C. R. Physique 16 (2015) 332–342

Contents lists available at ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

Coarsening dynamics / Dynamique de *coarsening*

Coarsening in inhomogeneous systems

Coarsening *dans les systèmes non homogènes*

Federico Corberi

Dipartimento di Fisica "E. R. Caianiello", and INFN, Gruppo Collegato di Salerno, and CNISM, Unità di Salerno, Università di Salerno, via Giovanni Paolo II 132, 84084 Fisciano SA, Italy

ARTICLE INFO

Article history:
Available online 15 April 2015

Keywords:
Coarsening
Scaling
Disorder

Mots-clés :
Coarsening
Échelle
Désordre

ABSTRACT

This article is a brief review of coarsening phenomena occurring in systems where quenched features—such as random field, varying coupling constants or lattice vacancies—spoil homogeneity. We discuss the current understanding of the problem in ferromagnetic systems with a non-conserved scalar order parameter by focusing primarily on the form of the growth law of the ordered domains and on the scaling properties.

RÉSUMÉ

Cet article constitue un bref survol des phénomènes de *coarsening* qui se produisent dans des systèmes où des inhomogénéités gelées, telles que champs aléatoires, constantes de couplage variables ou lacunes de réseau, détruisent l'homogénéité. Nous discutons la compréhension que l'on a actuellement de ce problème dans les systèmes ferromagnétiques avec un paramètre d'ordre scalaire non conservé, en se concentrant d'abord sur la loi de croissance des domaines ordonnés et sur les propriétés d'échelle.

1. Introduction

Coarsening can exhibit rather different features in systems where homogeneity holds or not. In order to keep the discussion at the simplest level I will focus on the class of ferromagnetic systems with a scalar order parameter (spin), such as those whose properties can be described by the Ising model or, in a continuum approach, by the usual Ginzburg–Landau free energy. Inhomogeneous systems with a vectorial order parameter are much less studied and will not be discussed here. Besides, the attention will be restricted to a dynamics that does not conserve the order parameter. These descriptions are in principle suited, e.g., for a magnetic solid. The aim of the article is not to provide a comprehensive description of the behavior of existing real systems, but rather to discuss at the simplest possible level the new features introduced by the presence of inhomogeneities.

In the kind of systems we are considering, coarsening is usually observed after a quench from a high-temperature equilibrium configuration to a temperature below the critical one. Relaxation toward the new equilibrium state then occurs by

E-mail address: corberi@sa.infn.it.

http://dx.doi.org/10.1016/j.crhy.2015.03.019
1631-0705/

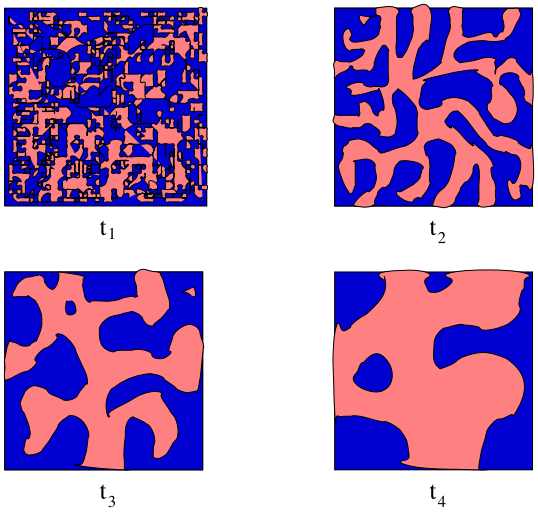

Fig. 1. (Color online.) A pictorial representation of a coarsening process. Blue and pink regions are the growing domains of the two low-temperature equilibrium ordered phases. The four panels correspond to snapshots of the system at subsequent times $t_1 < t_2 < t_3 < t_4$.

the formation and growth of domains inside which the system is basically equilibrated in one of the two symmetry-related low-temperature equilibrium phases (see the article by L.F. Cugliandolo in this dossier). For a magnet, these are the two possible ordered configurations with positive or negative magnetization $\pm m$. The non-equilibrium behavior occurs on the interfaces that move in order to increase the typical domain's size. A pictorial representation of the process is depicted in Fig. 1.

The distinguishing characteristic of coarsening in homogeneous systems is the scaling symmetry [1]. Accordingly, in the late stage of the process, configurations of the system at different times are statistically equivalent if lengths are measured in units of the characteristic size $L(t)$ of the domains, which grows algebraically like

$$L(t) = at^{1/z} \tag{1}$$

where a is a constant, and $z = 2$ in the case without order-parameter conservation considered here [1]. The scaling symmetry is mirrored by the form taken by different quantities, as for instance correlation functions. Using the language of spin systems the two-time spin–spin correlation $G(r,t) = \langle S_i(t) S_j(t_w)\rangle$, where $S_i = \pm 1$ are spin variables on sites i and j of a regular lattice at a distance r, obeys

$$G(r, t_w, t) = g\left[\frac{r}{L(t_w)}, \frac{L(t)}{L(t_w)}\right] \tag{2}$$

where $g(x, y)$ [with $x = r/L(t_w)$ and $y = L(t)/L(t_w)$) is a scaling function (and, similarly, $s(x)$ and $c(y)$ below)]. The form (2) contains, as special cases, the scaling $S(r,t) = s(x)$ of the equal-time correlation function $S(r,t) = G(r,t,t)$ and that $C(t,t_w) = c(y)$ of the autocorrelation function $C(t,t_w) = G(0,t,t_w)$.

Eq. (2) is, in principle, strictly obeyed only in a quench to a vanishing final temperature $T_f = 0$. The effect of a finite quench temperature is to modify the behavior (2) of correlations at small distances [i.e. $r < \xi(T_f)$, where $\xi(T_f)$ is the coherence length of the equilibrium state at T_f (which is small except in the critical region)] and small time-differences $t - t_w < \xi(T_f)^z$]. For simplicity, in the following our discussion about scaling properties will always be restricted to sufficiently large space and time scales where such corrections can be discarded. A thorough discussion of the whole scaling form taken by G at finite T_f is contained in [2].

With the possible exception of liquid systems, whose comprehension however turns out to be more difficult (see article by S. Das, S. Roy and J. Midya in this volume), real systems are almost never homogeneous, due to the presence of quenched lattice defects or because of external disturbances as site-dependent applied fields. Experiments show that these sources of inhomogeneities deeply perturb the coarsening process and the scaling properties. Indeed, although some kind of scaling symmetry comparable to Eq. (2) is sometimes reported [3] for the correlation functions, the asymptotic growth law is never observed in the form (1) and, instead, a much slower increase of the ordered patterns, usually of a logarithmic type [4–6,3]), occurs.

In this article we will mainly review how and why the growth law and the dynamical scaling symmetry, expressed by Eqs. (1, 2), are modified in an inhomogeneous coarsening system.

2. Models

2.1. Homogeneous systems

As an optimal playground to study coarsening I consider the description of a homogeneous system in terms of the Ising model with Hamiltonian

$$\mathcal{H}(\{S_i\}) = -J \sum_{\langle ij \rangle} S_i S_j \tag{3}$$

where $\{S_i\}$ is a configuration of N spin variables $S_i = \pm 1$ and $i = 1, \ldots, N$ are sites of a regular d-dimensional lattice. By *homogeneous* it is meant that the Hamiltonian [for instance the one (3)] does not select any particular spatial position and all places are *a priori* equal. Alternatively one can say that, although a single realization of the system can be inhomogeneous (i.e. due to the presence of domains), local observables do not depend on space after thermal averaging.

A dynamics is introduced by means of a master equation with transition rates $w(\{S_i\} \to \{S_i'\})$ regulating the hopping between configurations. In the following detailed balance with respect to the equilibrium Boltzmann–Gibbs measure $\mathrm{e}^{-\beta\mathcal{H}}$ will be assumed for the w's, implying that the equilibrium state is spontaneously approached. I will also assume, again for simplicity, that the sequential evolution of single spins occurs and that the order parameter is not conserved.

For such a model, it is well known that, after a quench from a high-temperature equilibrium ensemble to a temperature $T_\mathrm{f} = (k_\mathrm{B}\beta_f)^{-1}$ below the critical one T_c, dynamical scaling holds together with Eqs. (1, 2) for any $T_\mathrm{f} < T_\mathrm{c}$ including $T_\mathrm{f} = 0$.

In the following, I will discuss how the most common sources of inhomogeneities, i.e. varying coupling constants, external fields and lattice defects can be included in the above model. I will restrict the discussion to the case in which such features are quenched, namely their position, strength, etc., are not thermalized and do not vary appreciably during the coarsening phenomenon.

2.2. Varying coupling constants

If the coupling constants are not uniform, the Ising Hamiltonian reads

$$\mathcal{H}(\{S_i\}) = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \tag{4}$$

where $J_{ij} = J_0 + \theta_{ij}$ and the θ's are usually assumed to be uncorrelated random numbers with $\overline{\theta_{ij}\theta_{k\ell}} = \mathcal{J}^2 \delta_{i,k}\delta_{k,\ell}$. Here and in the following an over-bar denotes an average taken over the realization of the disorder. The probability distribution of the θ's must be such that $J_{ij} \geq 0$ (otherwise frustration effects could be present, which may radically spoil the ferromagnetic character of the system, with a drastic modification of the equilibrium and non-equilibrium properties). This is known as the random-bond Ising model (RBIM). Notice that, since the distribution of the coupling constant is usually assumed to be symmetric around J_0, in this model there is a maximum value $J_{ij} = J_\mathrm{max} \leq 2J_0$ of the strength of the bonds.

2.3. External fields

Under the action of a site-dependent external magnetic field h_i the Ising Hamiltonian reads

$$\mathcal{H}(\{S_i\}) = -J \sum_{<ij>} S_i S_j - \sum_i h_i S_i \tag{5}$$

A useful model is the one where h_i is an uncorrelated random variable with expectations $\overline{h_i h_j} = h^2 \delta_{ij}$, extracted from a given distribution (the simplest case is a symmetric bimodal distribution $h_i = \pm h$). With these specifications one has the so-called random-field Ising model (RFIM).

2.4. Lattice vacancies

The effect of quenched vacancies can be taken into account by defining a dilution variable $\chi_{ij} = 0, 1$ on the lattice in such a way that the Hamiltonian reads

$$\mathcal{H}(\{S_i\}) = -J \sum_{\langle ij \rangle} \chi_{ij} S_i S_j \tag{6}$$

If the dilution variable is factorized as $\chi_{ij} = \chi_i \chi_j$, the system has site-dilution and the model is usually referred to as the site-diluted Ising model (SDIM). Another possibility is to have bond dilution: In this case, χ_{ij} is defined on the couples of neighboring spins.

In many cases, the χ's can be considered random numbers, usually uncorrelated in space, such that a fraction D of the lattice sites—or bonds—are diluted, namely they are associated with $\chi_i = 0$ or $\chi_{ij} = 0$, respectively. These models are usually denoted as diluted Ising models.

For dilutions larger than the limiting value $D_c = 1 - P_c$—where P_c is the critical percolation probability—a spanning network does not exist. This case is not interesting to us because coarsening implies spins to live on a connected graph extending throughout the system. In particular, since $D_c = 0$ in one dimension, these models will not be considered in $d = 1$.

For the description of other systems, like those where spins are defined on the edges of a fractal network, which can be either deterministic or random, the χ's are taken in order to reproduce the observed topology of such a network. I will briefly discuss this issue later.

Of course, possible modifications of the models introduced so far—i.e. those where the random variables are correlated or where more than one source of disorder is present—are possible, but I will not consider them here.

3. Downhill versus activated coarsening

In a homogeneous system described by the model of Section 2.1, coarsening occurs at any temperature, including $T_f = 0$. This implies that the dynamics, whose main mechanism is a displacement of the interfaces which is regulated by their curvature (see article by A.A. Nepomnyashchy in this volume), proceeds without thermal activation.

Quenched disorder or any other source of inhomogeneity in the system, instead, usually introduces preferred positions where interfaces get pinned in local energy minima. The dynamics can then proceed only by means of thermal activation.

A simple argument shows how this can slow down the kinetics with respect to the homogeneous case where Eq. (1) holds. In view of the dynamical scaling property, the typical pinning energy barrier ΔE at time t is expected to depend on the configuration through $L(t)$ alone, $\Delta E = f[L(t), K]$, where K specifies the set of model parameters, as for instance $K = (J/T, h/T)$ in the RFIM or $K = (J/T, d)$ in the SDIM. In the following, among these various parameters, we will explicitly write only the one, denoted by ϵ, which denotes the strength of the disorder (as h in the RFIM, or $\mathcal{J}$ in the RBIM).

The evolution is slowed down due to the Arrhenius time $t_{esc}(\Delta E) \propto e^{\beta \Delta E}$ needed to escape pinning barriers. Making the simplifying assumption that this effect can be taken into account by a simple rescaling of time ($t \to t/t_{esc}$) in Eq. (1) one has

$$L(t, \epsilon) = a t^{1/2} e^{-\frac{\beta}{2} \Delta E} \tag{7}$$

(In the presence of quenched disorder, quantities as $L(t)$ or $G(r, t, t_w)$ are defined throughout as averaged both over the thermal history and the disorder realizations.) The next point is to establish the form of $f[L, \epsilon]$. Although this cannot be done in general, most of the systems can be divided into three classes [7] according to the way ΔE depends asymptotically on $L(t)$, namely if (i) ΔE approaches a constant value or (ii) it diverges algebraically or (iii) logarithmically with $L(t)$. Let us detail below these cases:

- (i) $\lim_{L\to\infty} f[L, \epsilon] \leq c(\epsilon)$, where $c(\epsilon)$ is usually an increasing function of disorder (with $c = 1$ for $\epsilon = 0$), but it is constant with respect to L;
  This implies that there is an upper limit to the height of the barriers. Using the fact that barriers are approximately constant in Eq. (7), one finds the same growth law as that of a homogeneous system, but with a smaller pre-factor, of the order of $a e^{-\frac{\beta}{2} c(\epsilon)}$, which depends strongly on temperature;
- (ii) $\lim_{L\to\infty} f[L, \epsilon] = b^{-1}(\epsilon) L^{\psi}$, where $\psi > 0$ is an exponent and $b(\epsilon)$ is another constant. In this case, barriers grow algebraically with the domain's size. This leads to a logarithmic growth
  $$L(t) \simeq [b\beta^{-1} \ln t]^{1/\psi} \tag{8}$$
  for large times;
- (iii) $\lim_{L\to\infty} f[L, \epsilon] = z(\epsilon) \ln L$, with $z(\epsilon)$ a constant. Plugging into Eq. (7) one finds the large time behavior
  $$L(t) \propto t^{1/\zeta} \tag{9}$$
  with an exponent $\zeta = 2 + \beta z$ which depends on temperature and, possibly, on other parameters of the system.

3.1. Role of the lower critical dimension

The models introduced above [including the usual Ising model (3)] have a finite critical temperature (only in spatial dimension) larger than the lower critical one d_L. If $d \leq d_L$, a non-interrupted coarsening process can be observed only when quenching to $T_f = 0$ (in an infinite system), because for any $T_f > 0$ the ordered phase is not sustained.

Despite this, by quenching to a very low temperature T_f, if the dynamics does not require activation, one usually does observe coarsening up to a certain time $t_{eq}(T_f)$ such that $L(t_{eq}) \simeq \xi(T_f)$, where $\xi(T_f)$ is the final equilibrium coherence length. For instance, this is what happens in the usual Ising model in $d = d_L = 1$ where $\xi(T_f) \simeq e^{2\beta_f J}$. At low temperatures, $t_{eq}(T_f)$ grows very large, and for $t \ll t_{eq}(T_f)$, the kinetics is analogous to the one at $T_f = 0$.

When homogeneity is spoiled, the dynamics right at $T = 0$ is frozen due to the energetic barriers discussed above that pin the system in metastable states. The way out is to quench to a temperature high enough to drive activated coarsening, but low enough to inhibit the nucleation of equilibrium fluctuations. As for the clean case, as long as $L(t)$ is smaller than the equilibrium correlation length $\xi(T_f)$, one observes the same coarsening behavior as in the $T_f = 0$ quench.

These considerations apply to all the one-dimensional cases that will be discussed in Section 5 as well as to other systems that will be considered in the following for which $T_c = 0$, as for instance the $d = 2$ RFIM case or some models defined on fractal lattices.

3.2. Crossover structure

The growth laws (i), (ii), (iii) of Section 3 are asymptotic behaviors of $L(t)$. Usually, these laws are expected when $L(t)$ exceeds a certain characteristic length $\lambda(\epsilon)$ associated with the presence of the disorder (or of other sources on inhomogeneities). In order to understand the meaning of $\lambda(\epsilon)$ by means of a specific example, let us consider the SDIM of Section 2.4. For D sufficiently small, vacancies are isolated with a typical inter-distance $\lambda(D) = D^{-1/d}$. As long as $L(t) \ll \lambda(D)$, the domains do not feel the presence of the dilution and the system behaves as a homogeneous one (this will be further discussed in Section 7.2). Hence, whatever the asymptotic behavior of this model is [either in class (i), (ii) or (iii) of the previous Section 3], this can only show-up for sufficiently large times such that $L(t) \gg \lambda(D)$. One can show that the crossover phenomenon at $L(t) \simeq \lambda(\epsilon)$, where the behavior of the system changes, is a common feature, not restricted to the previous example of the SDIM, as we will discuss further.

On the other hand, the behavior of the systems in the pre-asymptotic regime $L(t) \ll \lambda(\epsilon)$ is generally model-dependent: There are cases where, as in the SDIM discussed above or in models with a fraction of spins frozen in a random configuration [8], inhomogeneities are initially ineffective and one recovers the growth law (1) of the clean case, and others where they produce a different behavior.

As an example of this different behavior, let us consider the RBIM. In this case, there are random couplings on *any* bond, and—if T_f is sufficiently low—this influences the process already at the level of the fast process of single spin flips. Due to this, in the pre-asymptotic regime $L(t) \ll \lambda(\mathcal{J})$, the disordered bonds alter the behavior of the system with respect to the clean case, thus spoiling Eq. (1), at variance with the SDIM [this will be further discussed in Sections 5.1, 7.3]. However, although determined by the disorder, this early behavior is not the asymptotic one, which sets in only when $L(t)$ is sufficiently large to allow for the simple scaling argument of the previous Section 3 to become valid. Translating *sufficiently large* in the condition $L(t) \gg \lambda(\mathcal{J})$ provides a definition of $\lambda(\mathcal{J})$ in this case (although physically not as transparent as the one above for the SDIM).

I conclude this section by mentioning that the slowing down of the growth law discussed above is not restricted to those systems that—when homogeneous—support a non-activated dynamics. The simplest example is provided by systems with a conserved order parameter. In the homogeneous case, coarsening proceeds asymptotically by means of the activated evaporation–condensation process. However, the presence of disorder [9] or other inhomogeneities usually introduces further energetic barriers slowing even more the kinetics.

4. Scaling form

Once the existence of a crossover length $\lambda(\epsilon)$ next to the ordering length $L(t)$ is established, the next step is to ascertain whether a scaling symmetry is induced by the presence of such lengths and, in this case, how the form (2) can be generalized.

A straightforward generalization of the form (2) keeping into account the crossover length is the following

$$G(r, t_w, t, \epsilon) = \mathcal{G}\left[\frac{r}{L(t_w)}, \frac{L(t)}{L(t_w)}, \frac{\lambda(\epsilon)}{L(t_w)}\right] \tag{10}$$

and similarly for other observable quantities. This amounts to the usual scaling prescription: with each parameter entering G, here r, t_w, t, ϵ, a characteristic length is associated, $r, L(t_w), L(t), \lambda(\epsilon)$ respectively, and these quantities enter as adimensional ratios a scaling function $\mathcal{G}$.

A debated question regards the dependence of the scaling functions, like $\mathcal{G}(x, y, z)$, on the strength of the noise, which occurs through the third entry $z = \lambda(\epsilon)/L(t_w)$. According to a hypothesis, the so-called *superuniversality* [10], the effect of disorder can be fully accounted for by the slower growth of $L(t)$. This implies that disorder does not alter all other properties of the coarsening process, among which—for instance—the geometry of the growing pattern, which is encoded in the form of $\mathcal{G}$ in Eq. (10). This would mean that no extra dependence on ϵ is left in the scaling functions and, for the correlations considered above, one should have $G(r, t_w, t, \epsilon) = \mathcal{G}\left[\frac{r}{L(t_w)}, \frac{L(t)}{L(t_w)}, 1\right] = g\left[\frac{r}{L(t,\epsilon)}, \frac{L(t)}{L(t_w)}\right]$, where $g(x)$ is the same as that in (2) for the clean system.

Presently the superuniversality hypotheses has received confirmation in some specific cases [11–16], but it has also been shown not to be obeyed in other system [17–21], as we will discuss below.

5. Coarsening in one dimension

The effect of inhomogeneities on the coarsening process can be more easily understood in one-dimensional systems. In this case the task is simplified by the fact that interfaces are point-like.

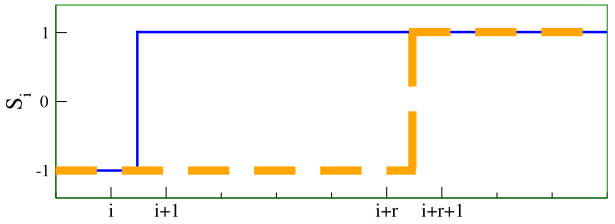

Fig. 2. (Color online.) An interface initially located between sites i and $i+1$ (plotted in a continuous blue line) moves to a new position r sites away (dashed orange line).

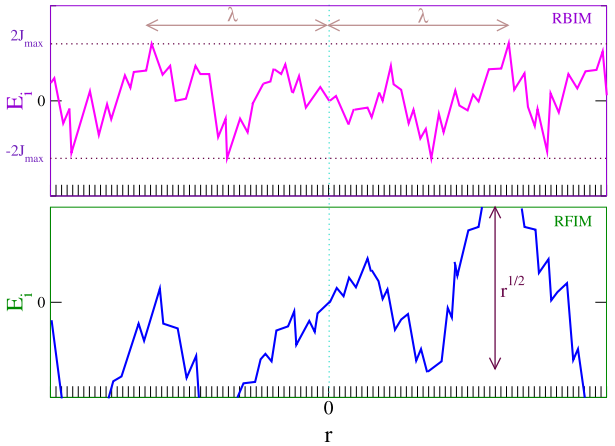

Fig. 3. (Color online.) The energy of a single interface is sketched against its displacement r for the RBIM (upper panel) and the RFIM (lower panel) in $d=1$ (we set $E=0$ for $r=0$). For the RBIM, the distance λ where the maximum energy barrier $4J_{\max}$ is felt is also indicated. For the RFIM, the typical energy barrier increases as $r^{1/2}$.

5.1. RBIM

Let us suppose to have a single interface in the system originally located between sites i and $i+1$ (namely $S_iS_{i+1}=-1$), as pictorially sketched in Fig. 2 (continuous blue line).

Let us suppose that at a later time the interface moves a distance r away from the original position, namely between sites $i+r$ and $i+r+1$ (dashed orange line in Fig. 2). The energy change in this process is $\Delta E_{i,i+r}=2[J_{i+r,i+r+1}-J_{i,i+1}]$. Since the coupling constants are random numbers this form of $\Delta E_{i,i+r}$ induces local minima and maxima in the potential landscape felt by an interface, namely the energy cost of an interface as a function of its position, which is sketched in Fig. 3.

Recalling the discussion at the end of Section 2.2, there exists a maximum value $c(\epsilon)=4J_{\max}$ for the height of the barriers and hence the RBIM is an example of class i) of Section 3. Therefore Eq. (1) holds for long times (but with $t<t_{\rm eq}$) at low $T_{\rm f}$.

Clearly, interfaces must travel a certain distance—or, equivalently, domains must grow up to a certain crossover size $L(t)=\lambda$—in order to sweep a region containing the barrier of maximal height c. According to the argument of Section 3.2, pinning is effective from the very beginning of the process, but the difference with the asymptotic stage is that, for $L(t)<\lambda$, ΔE may depend on $L(t)$. Accordingly the growth law of the homogeneous system (1) is not observed pre-asymptotically and, instead, one finds a $\mathcal{J}/T$-dependent power law, as in Eq. (9) [17].

Regarding the superuniversality hypotheses, it has been convincingly shown not to hold in [17].

5.2. RFIM

In this case, one has

$$\Delta E_{i,i+r}=-\sum_{j=i}^{i+r}h_j\sim hr^{1/2} \tag{11}$$

where in the last passage we have assumed a bimodal distribution $h_i=\pm h$ of the random field (but other possible choices do not make significative differences) and we have used the central limit theorem. Also in this case, there are local minima and maxima, but the potential landscape where the point defect moves increases with r, as it is pictorially shown in Fig. 3. Indeed, as Eq. (11) shows, the energy of the interface, which has moved to a site at a distance r, can be though itself as

the position of a Brownian walker after r steps: the energy landscape is therefore of the Sinai type [22]. As a consequence, the pinning energy is not bounded as in the RBIM and increases algebraically with the typical distance $x(t)$ traveled by the interface, as indicated by Eq. (11). Since it can be shown that also here, as in the clean case, the average size of domains $L(t)$ grows proportional to $x(t)$ [18], one concludes that the $1d$ RFIM belongs to class ii) of Section 3. One indeed has $L(t) \propto (\ln t)^2$ in the large time domain [18,19].

Notice that monotonously increasing barriers mean that they are bound to prevail over the thermal energy scale $k_B T_f$ if $L(t)$ is large enough, namely for $L(t) \gg \lambda$, where $\lambda(h/T_f) \sim (h/T_f)^{-2}$ is the crossover length [19] that can be obtained balancing the two energetic contributions.

At early times $L(t) \ll \lambda$, the Sinai potential is washed out by thermal fluctuations and one recovers the growth law (1) characteristic of the non-disordered system. The $1d$ RFIM is therefore another example of a crossover with an initial homogeneous-like behavior.

Also in this model superuniversality is not obeyed [18,19].

6. Coarsening on fractal substrates

Coarsening occurring on fractal substrates that can be embedded in a Euclidean space can be still described by a model with dilution, like the ones of Section 2.4 [see Eq. (6)], with suitable prescriptions for the $\chi's$. In the following I will focus on the case with site-dilution. The problem has been studied numerically [23,24] both for spins defined on random fractal networks, like the percolation cluster, and for deterministic ones, such as the Sierpinski fractal and others. These deterministic structures are built recursively upon replicating a native geometry—e.g. a triangle for the Sierpinski gasket—of a certain size ℓ.

After the quench, the typical signatures of the coarsening phenomenon are observed, with the growth of an ordering-length and dynamical scaling. Correlation functions can be cast in scaling form, as in Eq. (2): the scaling function is different from the homogeneous case and depends on the fractal considered. Regarding the growth law, one observes either a logarithmic increase of $L(t)$, as for a system in class (ii) [Section 3], or a temperature-dependent power growth law, as for class (iii). This shows that some topological aspect of the substrate—namely the network on whose edges the spin variables are defined—is relevant to determining the growth law and other universal properties. However, this aspect cannot be traced back to usual indicators such as the fractal or the spectral dimension.

Despite a precise identification of this topological feature is currently missing, in [23] the conjecture was proposed that the same property might determine both the existence of the equilibrium phase transition and the non-equilibrium asymptotic growth law. Specifically, systems that do not sustain a (finite-temperature) para-ferromagnetic transition (examples are the percolation network and the Sierpinski gasket) should belong to class (iii) and the others (i.e. the Sierpinski carpet) to class (ii).

Concerning the pre-asymptotic behavior, let us stress that, due to the recursive construction of the deterministic fractals discussed above, the fractal properties are limited to distances larger than ℓ, while for shorter distances the structure is compact. This fixes the crossover length $\lambda = \ell$. For $L(t) \ll \ell$, one recovers the characteristic growth law (1) of a homogeneous substrate.

7. Coarsening in $d > 1$

Understanding coarsening in inhomogeneous systems in dimension larger than one is much harder than in $d = 1$. On the one hand, this is due to the fact that analytical approaches are much more difficult, on the other hand because physical intuition is less straightforward when interfaces are lines or surfaces that in the presence of the pinning centers can bend and stretch. Moreover, numerical simulations—crucial in the absence of analytical tools—are very demanding due to the slow growth of $L(t)$.

Despite many similarities with the one-dimensional models, the pattern of behaviors exhibited in higher dimensions is different and in some cases quite enriched with respect to the $1d$ case, as we will discuss below.

7.1. RFIM

There is a general consensus on the fact that the RFIM is characterized by a logarithmic growth both in $d = 2$ [25,21,9] and in $d = 3$ [11,21], although the value of the exponent ψ defined in Section 3 (ii) cannot be precisely determined numerically, due to the very slow increase of $L(t)$. Recalling also the behavior in $d = 1$, this shows that the barriers encountered at late times have a similar nature in all the spatial dimensions considered. Experiments on real systems [4] confirm the logarithmic growth.

The behavior of the model in the pre-asymptotic stage is, instead, more rich. In [11], a pure-like early stage with Eq. (1) was found. However, in [21], it was shown that, by taking the small T_f limit with fixed h/T, a different power law, with an h/T_f dependent exponent ζ is observed, as in Eq. (9). A similar behavior is observed in [26].

These apparently contrasting observations can be reconciled in a *double crossover* scenario. With the two independent parameters $\epsilon = h/T_f$ and $\tau = T_f/J$ of the model, one can build two characteristic lengths $\lambda(\epsilon)$ and $\Lambda(\tau)$. Then, in principle, one may observe a couple of crossovers as $L(t)$ meets these two lengths, with different behaviors in the three time sectors

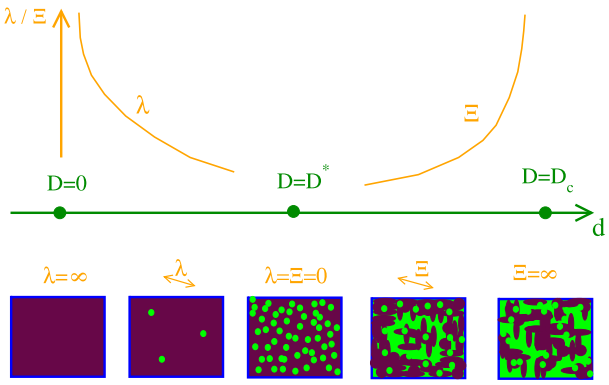

Fig. 4. (Color online.) A pictorial representation of the substrate (plum boxes below) as the density D of vacancies is changed (according to the green axis in the middle). The behavior of the two lengths λ and Ξ is shown in orange.

separated by them [furthermore, Eq. (2) should be upgraded to contain the extra dependence of the scaling function g on $\Lambda(\tau)/L(t_w)$].

In the low-T_f limit considered in [21], where h/T_f is kept fixed, λ is finite and $\tau \simeq 0$, implying also $\Lambda \simeq 0$. This means that there is a single crossover at $L(t) \simeq \lambda$ from a pre-asymptotic to a late regime where the barriers seeded by the disorder act differently as to produce a change from an initial homogeneous-like growth [Eq. (1)], to the disorder-dependent power law of Eq. (9). In the opposite situation of a large T_f (actually as large as possible to keep the ordering of the system), Λ is rather large and λ is so small that the crossover associated with it, occurring at very early times, is not observable. One has again a single crossover, when $L(t) \simeq \Lambda$, but now it occurs similarly to the $1d$ case, namely from a pre-asymptotic behavior where barriers are tamed by thermal fluctuations, when one recovers Eq. (1), to the asymptotic logarithmic behavior.

In the articles cited above, the authors observe alternatively one or the other crossover, and this is possibly due to the choice of the model parameters. If the *double crossover* interpretation is correct, it should in principle be possible, by tuning the parameters opportunely, to observe the two crossovers in a single quench history. To the best of my knowledge, however, this has not yet been reported.

There is no consensus on the validity of superuniversality in this model. Indeed in [11,12] it was found to hold, while it was clearly shown not to be obeyed in [21]. However, it must be recalled that the evidences in favor and against superuniversality are often obtained with different parameter settings. For instance, the two situations with $T_f \simeq 0$ and T_f finite are considered where, as discussed above, also a different crossover pattern is observed. This issue, therefore, is currently not well understood.

7.2. SDIM

Numerical simulations of this system in $d = 2$ have been interpreted both in terms of a power-law increase of $L(t)$ [13,27], or as a logarithmic growth law [28]. A reconciliation between these interpretation is provided in [29] in terms again of a rich crossover interplay, as it is briefly explained below.

Coarsening in this system is deeply related to the geometry of the diluted lattice. Let us start, therefore, with a brief review of the geometrical properties of the substrate, as dilution is varied from high to low values. According to percolation theory [30], at $D = D_c$ a spanning fractal cluster of non-diluted sites is present, with which an infinite percolative coherence length $\Xi(D_c)$ is associated (not to be confused with the coherence length ξ of the Ising model defined on top of the substrate). This situation is pictorially represented on the far right of Fig. 4.

As D is slightly lowered below D_c (i.e. moving a bit to the left in Fig. 4), the infinite cluster is fractal over distances up to $\Xi(D) \sim (D_c - D)^{-\nu}$, where $\nu = 4/3$ is a critical exponent, and it becomes compact over larger distances. Besides, finite clusters are also present. Upon lowering further D, Ξ keep decreasing until, for a certain value D^* of D, it becomes comparable to the lattice spacing. The behavior of $\Xi(D)$ is sketched as an orange line in the upper part of Fig. 4 [i.e. for $D > D^*$]. For $D < D^*$, the infinite cluster is compact over all length scales and there are no finite droplets. There only remain isolated vacancies and, as already discussed in Section 3.2, their typical distance defines another length $\lambda(D)$, which keeps increasing upon lowering D up to $\lambda(D = 0) = \infty$, as shown in Fig. 4 for $D < D^*$.

In the following, we show how the conjecture put forth in [23], which was reviewed in Section 6, may help in understanding the kinetics of this model. Exactly at D_c, coarsening occurs on a percolation cluster. Since there is no ferromagnetic transition on this network, that is to say that $T_c = 0$, a power-law growth as in Eq. (9) is expected. This behavior is drawn in bold green in the schematic picture of Fig. 5. Away from the percolation point $D = D_c$, the large-scale properties of the network provide an equilibrium phase transition at a critical temperature $T_c(D) > 0$. In this case, therefore, one expects an asymptotic logarithmic growth law. This is recognized in Fig. 5 by the asymptotic downward bending of the curves with $0 < D < D_c$ for large times. Clearly, right at $D = 0$, the homogeneous case is recovered with the usual law (1), which is shown in red in Fig. 5.

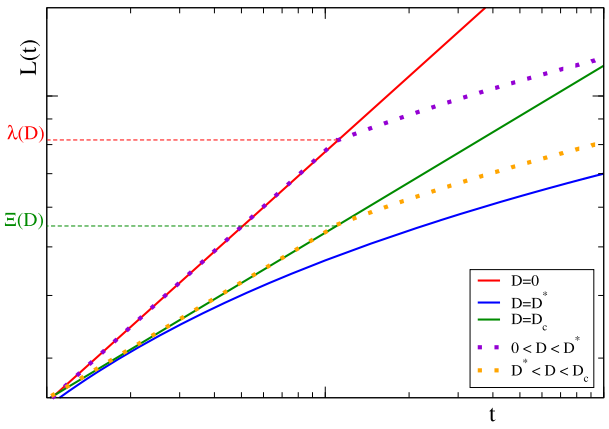

Fig. 5. (Color online.) A pictorial representation of the behavior of $L(t)$ for different choices of D, on a double logarithmic scale.

This pattern of behaviors is indeed observed in the numerical study [29] of the model in the small T_f-limit. This supports the conjecture discussed in Section 6.

Not only the asymptotic laws, but also the early-stage behaviors can be understood. Let us start from the large dilution regime $D \lesssim D_c$. In this case, the properties of the spanning cluster are the same as those at D_c up the length $\Xi(D)$. Hence a crossover phenomenon is expected from an early stage regulated by Eq. (9) for $L(t) \ll \Xi(D)$ to the asymptotic logarithmic growth when $L(t) \gg \Xi(D)$, as it is pictorially represented with a dotted orange line in Fig. 5. This shows that the growth law may be interpreted as algebraic or logarithmic, depending on the choice of D and on the time-scale of the observations and might explain why different findings are reported in the literature.

Let us turn now to consider the small dilution regime, which behaves differently. As discussed in Section 3.2, a crossover occurs here between an early homogeneous-like behavior (1) and a late logarithmic growth when $L(t)$ meets $\lambda(D) \sim D^{-1/d}$. This is shown in Fig. 5 with a dotted-violet line.

This whole pattern of crossovers is actually observed in [29] suggesting—once again—that an interpretation in terms of the scaling symmetry and the related crossover structure is a groundbreaking tool.

Notice also that, although two lengths are present also in the SDIM, the crossover mechanism is very different from the one—denoted as *double crossover* in Section 7.1—invoked for the RFIM. Indeed, in that case, the two lengths $\lambda(\epsilon)$ and $\Lambda(\tau)$ are built from two independent parameters of the model. This allows one, in principle, to have a double crossover, namely three consecutive regimes.

For the SDIM in the small-T_f limit considered in [29], both Ξ and λ depend on the same parameter D and this occur in such a way that either $\lambda \simeq 0$ (for $D > D^*$) or $\Lambda \simeq 0$ (for $D < D^*$). Hence, for any choice of $0 < D < D_c$, only one between the two lengths λ and Λ can play a role, and a single crossover between an early and a late stage can be observed, as it is clarified in Fig. 5.

Finally, let us consider the issue of superuniversality. In Refs. [28] [27], the authors succeed in collapsing the curves for correlation functions using the simple scaling (2). In [28], this is interpreted as a possible confirmation of the superuniversality hypotheses. However, in [29], a study with a more ample choice of values of D and different timescales clearly shows that (2) is not sufficient to collapse all the curve and, instead, a scaling form with an extra argument as Eq. (10) is needed, suggesting that superuniversality is violated also in this model.

7.3. RBIM

For the RBIM, Huse and Henley (HH) [31], elaborating on previous ideas due to J. Villain [32], predicted the logarithmic law (8) with $\psi = 1/4$. Their argument, in $d = 2$, can be summarized as follows: according to roughening theory, a piece of interface of linear size r in equilibrium deviates from flatness by a typical quantity $w(r) \propto r^{\zeta_r}$, where the *roughening exponent* ζ_r is a universal quantity. If disorder is present, the pinning energy of the interfaces scales as $E_p \sim r_r^{\chi}$, where χ_r is another exponent.

On the other hand, simply due to geometry, moving a distance r along the surface of a coarsening curved domain one also deviates from flatness by a certain quantity $a(r, R)$, where R is the radius of curvature. Hence, on scales r up to r_c—such that $a(r, R) \simeq w(r)$—the domain's wall is basically equilibrated. A simple calculation shows that $a(r, R)$ is at least of order $a = r^2/R$ (this can be very easily proved for a circular shape in the small r limit) and hence one has $r_c \simeq R^{1/(2-\zeta_r)}$.

The next smallest scale to evolve towards equilibrium is precisely r_c and, recalling the discussion above, the pinning energy associated with this is $E_p \sim r_c^{\chi_r} \simeq R^{\psi}$, with $\psi = \chi_r/(2 - \zeta_r)$. For random bond disorder in $d = 2$, it is known [31,33] that $\zeta_r = 2/3$ and $\chi_r = 1/3$, leading to $\psi = 1/4$. Assuming scaling by the identification $R \sim L(t)$ and plugging this result into ii) of Section 3, one arrives at Eq. (8) with $\psi = 1/4$.

The correctness of the HH prediction has remained controversial for quite a long time since numerical results for the $d = 2$ RBIM showed a steady algebraic growth with a disorder-dependent dynamical exponent [12,34] as in Eq. (9). This would imply an asymptotic logarithmic increase of the pinning barriers at variance with the prediction of HH. Recently, more extensive numerical simulations [20] have produced sufficient numerical evidence for the existence, after a long-lasting algebraic regime, of a crossover to an asymptotic logarithmic growth. This supports the HH scenario, although a precise determination of the ψ exponent is, also in this case, not possible. Experiments on real two-dimensional systems also find [5,6,3] a logarithmic growth, and the HH exponent $\psi = 4$ was reported in [6,3].

The very nature of the long-lasting pre-asymptotic regime with a power-law increase of the domain's size, which is observed also in some experiments [3], is not presently clarified. In view of the discussion of Section 3, it could be due to a genuine regime where the scaling of barriers is only logarithmic—according to iii)—different therefore from the asymptotic behavior described by HH. This is not necessarily the case, since even within the HH scenario long-lasting pre-asymptotic corrections might yield an effective algebraic growth law at intermediate times, as noticed in different contexts in [35].

Interestingly enough, the algebraic growth law can be made truly asymptotic by considering a symmetric bimodal distribution $J_{ij} = J_0 \pm J_0$ of the coupling constants, as it was shown numerically in [36]. This choice amounts to have a fraction $1/2$ of bonds $J_{ij} = 0$ and an equal quantity of $J_{ij} = 2J_0$. This realizes a percolation cluster of bonds, since the bond percolation threshold is $D_c = 1/2$. According to the conjecture mentioned in Section 6, one expects an algebraic increase of $L(t)$ as in Eq. (9), as indeed it is observed.

Regarding superuniversality, also in this model the situation is controversial: while it was observed to hold true in [12,16], clear violations have been observed in [20]. Some experimental evidences in favor of a superuniversal scaling have been reported in [3].

8. Conclusions

In this article, I briefly reviewed the coarsening phenomena occurring in simple model systems without space-translation invariance. Following the footsteps of early studies of phase ordering in homogeneous ferromagnets, most efforts in this field have been devoted to the determination of the growth law and of the dynamical scaling properties. Apart from some instances of one-dimensional systems, analytical approaches are scarce, and the current understanding of the problem rests mostly upon heuristic arguments and slowly converging numerical simulations. This fact, together with the observation of a broad spectrum of distinct behaviors, makes the problem far from being settled.

Our poor understanding hinders the development of a general framework for growth kinetics in disordered and inhomogeneous ferromagnetic systems, which is a subject of noteworthy technological relevance. This, in turn, encumbers the discussion on the very nature of more complex disordered systems, such as spin glasses, which according to a debated interpretation [37] could also be described as a disordered ferromagnet.

As discussed in this paper, despite this state of affairs, the rich (and sometimes apparently contrasting) behavior of different systems—or even of the same system in different temporal or parameter regions—can be interpreted in terms of an *upgraded* scaling framework where, next to the typical size L of ordered regions, there is another (or, in some cases, others) relevant characteristic length(s) λ associated with the inhomogeneities. This fosters the overview of the phenomenon and could help to set the basis for a general renormalization group scheme where diverse *universal* behaviors—e.g., power-law or logarithmic growth laws—are connected by a crossover pattern. However, the development of such a controlled theory is a matter for future research.

Acknowledgements

I acknowledge financial support from MURST PRIN 2010HXAW77 005.

Part of the project of this dossier on Coarsening took place at the Galileo Galilei Institute for Theoretical Physics, Arcetri, Tuscany, Italy, during the workshop *Advances in Nonequilibrium Statistical Mechanics*, in the summer of 2014.

References

[1] For reviews see A.J. Bray, Adv. Phys. 43 (1994) 357;
M. Zannetti, Aging in domain growth, in: S. Puri, V. Wadhawan (Eds.), Kinetics of Phase Transitions, CRC Press, Boca Raton, FL, USA, 2009, p. 153;
F. Corberi, L.F. Cugliandolo, H. Yoshino, Growing length scales in aging systems, in: L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipeletti, W. van Saarloos (Eds.), Dynamical Heterogeneities in Glasses, Colloids, and Granular Media, University Press, Oxford, UK, 2010.

[2] A. Crisanti, F. Ritort, J. Phys. A, Math. Gen. 36 (2003) R181, arXiv:cond-mat/0212490;
J.-P. Bouchaud, L.F. Cugliandolo, J. Kurchan, M. Mézard, in: A.P. Young (Ed.), Spin Glasses and Random Fields, World Scientific, Singapore, 2003, arXiv:cond-mat/9702070, 1997;
F. Corberi, E. Lippiello, M. Zannetti, J. Stat. Mech. (2007) P07002;
F. Corberi, E. Lippiello, M. Zannetti, J. Stat. Mech. (2004) P12007;
F. Corberi, E. Lippiello, M. Zannetti, Phys. Rev. E 68 (2003) 046131;
F. Corberi, E. Lippiello, M. Zannetti, Phys. Rev. E 65 (2002) 046136;
F. Corberi, E. Lippiello, M. Zannetti, Eur. Phys. J. B 24 (2001) 359;
F. Corberi, E. Lippiello, M. Zannetti, Phys. Rev. E 63 (2001) 061506;
F. Corberi, N. Fusco, E. Lippiello, M. Zannetti, Int. J. Mod. Phys. B 18 (2004) 593.

[3] V. Likodimos, M. Labardi, X.K. Orlik, L. Pardi, M. Allegrini, S. Emonin, O. Marti, Phys. Rev. B 63 (2001) 064104;
V. Likodimos, M. Labardi, M. Allegrini, Phys. Rev. B 61 (2000) 14440.
[4] H. Ikeda, Y. Endoh, S. Itoh, Phys. Rev. Lett. 64 (1990) 1266.
[5] A.G. Schins, A.F.M. Arts, H.W. de Wijn, Phys. Rev. Lett. 70 (1993) 2340.
[6] D.K. Shenoy, J.V. Selinger, K.A. Grüneberg, J. Naciri, R. Shashidhar, Phys. Rev. Lett. 82 (1999) 1716.
[7] Z.W. Lai, G.F. Mazenko, O.T. Valls, Phys. Rev. B 37 (1988) 9481.
[8] G.S. Grest, D.J. Srolovitz, Phys. Rev. B 32 (1985) 3014.
[9] M. Rao, A. Chakrabarti, Phys. Rev. E 48 (1993) R25(R).
[10] D.S. Fisher, D.A. Huse, Phys. Rev. B 38 (1988) 373;
L.F. Cugliandolo, Physica A 389 (2010) 4360, for an explanation of the super universality concept see Sect. 5.2 and references quoted therein.
[11] M. Rao, A. Chakrabarti, Phys. Rev. Lett. 71 (1993) 3501;
C. Aron, C. Chamon, L.F. Cugliandolo, M. Picco, J. Stat. Mech. (2008) P05016.
[12] R. Paul, S. Puri, H. Rieger, Europhys. Lett. 68 (2004) 881;
R. Paul, S. Puri, H. Rieger, Phys. Rev. E 71 (2005) 061109.
[13] S. Puri, D. Chowdhury, N. Parekh, J. Phys. A 24 (1991) L1087;
S. Puri, D. Chowdhury, N. Parekh, J. Phys. A 24 (1991) L1087;
S. Puri, N. Parekh, J. Phys. A 25 (1992) 4127;
A.J. Bray, K. Humayun, J. Phys. A 24 (1991) L1185;
B. Biswal, S. Puri, D. Chowdhury, Physica A 229 (1996) 72.
[14] A. Sicilia, J.J. Arenzon, A.J. Bray, L.F. Cugliandolo, Europhys. Lett. 82 (2008) 10001.
[15] M.P.O. Loureiro, J.J. Arenzon, L.F. Cugliandolo, A. Sicilia, Phys. Rev. E 81 (2010) 021129.
[16] M.F. Gyure, S.T. Harrington, R. Strilka, H.E. Stanley, Phys. Rev. E 52 (1995) 4632.
[17] E. Lippiello, A. Mukherjee, S. Puri, M. Zannetti, Europhys. Lett. 90 (2010) 46006.
[18] D.S. Fisher, P. Le Doussal, C. Monthus, Phys. Rev. E 64 (2001) 066107;
D.S. Fisher, P. Le Doussal, C. Monthus, Phys. Rev. Lett. 80 (1998) 3539.
[19] F. Corberi, A. de Candia, E. Lippiello, M. Zannetti, Phys. Rev. E 65 (2002) 046114.
[20] F. Corberi, E. Lippiello, A. Mukherjee, S. Puri, M. Zannetti, J. Stat. Mech. Theory Exp. (2011) P03016.
[21] F. Corberi, E. Lippiello, A. Mukherjee, S. Puri, M. Zannetti, Phys. Rev. E 85 (2012) 021141.
[22] Y.G. Sinai, Theory Probab. Appl. 27 (1982) 256.
[23] R. Burioni, F. Corberi, A. Vezzani, Phys. Rev. E 87 (2013) 032160.
[24] R. Burioni, F. Corberi, A. Vezzani, J. Stat. Mech. (2010) P12024;
R. Burioni, F. Corberi, A. Vezzani, J. Stat. Mech. (2009) P02040;
R. Burioni, D. Cassi, F. Corberi, A. Vezzani, Phys. Rev. E 75 (2007) 011113;
R. Burioni, D. Cassi, F. Corberi, A. Vezzani, Phys. Rev. Lett. 96 (2006) 235701;
U. Marini Bettolo Marconi, A. Petri, Phys. Rev. E 55 (1997) 1311;
U. Marini Bettolo Marconi, Phys. Rev. E 57 (1998) 1290;
C. Castellano, F. Corberi, U. Marini Bettolo Marconi, A. Petri, J. Phys. IV 8 (1998) 93.
[25] S. Puri, N. Parekh, J. Phys. A 26 (1993) 2777;
E. Oguz, A. Chakrabarti, R. Toral, J.D. Gunton, Phys. Rev. B 42 (1990) 704;
E. Oguz, J. Phys. A 27 (1994) 2985.
[26] S. Sinha, P.K. Mandal, Phys. Rev. E 87 (2013) 022121;
P.K. Mandal, S. Sinha, Phys. Rev. E 89 (2014) 042144.
[27] R. Paul, G. Schehr, H. Rieger, Phys. Rev. E 75 (2007) 030104(R).
[28] H. Park, M. Pleimling, Phys. Rev. B 82 (2010) 144406.
[29] F. Corberi, E. Lippiello, A. Mukherjee, S. Puri, M. Zannetti, Phys. Rev. E 88 (2013) 042129.
[30] D. Stauffer, Phys. Rep. 54 (1979) 1;
D. Stauffer, A. Aharony, Introduction to Percolation Theory, revised second edition, Taylor and Francis, London, 1994.
[31] D.A. Huse, C.L. Henley, Phys. Rev. Lett. 54 (1985) 2708.
[32] J. Villain, Phys. Rev. Lett. 52 (1984) 1543.
[33] M. Kardar, Phys. Rev. Lett. 55 (1985) 2923;
D.A. Huse, C.L. Henley, D.S. Fisher, Phys. Rev. Lett. 55 (1985) 2924.
[34] M. Henkel, M. Pleimling, Europhys. Lett. 76 (2006) 561;
M. Henkel, M. Pleimling, Phys. Rev. B 78 (2008) 224419;
J.H. Oh, D. Choi, Phys. Rev. B 33 (1986) 3448.
[35] L.F. Cugliandolo, Topics in coarsening phenomena, arXiv:0911.0771v1 [cond-mat.stat-mech];
J.L. Iguain, S. Bustingorry, A.B. Kolton, L.F. Cugliandolo, Phys. Rev. B 80 (2009) 094201;
J.-P. Bouchaud, V. Dupuis, J. Hammann, E. Vincent, Phys. Rev. B 65 (2001) 024439.
[36] F. Corberi, R. Burioni, E. Lippiello, A. Vezzani, M. Zannetti, Phys. Rev. E (2015), submitted.
[37] D.S. Fisher, D.A. Huse, Phys. Rev. Lett. 56 (1986) 1601;
D.S. Fisher, D.A. Huse, Phys. Rev. B 38 (1988) 386;
D.S. Fisher, D.A. Huse, J. Phys. A 20 (1987) L1005.