photoemission electron microscopy and Lorentz transmission electron microscopy are used to visualise the magnetic moment configurations in artificial magnetic materials, such as the 2d spin-ice samples [14,15] that undergo two kinds of order–disorder phase transitions across which stripe magnetic ordering develops [16–19].

New experimental techniques make now possible the direct visualisation of the domain structure of *three-dimensional* coarsening systems as well. In earlier studies, the domain structure was usually observed *post mortem* at a given age, and only on two-dimensional slices of the samples with optic or electronic microscopy. Nowadays, it is possible to observe the full 3d micro structure *in situ*, in the course of evolution.

Three-dimensional images give, in principle, access to a complete topological characterisation of the interfaces via the calculation of quantities such as the Euler characteristics and the local mean and Gaussian curvatures. On top of these very detailed analyses, one can also extract the evolution of the morphological domain structure on different planes across the samples and investigate up to which extent the third dimension has an effect on what occurs in strictly two dimensions. These methods, and the study of the structures in real space, are becoming more and more popular. We here mention just a few applications to different kinds of domain growth systems.

In the context of soft-matter systems, laser scanning confocal microscopy was applied to phase separating binary liquids [20] and polymer blends [21–23]. For example, from images at a very late stage of phase separating bicontinuous polymer blend made of 50% polybutadiene and 50% polyisoprene, Jinnai et al. [24] observed saddle-shaped surfaces with the statistical averages, $K_{\rm M} \simeq 0$ for the mean curvature, and $K_{\rm G} \simeq -6.2 \times 10^{-2}$ μm^2 for the Gaussian curvature. Therefore, the interfaces resemble, on average, minimal surfaces (where $K_{\rm M} = 0$ at each surface point), although considerable deviations were also reported.

X-ray tomography, a way to observe slices of the sample in a progressive and non-destructive manner, was used to quantify phase separating glass-forming liquid binary mixtures [25]. Among many other interesting features, this study showed that, unexpectedly, this system evolves in a diffusive hydrodynamic regime.

In the realm of magnetic systems, the Talbot–Lau neutron tomography non-destructive method presented in [26] looks very promising.

3. Models

Phase ordering kinetics are modelled with kinetic Ising or Potts models [9]; they are studied numerically with lattice Boltzmann [6,27] and Monte Carlo simulations, and, mostly analytically, with stochastic equations on coarse-grained fields [3,5]. These approaches give access to different aspects of the processes, and which one is easier to deal with analytically depends upon the issue to explore.

Interacting two-state systems were originally introduced to model magnetic phase transitions in easy axis magnets. The two-state variables, Ising spins, can be suitably transformed into occupation numbers and one can model with them, *e.g.*, binary mixtures; they can be generalised to take many values and one can describe with them, *e.g.*, soap froths; still, the updates between different states can be chosen in very different ways to capture various kinds of microscopic dynamics.

The conservation laws, if any, should be respected by the microscopic updates. The two main universality classes one encounters are:

- non-conserved order parameter dynamics, with its most prominent example being the ordering dynamics of a ferromagnet with scalar of vectorial order parameter;
- conserved order parameter dynamics, its typical example being the kinetics of phase separation of a binary mixture.

The master equation is the basic analytic tool to deal with the stochastic dynamics of discrete variables. This route was opened by R. Glauber in his analysis of the stochastic dynamics of the Ising chain and it has been followed by numerous authors in varied contexts [9,28–30]. However, in general, the master equations lead to a hierarchy of coupled equations for the correlation functions that can neither be disentangled nor solved analytically. Numerical simulations using Monte Carlo methods are, instead, easy to implement and provide us with valuable information.

The other popular way to describe coarsening phenomena is phenomenological and it has proven to be very successful and quite complete. It is based on the identification of a coarse-grained order parameter, the definition of a deterministic force (that may or may not derive from a thermodynamic Landau free-energy density functional), and the proposal that the order parameter evolution is governed by a Langevin equation that respects the conservation laws. This leads to the time-dependent Ginzburg–Landau equation for non-conserved order parameter dynamics, the Cahn–Hilliard equation for the conserved order parameter case, and other stochastic differential equations.

This approach allows one to reach many interesting conclusions. For instance, with the field theory one proves that there is no pinching-off of protrusions in the 2d continuous-curvature-driven dynamics at zero temperature. Instead, an interface in 3d that has both convex and concave parts can undergo fission if the concave portion is thin enough [9]. However, a full solution to these problems is hard to develop as they are set into the form of a non-linear functional Langevin equation with no small parameter.

The microscopic discrete models as well as the field theoretical ones can be extended to include energy injection via non-conservative forces, or special microscopic dynamic rules.

A few completely or partially solvable models for coarsening exist, but they are confined to one dimension, as the Glauber Ising chain or the one-dimensional Ginzburg–Landau equation, or are mean-field approximations that can be very simple or quite refined as the large N limit of a vector field stochastic dynamics, see, *e.g.*, [9,10]. The analysis of the effect of quenched disorder in the form of random fields can be done in 1d thanks to the strong disorder renormalisation group method [31]. In the more realistic finite space and internal dimension problem, one cannot find analytical solutions to the discrete or continuous variable models.

Several approximation schemes, including self-consistent approximations to the perturbation series and auxiliary field methods, have been developed over the years to characterise the correlation functions. These methods, notably the ones due to K. Kawasaki and collaborators and G. Mazenko and collaborators, are well explained in the literature, see, *e.g.*, [3,5,7]. They are quite successful for non-conserved order parameter dynamics as they predict scaling functions for the space–time correlations that are very close to the ones obtained with numerical or experimental methods. They are, though, less precise for scalar conserved order parameters [7]. The problem of finding a good analytic approach to this phenomenon remains, thus, open.

4. Dynamic scaling

Domain growth [3,5] in systems with scalar order parameter is characterised by a patchwork of large domains, the interior of which is basically thermalised in one of the equilibrium phases while their boundaries slowly move and tend to become smoother due to their elastic energy. The patterned structure is not quiescent, ordered regions grow on average with a linear length $R(t)$, but the time needed to fully order the sample diverges with the system size.

The *dynamic scaling hypothesis* states that at late times and in the scaling limit $r \gg \xi$ (r is a distance and ξ the equilibrium correlation length), the system is characterised by a single length-scale $R(t)$ such that the domain structure looks similar at different times if one rescales lengths by $R(t)$. In practice, this means that all time and space dependencies in correlation functions appear as ratios between distances and $R(t)$. The time-dependence of the length scale is not sensitive to microscopic details, but it is to the dimension of the order parameter, the conservation laws and the presence of quenched randomness. In clean systems the characteristic length grows algebraically in time, $R(t) \simeq t^{1/z}$, with z a dynamic exponent that defines the dynamic universality class [2,3]. In scalar systems with non-conserved order parameter dynamics, $z = 2$. In scalar systems with conserved order parameter dynamics, bulk diffusion dominates on long length scales and $R(t) \simeq t^{1/3}$ while surface diffusion can be important when the scales are small and $R(t) \simeq t^{1/4}$. Surface diffusion, negligible as $t \to \infty$, often dominates in the experimental range of interest.

Hydrodynamic flows make the dynamics of liquids more complicated [7,27]. Indeed, in a phase-separating system, the mean local curvature of the interfaces induces a pressure difference that produces flow. Therefore, in a binary mixture, fluid flow also contributes to the transport of the order parameter. When diffusion dominates domain growth, the growth law reduces to the one of a binary alloy, $R(t) \simeq t^{1/3}$. In the low Reynolds number limit, a simple argument whereby the friction and Laplace pressure terms are asked to be of the same order yields $R(t) \simeq t$ (viscous hydrodynamic regime). For later times and/or large Reynolds numbers, inertial effects on the flow velocity can no longer be neglected and a different order of magnitude argument leads to $R(t) \simeq t^{2/3}$ (inertial hydrodynamic regime). The cross-over between these growth laws was checked with lattice Boltzmann simulations [27]. A discussion of the subtle cross-over between these regimes can be found in [4,6]. Coarsening in fluids will be further discussed in the chapter by S. Das, S. Roy, and J. Midya.

Hydrodynamic effects of this kind are especially important in polymer systems. In symmetric polymer blends, where constituent polymers have nearly identical molecular weights and viscoelastic properties, the hydrodynamic interaction eventually governs the late-stage phase separation in the same manner as in usual binary fluid mixtures. In asymmetric polymer blends, however, viscoelastic effects unique to polymers can drastically influence phase separation, see [7,32] for more details.

In systems with *competing interactions*, for instance, in problems in which a spin has ferromagnetic and antiferromagnetic exchanges with different spins in the same sample, the growth of $R(t)$ can be much slower and logarithmic growth has been obtained in some of these cases. These models have been used to mimic the slow down in glassy systems [33,34].

Weak quenched disorder slows down the dynamics of macroscopic systems with respect to their clean limit. Take, for instance, the random-field Ising model. For probabilistic reasons, the fields can be very strong and positive in some region of the sample and favour positive magnetisation, and very strong and negative in a neighbouring region and favour negative magnetisation. It will then be very hard to displace the phase boundary and let one of the two states conquer the full spatial region originally covered by the two phases. Indeed, in 2d the random fields destroy the finite T transition, while in 3d they just deplete the ordered phase. The probability of finding such rare regions can be quantified. The relaxation of a small perturbation that takes the system slightly away from equilibrium becomes slower than exponential, the dynamic counterpart of the Griffiths essential singularities of free energy. The assumption of a power-law dependence of free-energy barriers with size combined with an Arrhenius argument suggests $R(t) \simeq \ln^{1/\psi} t$ for the growing length.

Strong quenched disorder renders the dynamics still more complex. Spin-glasses are the archetype of such systems. After a quench from high to low temperatures, they not only show very slow out of equilibrium relaxation, never reaching thermal equilibrium, but they also display very intriguing memory effects under complicated paths in parameter space [35]. Although it is not clear whether the dynamics occur *via* the growth of domains, scaling of dynamic correlation functions describe numerical data quite precisely and, somehow surprisingly, with a power law $R(t) \simeq t^{1/z}$. This fact is not compatible with mean-field predictions of a much complex time-dependence that could perhaps only establish at much longer time scales. The power-law growth of $R(t)$ is not compatible either with the droplet picture predictions. In this model, static order is assumed to grow as in standard coarsening systems with two equilibrium states related by symmetry. The evolution at low-enough temperatures should then be dominated by thermal activation with the typical free-energy barrier to nucleate a droplet assumed to scale as L^{ψ} with ψ a non-trivial exponent. Dynamical observables should then follow scaling laws in terms of a growing length $R(t) \simeq \ln^{1/\psi} t$. Before drawing conclusions one must keep in mind that the analysis of experimental and numerical data is difficult given the limited range of time scales available in both cases. In [36], an efficient strategy for data analysis with the goal of finding the best $R(t)$ for dynamic scaling is discussed and might be of help in the future analysis of this question.

The chapter by Federico Corberi presents a discussion of quenched randomness effects on coarsening systems.

We have already stressed that the dynamics of coarsening systems with scalar order parameter is very much determined by the dynamics of the interfaces between domains. Coarsening is therefore closely related to the dynamics of elastic manifolds (with or without quenched disorder) [37–41]. The dynamics of *directed manifolds* is relevant, for instance, to the understanding of crystal or other surface growth and is also a problem of great interest. It will be discussed in the chapter by Chaouqi Misbah and Paolo Politi.

In systems with vector order parameter, spin waves are typically accompanied by topological defects and these may have an influence on the growing length [3,5]. The latter is identified from the scaling of the correlation functions. In the 2d xy model with non-conserved order parameter, $R(t) \simeq (t/\ln t)^{1/2}$, and the logarithmic correction is due to the vortices, while in 3d the simple $R(t) \simeq t^{1/2}$ is recovered.

The time-dependent scaling properties of linear response functions at criticality [42] and in the ordering phase [43] have been investigated by many authors. The motivation for these studies was to compare their scaling behaviour to the one of the associated correlation functions and to analyse how the equilibrium fluctuation–dissipation relation is modified out of equilibrium [44]. The ultimate purpose being the comparison to the fluctuation–dissipation relations in glassy systems with no obvious coarsening process. I will not pursue the description of the scaling forms found as these have been reported in detail in the three review articles mentioned above.

5. Morphologies

The characterisation of coarsening in samples and models has been done, mostly, in terms of correlation functions. These are easy to access experimentally with different kinds of scattering methods. The more recent experimental techniques briefly listed in Section 2 give us access to the real-space structure, even in three dimensions. In parallel, an ensemble of numerical and analytic works focused recently on the geometry of the time-dependent mosaic domain structure in Ising models and the eventual metastable states reached at zero temperature. We here summarise the main questions posed and the results obtained. I give references to the original articles as these results have neither been collected nor explained in reviews yet.

Influence of the initial state Two kinds of initial conditions should be distinguished and have a different subsequent low-temperature dynamics. Initial conditions drawn from equilibrium at super-critical temperature ($T_0 > T_c$) have finite correlation length. This correlation length diverges in the initial configurations that are typical of the critical point ($T_0 = T_c$). The evolution of these states is rather different from the one of completely disordered high temperature initial configurations. This difference can be quantified with the study of correlation functions [45], the time-dependent pattern of domains [46–48] and the probability of getting stuck in metastable states at zero temperature [49–53].

Percolation in the 2d kinetic Ising model It was shown in [46,47] that, after a very short time span (a few Monte Carlo steps for the simulated cell), the morphological and statistical properties of the large structures in a 2d Ising model quenched from high to low temperature (areas of domains, lengths of interfaces, etc.) look like the ones of site percolation at its threshold. As the occupation probability for up and down spins in a high-T equilibrium configuration is smaller than the

one at critical percolation, this fact suggests that the system must have reached critical percolation at some point. A careful study of this approach demonstrated that the time needed to reach critical percolation scales as $t_{\rm p}(L) \simeq L^{\alpha_{\rm p}}$ with $\alpha_{\rm p}$ an exponent that is smaller than one, depends upon the lattice structure and, presumably, the microscopic dynamics [54]. This time-scale can be transformed into a length scale $L_{\rm p} \simeq t_{\rm p}^{1/z} \simeq L^{\alpha_{\rm p}/z}$, and with it a new scaling variable can be added to correct the scaling of correlation functions and other observables. This issue has been discussed in this work as well and should be important for the description of experimental data.

Zero-temperature metastable states In $d = 1$, both the kinetic Ising chain with Glauber dynamics and the time-dependent Ginzburg–Landau equation with non-conserved order parameter ultimately reach complete alignment in a time scale that grows algebraically with the system size in the former case and exponentially in the system size in the latter. However, in the absence of thermal fluctuations, the $d \geq 2$ ferromagnetic coarsening may freeze into metastable states.

In $d = 2$ the discrete model reaches the ground state or a stripe state with probabilities that depend upon the initial condition ($T_0 > T_{\rm c}$ or $T_0 = T_{\rm c}$) and are equal, with high numerical precision, to the ones of having a spanning cluster at critical percolation in the former case [49,52] and different kinds of critical spanning clusters in the latter case [53].

In a time scale of the order of L^2, the continuous model may also reach a stripe state. However, this one will eventually disappear due to the domain–wall interaction, and the system will condense into the ground state. This second process takes place on a much longer time scale, which diverges exponentially with the system size and lies well beyond experimentally relevant time scales for typical sample sizes.

For $d = 3$, the discrete model never reaches the ground state and several aspects of the metastable states have been discussed in [50,51]. In the continuous case, the Allen–Cahn equation implies that the long-lived metastable states are minimal surfaces (with vanishing local mean curvature) compatible with the boundary conditions.

For the Kawasaki spin-exchange dynamics, because of the conservation of the magnetisation, a system quenched to $T = 0$ always gets stuck.

Morphology of the time-dependent mosaic structure The distributions of domain lengths in Glauber Ising and Potts chains have been derived analytically [55–57,31]. The methods used in these papers are not adapted to be applied in higher dimensions. Still, many statistical and geometric properties of the hull-enclosed areas and domains in two-dimensional coarsening of Ising and Potts models with non-conserved and conserved-order-parameter dynamics have been elucidated [46–48,58,59]. The results for hull-enclosed areas in the non-conserved order scalar parameter universality class are, quite surprisingly, exact [46,47]. Similar studies of the ferromagnetic Ising universality class on two-dimensional slices of the 3d lattice are on their way [60]. Analysis of the full 3d structure obtained numerically appeared in [61,62] for non-conserved-order-parameter dynamics and in [63] for viscoelastic phase separation.

6. Cooling rate dependencies

In condensed-matter systems, most theoretical studies of phase ordering kinetics focus on the dynamics after infinitely rapid quenches, as finite quench timescales are expected to alter the dynamics only at short times and eventually become irrelevant. Cooling-rate dependencies in defect dynamics are, instead, central in cosmological theories as their time-scales are very different from the ones of the laboratory. Although the scenario whereby topological defects would act as seeds for matter clustering seems to be excluded by observation, the interest in predicting their density remains due to other possible effects of these objects. In the late 1980s, Zurek assumed that defect dynamics, once the systems has gone through the phase transition, are negligible and used critical scaling to derive a quantitative prediction for the density of defects; he also proposed to check these ideas in condensed-matter systems [64,65]. A large experimental and numerical activity followed with variable results, summarised in [66]. In two recent papers, we revisited the Kibble–Zurek mechanism in infinitely slow annealing across classical thermal second-order [67] and Kosterlitz–Thouless [68] phase transitions, and we elucidated the time and cooling-rate dependencies of the density of topological defects (domain walls or vortices) left over in the low-T phases. Our results proved that the dynamics in the low-T phase contribute significantly to the reduction of the density of topological defects in quenches with dissipative dynamics. Some other studies along these lines appeared recently in, *e.g.*, [69,70].

7. External drive

The influence of an external drive such as an imposed shear flow, turbulence, gravity or temperature gradients have been the object of intensive study due to their relevance to applications. In spite of these efforts, basic questions about their effect on coarsening remain still open.

In *binary fluids under an external shear flow*, two interesting open questions are: which is the late-time behaviour and whether coarsening continues indefinitely or is ultimately arrested. This problem has been treated in some simplified cases. For example, it was shown that flow imposed by weak shear on two opposite surfaces of a two-dimensional binary fluid makes coarsening in the direction of flow more rapid. Instead, the behaviour of a binary fluid under strong shear is still an open question, and whether growth saturates in the transverse direction or not is still a matter of debate [4,7].

Another externally driven system where coarsening effects appear are *vibrated granular matter*. Granular matter is a generic term that encompasses athermal systems of remarkable importance in industry and engineering. Grains can be actual sand grains but also pills, seeds or many other. In these athermal systems, a variation of the input energy as realised, for instance, by an external shake or tap, can change the internal ordering of the material. A common phenomenon studied in these systems is compaction, or the increase of the bulk density of, say, loosely packed sand under the effect of gravity when it is shaken at low amplitude. But compaction can be heterogeneous, with the formation and growth of grain clusters. Coarsening in granular matter will be discussed in the chapter by A. Baldassarri, A. Puglisi, and A. Sarracino.

The constituents of *active matter* [71] absorb energy from their environment or internal fuel tanks and use it to carry out motion. Energy is partially transformed into mechanical work and partially dissipated to the environment in the form of heat. The effect of the motors can be dictated by the state of the particle and/or its immediate neighbourhood and it is not necessarily fixed by an external field. The units interact directly or through disturbances propagated in the medium. Active matter exists at very different scales including bacterial suspensions and swarms of different animals.

Active matter displays out of equilibrium phase transitions that may be absent in their passive counterparts. Much theoretical effort has been recently devoted to the description of different aspects, such as the self-organisation of living microorganisms, the identification and analysis of states with spatio-temporal structure, and the study of the rheological properties of active particle suspensions.

A particularly interesting feature of active matter, in the context of this dossier, is the spatial phase separation into an aggregated phase and gas-like regions for sufficiently large packing fractions in the complete absence of attractive interactions. The current research in phase separation in active fluids will be developed in the chapter by A. Tiribocchi, G. Gonnella, and D. Marenduzzo.

Stochastic Markov processes in which the noise acts multiplicatively on a function of the variable of interest are manifold. Examples of these processes appear in physics, chemistry and even economy: the diffusion of a colloidal particle close to a wall, the autocatalytic chemical reactions in which the production of a molecule is enhanced by the presence of the same molecules already produced, and the Black and Scholes model for option pricing. *White multiplicative noise* can induce highly nontrivial effects in one variable [72] as well as in extended systems. On the one hand, convective patterns are predicted by the Swift–Hohenberg equation near deterministic points where no pattern would exist without the external multiplicative white noise [73]. On the other hand, phase transitions in problems in which the deterministic part does not exhibit any symmetry breaking have also been exhibited [74,75]. Indeed, a short time instability is generated owing to the noise, and the generated nontrivial state is afterwards rendered stable by the spatial coupling. The transition shows a divergence of the correlation length, critical slowing down, and scaling properties, similarly to what occurs at a conventional second-order equilibrium phase transition. The dynamics display coarsening effects [75].

Godrèche and collaborators (see, *e.g.*, [76] and refs. therein) have recently focused on the role played by *spatial asymmetry* in the dynamics of phase ordering systems. In the context of Glauber dynamics, asymmetry means that the flipping spin is more influenced by some of its neighbours than by other ones. Such an asymmetric dynamics is therefore irreversible, because the principle of action and reaction is violated and detailed balance no longer holds. However, for some choices of the updating rules, the dynamics still take the system to the Gibbs–Boltzmann distribution function asymptotically and a coarsening process can establish below some critical value of the control parameter.

Coarsening arises also in the approach to condensation in driven diffusive systems. This phenomenon occurs in, *e.g.*, zero-range processes, models in which equivalent particles hop from sites to sites on a lattice, with prescribed rates that only depend on the occupation of the departure site. The asymmetric Ising model, Urn models, models of mass transport, and other driven dissipative models also show this phenomenology [77]. The condensate manifests itself by the macroscopic occupation of a single site of a thermodynamically large system by a finite fraction of the whole mass in the first models mentioned, while for driven diffusive systems, the condensate manifests itself as a domain of macroscopic size.

8. Pattern formation

Pattern formation is the spontaneous formation of macroscopic spatial structures in open systems constantly driven far from equilibrium. Patterns can be stationary in time and periodic in space, periodic in time and homogeneous in space or periodic in both space and time. The keywords ‘dissipative structures’ or ‘self-organisation’ are also attached to this phenomenon that was initially studied in fluid dynamics and chemical reactions but also appears in solid-state physics, soft condensed matter and nonlinear optics, and is now central to biology with, for instance, morphogenesis and the dynamics of active matter playing a predominant role. The traditional example is Rayleigh–Bénard convection: a fluid placed between two infinite horizontal plates that are perfect heat conductors at different temperatures. At a threshold value of the temperature difference, the uniform state with a linear temperature profile becomes unstable towards a state with convective flow. In the context of chemistry the paradigm are systems with competition between temporal growth rates and diffusivity of the different species.

Systems that form patterns are usually described by over-damped dissipative deterministic partial differential equations. The exact form of the equations depends on the problem at hand, ranging from Navier–Stokes equations for fluid dynamics to reaction–diffusion equations for chemical systems. A well-studied case is the non-linear Schrödinger equation.

The spatio-temporal structures are found from the growth and saturation of modes that are unstable when a control parameter is increased beyond threshold. A parallel between the kind of bifurcation of fixed-point solutions and the order

of a phase transition can be established. Concretely, a linear stability analysis of the uniform state reveals the mechanism leading to the pattern formation. The analysis of non-linear effects is often realised numerically.

Noisy fluctuations can interact with a system's nonlinearities or effectively produce non-linearities to enhance regular behaviour. For simple, typically one-variable systems, this counterintuitive noise effect is well known. On the one hand, noise can induce a potential barrier crossing in a multi-stable system. If, for an optimal noise level, the stochastic crossing times statistically match a deterministic time scale either internal or external of the system, a more regular behaviour may arise, in the form of a higher periodicity, for instance. Realisations of this mechanism are stochastic resonance, coherence resonance and noise-induced transport. On the other hand, noise can destabilise existing steady states and induce new ones with, possibly, higher regularity, via noise-induced transitions. In extended systems, noise can induce phase transitions between disordered and ordered phases. Noise driven spatio-temporal order has been observed experimentally, mainly in chemical and electronic systems, but more recently also in the biophysical context, and numerically in many models [78].

The theory of pattern formation was reviewed in [1] and the noise effects in [78]. More recent developments in this field will be covered in the chapter by A. Nepomnyashchy.

9. Quantum fluctuations

In the last decade, quantum non-equilibrium phenomena have grown in importance. In condensed matter systems, transport through ever-smaller nanostructures, even single molecules, has been realised. On a different front, significant advances in the field of ultra cold atoms have allowed one to engineer quantum many-body systems in almost perfect isolation from the environment. Thanks to the ability to rapidly tune different parameters, e.g. the interaction strength between the atoms or the creation of controlled excitations, the non-equilibrium many-body physics of isolated quantum systems has thus been accessed. Besides their current experimental relevance and their possible technological applications, the physical behaviour of dissipative and isolated quantum systems out of equilibrium is clearly a topic of fundamental interest, and its understanding is still at an early stage.

A quantum system can undergo a quantum phase transition as a function of a parameter (say, external field or pressure) at strictly zero temperature. This critical point can continue into a critical line in the temperature – parameter-of-choice phase diagram.

The relaxation dynamics of systems quenched to a quantum critical point were analysed in [79,80], where various spin chains were considered, and in [81] where a vector field theory was studied with renormalisation and large N methods.

An abrupt quench from the disordered into the ordered phase following some path in the, say, two-dimensional parameter space will set the system into a non-equilibrium initial condition and the equilibrium process can then occur via quantum coarsening. There is no reason to believe that the dynamic scaling hypothesis will not hold under quantum fluctuations. The reasonable assumption whereby fluctuations within domains should be determined by quantum and thermal fluctuations while the dynamics of interfaces should be close to the ones of their classical counterparts has been demonstrated in some solvable mean-field models [82,83]. A scenario for the time-dependence of a growing length with several crossovers, in the coarsening dynamics of 3d itinerant ferromagnets was put forward in [84].

Spin textures, ferromagnetic domains and vortices have been observed experimentally *in situ* in quenched Bose–Einstein condensates of atoms with non-zero internal angular momentum [85]. The temporal and spatial evolution of these structures was also reported in this reference. The dynamics of this system is usually described with a Gross–Pitaevskii equation, and numerical simulations confirm the domain growth observed experimentally.

10. Beyond physics

When modelling dynamical systems in fields of science other than physics, there is, basically, no constraint on the dynamic rules that can be used.

The voter model is a particular reaction model, used to describe the spatial spreading of opinions [86] or populations [87] in terms of coarsening or segregation. A q-valued opinion variable is initially assigned, with some rule, to each site on a lattice or a graph. The variables have no conviction and the microscopic dynamics is very simple: at each time step, a variable chosen at random adopts the opinion of a randomly-chosen neighbour. This parameter free model approaches one of the q absorbing states with complete consensus in a time that depends on the spatial dimensionality, the value of q and the system's size. The approach occurs via a coarsening process in $d \leq 2$ and because of a large random fluctuation in $d > 2$. The coarsening process in $d = 2$ is very different from the curvature-driven one, demonstrating that symmetry alone does not specify the asymptotic dynamics. A large bubble does not shrink, but slowly disintegrates as its boundary roughens diffusively while the radius of the droplet remains statistically constant. These facts are associated with the absence of surface tension [88]. The evolution of a random initial condition shows the growth of ordered spatial regions leading to growing length $R(t) \simeq t^{1/2}$ and the coarsening process is driven by interfacial noise. In generalised versions of this models, with a temperature, a dynamic phase transition is defined as the critical line between a low-temperature region where clusters (or domains) grow indefinitely, and a high-temperature region where one only observes fluctuations at a finite scale [89].

11. Wrap-up

I have presented a short description of various systems evolving through phase-ordering dynamics. In the rest of this volume, some of these problems will be discussed in more detail.

Acknowledgements

L. F. C. is a member of the Institut universitaire de France.

References

[1] M.C. Cross, P.C. Hohenberg, Rev. Mod. Phys. 65 (1993) 851.
[2] P.C. Hohenberg, B.I. Halperin, Rev. Mod. Phys. 49 (1977) 435.
[3] A.J. Bray, Adv. Phys. 43 (1994) 357.
[4] A.J. Bray, Philos. Trans. R. Soc. Lond. 361 (2003) 781.
[5] S. Puri, Kinetics of phase transitions, in: S. Puri, V. Wadhawan (Eds.), Kinetics of Phase Transitions, Taylor and Francis, 2009.
[6] M.E. Cates, Complex fluids: the physics of emulsions, in: D. Queré, L. Bocquet, T. Witten, L.F. Cugliandolo (Eds.), Soft Interfaces, Les Houches XCVIII, Oxford University Press, 2015.
[7] A. Onuki, Phase Transition Dynamics, Cambridge University Press, 2004.
[8] S. Puri, V. Wadhawan (Eds.), Kinetics of Phase Transitions, Taylor and Francis Group, 2009.
[9] P.L. Krapivsky, S. Redner, E. Ben-Naim, A Kinetic View of Statistical Physics, Cambridge University Press, 2010.
[10] M. Henkel, M. Pleimling, Non-Equilibrium Phase Transitions: Ageing and Dynamical Scaling Far from Equilibrium, Springer-Verlag, 2010.
[11] I. Dierking, J. Phys. Condens. Matter 12 (2000) 8035.
[12] A. Sicilia, et al., Phys. Rev. Lett. 101 (2008) 197801.
[13] C. Haas, J.M. Torkelson, Phys. Rev. E 55 (1997) 3191.
[14] C. Nisoli, R. Moessner, P. Schiffer, Rev. Mod. Phys. 85 (2013) 1473.
[15] L.J. Heyderman, R.L. Stamps, J. Phys. Condens. Matter 25 (2013) 363201.
[16] Z. Budrikis, et al., New J. Phys. 14 (2012) 035014.
[17] D. Levis, L.F. Cugliandolo, Europhys. Lett. 97 (2012) 30002.
[18] G.M. Wysin, W.A. Moura-Melo, L.A.S. Mol, A.R. Pereira, New J. Phys. 15 (2013) 045029.
[19] D. Levis, L.F. Cugliandolo, L. Foini, M. Tarzia, Phys. Rev. Lett. 110 (2013) 207206.
[20] W.R. White, P. Wiltzius, Phys. Rev. Lett. 75 (1995) 3012.
[21] H. Jinnai, Y. Nishikawa, T. Koga, T. Hashimoto, Macromolecules 28 (1995) 4782.
[22] H. Jinnai, T. Koga, Y. Nishikawa, T. Hashimoto, S.T. Hyde, Phys. Rev. Lett. 78 (1997) 2248.
[23] H. Jinnai, Y. Nishikawa, T. Hashimoto, Phys. Rev. E 59 (1999) R2554.
[24] H. Jinnai, Y. Nishikawa, T. Ikehara, T. Nishi, Adv. Polym. Sci. 170 (2004) 115.
[25] D. Bouttes, E. Gouillart, E. Boller, D. Dalmas, D. Vandembroucq, Phys. Rev. Lett. 112 (2014) 245701.
[26] I. Manke, et al., Nat. Commun. 1 (2010) 125.
[27] G. Gonnella, J. Yeomans, Using the lattice Boltzmann algorithm to explore phase ordering in fluids, in: S. Puri, V. Wadhawan (Eds.), Kinetics of Phase Transitions, Taylor and Francis, 2009.
[28] B. Schmittmann, R.K.P. Zia, Statistical mechanics of driven diffusive systems, in: C. Domb, J.L. Lebowitz (Eds.), Phase Transition and Critical Phenomena, vol. 17, Academic Press, London, 1995.
[29] R. Stinchcombe, Adv. Phys. 50 (2001) 431.
[30] G.M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in: C. Domb, J.L. Lebowitz (Eds.), Phase Transition and Critical Phenomena, vol. 19, Academic Press, London, 2001.
[31] P. Le Doussal, Exact results and open questions in first principle functional RG, arXiv:0809.1192.
[32] H. Tanaka, Phase separation in soft matter: concept of dynamic asymmetry, in: D. Queré, L. Bocquet, T. Witten, L.F. Cugliandolo (Eds.), Soft Interfaces, Les Houches XCVIII, Oxford University Press, 2015.
[33] L.F. Cugliandolo, Dynamics of glassy systems, in: J.-L. Barrat, M.V. Feilgel'man, J. Kurchan, J. Dalibard (Eds.), Slow Relaxations and Nonequilibrium Dynamics in Condensed Matter, Les Houches LXXVII, Springer-Verlag, 2003.
[34] G. Tarjus, An overview of the theories of the glass transition, in: L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, W. van Saarloos (Eds.), Dynamical Heterogeneities in Glasses, Colloids, and Granular Media, Oxford University Press, 2011.
[35] E. Vincent, J. Hammann, M. Ocio, J.-P. Bouchaud, L.F. Cugliandolo, Slow dynamics and aging in spin-glasses, in: Complex Behaviour of Glassy Systems, in: Lect. Notes Phys., vol. 492, Springer-Verlag, Berlin, 1997.
[36] F. Corberi, L.F. Cugliandolo, H. Yoshino, Growing length scales in aging systems, in: L. Berthier, J.-P. Bouchaud, G. Biroli, L. Cipelletti, W. van Saarloos (Eds.), Dynamical Heterogeneities in Glasses, Colloids, and Granular Media, Oxford University Press, 2011.
[37] G. Blatter, M.V. Feigelman, V.B. Geshkenbein, A.I. Larkin, V.M. Vinokur, Rev. Mod. Phys. 66 (1994) 1125.
[38] T. Giamarchi, P.L. Doussal, Statics and dynamics of disordered elastic systems, in: A.P. Young (Ed.), Spin Glasses and Random Fields, World Scientific, 1997.
[39] T. Nattermann, S. Scheidl, Adv. Phys. 49 (2000) 607.
[40] A.B. Kolton, A. Rosso, T. Giamarchi, J. Phys. IV Proc. 131 (2005) 189.
[41] J.L. Iguain, S. Bustingorry, A.B. Kolton, L.F. Cugliandolo, Phys. Rev. B 80 (2009) 094201.
[42] P. Calabrese, A. Gambassi, J. Phys. A 38 (2005) R133.
[43] F. Corberi, E. Lippiello, M. Zannetti, J. Stat. Mech. (2007) P07002.
[44] L.F. Cugliandolo, J. Phys. A 44 (2011) 483001.
[45] A.J. Bray, K. Humayun, T.J. Newman, Phys. Rev. B 43 (1991) 3699.
[46] J.J. Arenzon, A.J. Bray, L.F. Cugliandolo, A. Sicilia, Phys. Rev. Lett. 98 (2007) 145701.
[47] A. Sicilia, J.J. Arenzon, A.J. Bray, L.F. Cugliandolo, Phys. Rev. E 76 (2007) 061116.
[48] A. Sicilia, Y. Sarrazin, J.J. Arenzon, A.J. Bray, L.F. Cugliandolo, Phys. Rev. E 80 (2009) 031121.
[49] K. Barros, P.L. Krapivsky, S. Redner, Phys. Rev. E 80 (2009) 040101.
[50] J. Olejarz, P.L. Krapivsky, S. Redner, Phys. Rev. E 83 (2011) 051104.
[51] J. Olejarz, P.L. Krapivsky, S. Redner, Phys. Rev. E 83 (2011) 030104.

[52] J. Olejarz, P.L. Krapivsky, S. Redner, Phys. Rev. Lett. 109 (2012) 195702.
[53] T. Blanchard, M. Picco, Phys. Rev. E 88 (2013) 032131.
[54] T. Blanchard, F. Corberi, L.F. Cugliandolo, M. Picco, Europhys. Lett. 106 (2014) 66001.
[55] B. Derrida, R. Zeitak, Phys. Rev. E 54 (1996) 2513.
[56] P. Krapivsky, E. Ben-Naim, Phys. Rev. E 56 (1997) 3788.
[57] K. Ouchi, N. Tsukamoto, T. Horita, H. Fujisaka, Phys. Rev. E 76 (2007) 041129.
[58] M.P.O. Loureiro, J.J. Arenzon, L.F. Cugliandolo, A. Sicilia, Phys. Rev. E 81 (2010) 021129.
[59] M.P.O. Loureiro, J.J. Arenzon, L.F. Cugliandolo, Phys. Rev. E 85 (2012) 021135.
[60] J.J. Arenzon, L.F. Cugliandolo, M. Picco, Slicing the 3d Ising model: critical equilibrium and coarsening dynamics, arXiv:1412.7456.
[61] M. Fialkowski, A. Aksimentiev, R. Holyst, Phys. Rev. Lett. 86 (2001) 240.
[62] M. Fialkowski, R. Holyst, Phys. Rev. E 66 (2002) 046121.
[63] T. Araki, H. Tanaka, Macromolecules 34 (2001) 1953.
[64] W.H. Zurek, Nature 317 (1985) 505.
[65] W.H. Zurek, Phys. Rep. 276 (1996) 177.
[66] T.W.B. Kibble, Phys. Today 60 (2007) 47.
[67] G. Biroli, L.F. Cugliandolo, A. Sicilia, Phys. Rev. E 81 (2010) 050101.
[68] A. Jelić, L.F. Cugliandolo, J. Stat. Mech. (2011) P02032.
[69] P.L. Krapivsky, J. Stat. Mech. (2010) P02014.
[70] C.W. Liu, A. Polkovnikov, A.W. Sandvik, Phys. Rev. B 89 (2014) 054307.
[71] D.A. Fletcher, P.L. Geissler, Annu. Rev. Phys. Chem. 60 (2009) 469.
[72] W. Horsthemke, R. Lefever, Noise Induced Transitions, Springer-Verlag, Berlin, 1984.
[73] J. García-Ojalvo, A. Hernández-Machado, J.M. Sancho, Phys. Rev. Lett. 71 (1993) 1542.
[74] C.V. den Broeck, J.M.R. Parrondo, R. Toral, Phys. Rev. Lett. 73 (1994) 3395.
[75] C. Van den Broeck, J.M.R. Parrondo, R. Toral, R. Kawai, Phys. Rev. E 55 (1997) 4084.
[76] C. Godrèche, M. Pleimling, Dynamics of the two-dimensional directed Ising model in the paramagnetic phase, arXiv:1401.1988.
[77] C. Godrèche, Lect. Notes Phys. 716 (2007) 261.
[78] F. Sagués, J.M. Sancho, J. García-Ojalvo, Rev. Mod. Phys. 79 (2007) 829.
[79] D. Patane, A. Silva, F. Sols, L. Amico, Phys. Rev. Lett. 102 (2009) 245701.
[80] M. Znidaric, T. Prosen, G. Benenti, G. Casati, D. Rossini, Phys. Rev. E 81 (2010) 051135.
[81] P. Gagel, P.P. Orth, J. Schmalian, Universal post-quench dynamics at a quantum critical point, arXiv:1406.6387.
[82] L.F. Cugliandolo, G. Lozano, Phys. Rev. Lett. 80 (1998) 4979.
[83] C. Aron, G. Biroli, L.F. Cugliandolo, Phys. Rev. Lett. 102 (2009) 050404.
[84] D. Belitz, T.R. Kirkpatrick, R. Saha, Phys. Rev. B 75 (2007) 144418.
[85] L.E. Sadler, J.M. Higbie, S.R. Leslie, M. Vengalattore, D.M. Stamper-Kurn, Nature 443 (2006) 312.
[86] C. Castellano, S. Fortunato, V. Loreto, Rev. Mod. Phys. 81 (2009) 591.
[87] D. Tilman, P. Kareiva (Eds.), Spatial Ecology – The Role of Space in Population Dynamics and Interspecific Interactions, Princeton University Press, Princeton, 1997.
[88] I. Dornic, H. Chaté, J. Chave, H. Hinrichsen, Phys. Rev. Lett. 87 (2001) 045701.
[89] J.-M. Drouffe, C. Godrèche, J. Phys. A 32 (1999) 249.