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The imprint of inflation on the cosmic microwave background

Découvrir l'inflation dans le fond fossile micro-onde

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A B S T R A C T

The cosmic microwave background is the most precise and the most simple cosmological dataset. This makes it our most prominent window to the physics of the very early Universe. In this article I give an introduction to the physics of the cosmic microwave background and show in some detail how primordial fluctuations from inflation are imprinted in the temperature anisotropy and polarisation spectrum of the CMB. I discuss the main signatures that are suggesting an inflationary phase for the generation of initial fluctuations.

R É S U M É

Le fond fossile micro-onde est l'ensemble de données cosmologiques les plus précises et les plus simples à interpréter. Ceci en fait notre fenêtre la plus directe sur la physique de l'univers primordial. Dans cet article, je présente une introduction à la physique du fond fossile micro-onde et je démontre comment les fluctuations primordiales de l'inflation se manifestent dans les anisotropies de la température et dans la polarisation du fond fossile. Je discute les principales observables qui présentent des indices importants vers une attribution des fluctuations initiales à une phase inflationnaire.

1. Introduction

As you can see from the contributions by most other authors to this volume, inflation is presently well established. It was originally introduced by Guth [1] to explain the flatness and the large entropy of the present Universe and to solve the horizon problem. Somewhat earlier, Starobinsky [2] had shown that a quasi de Sitter phase of expansion leads to the generation of gravitational waves from quantum fluctuations of the metric and somewhat later Mukhanov and Chibisov [3,4] found that also scalar fluctuations are inevitably generated during an inflationary phase.

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Contrary to the flatness, the homogeneity and isotropy and the large entropy of the Universe, which have been observed before they were explained and which therefore have to be regarded as 'post-dictions' of inflation,[1] the generation of perturbations was a prediction that has been verified for the first time by the COBE satellite [5] in 1992 and led to the Nobel Prize awarded to G. Smoot in 2006.

The COBE satellite observed the cosmic microwave background (CMB). The CMB is a background of thermal photons that, during the hot early phase of the Universe, were tightly coupled to baryons. As the Universe expands and cools, baryons eventually combine with electrons first to neutral helium and finally to neutral hydrogen. At an age of the Universe of about $\tau_{\rm dec} \sim 3\times 10^5$ years and a redshift $z_{\rm dec} \simeq 1090$, the temperature drops below $T_{\rm dec} \simeq 3000$ K and there are no longer sufficiently many high energy photons around to keep the Universe ionized, most electrons are bound in neutral atoms. After that time, photons propagate freely into our antenna to be detected by COBE and other experiments. CMB experiments literally take a photo of the Universe when it was about 3×10^5 years young. This early time is not much after matter and radiation equality and since in a radiation dominated Universe fluctuations cannot grow, they are still very simply and linearly related to their value after a phase of primordial inflation. This renders the CMB a unique pristine probe of the physics of the very early Universe.

Therefore, inflation and observations of the CMB are intimately related. In this paper, I want to review this relation. For this, I introduce in the next section linear cosmological perturbations. Then I briefly indicate how quantum fluctuations are amplified during an inflationary phase and lead to classical fluctuations in the spacetime geometry. This topic is elaborated in much more detail in the contribution by A. Starobinsky. In Section 3, the heart of this paper, I explain how inflationary perturbations are imprinted in the CMB. In Section 4, I conclude.

For simplicity, I shall concentrate on a spatially flat Friedmann metric given by

$$\mathrm{d}s^2 = a^2(t)\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) = -\mathrm{d}\tau^2 + a^2(\tau)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1}$$

Here a denotes the cosmic scale factor, t is conformal time and τ is cosmic time. We denote the conformal Hubble parameter by $\mathcal{H}$ and the physical one by H,

$$\mathcal{H} = \frac{\mathrm{d}a/\mathrm{d}t}{a}, \quad H = \frac{\mathrm{d}a/\mathrm{d}\tau}{a} = a^{-1}\mathcal{H}$$

Latin indices run from 1 to three while greek indices run from 0 to 3, spatial vectors are indicated in boldface. Both, the speed of light and Planck's constant are set to unity, $c=\hbar=1$. $M_{\rm p} = (8\pi G)^{-1/2}$ denotes the reduced Planck mass.

2. The generation of fluctuations during inflation

2.1. Linear cosmological perturbations

The fluctuations in the cosmic microwave background are small. It is therefore a good strategy to compute them with linear cosmological perturbation theory. We consider a linearly perturbed Friedmann metric,

$$\mathrm{d}s^2 = a^2(t)\left[\eta_{\mu\nu} + h_{\mu\nu}\right]\mathrm{d}x^\mu\mathrm{d}x^\nu \quad \text{with} \tag{2}$$

$$h_{\mu\nu} = 2\begin{pmatrix} -\Psi & 0 \\ 0 & -\Phi\delta_{ij} + H_{ij}\end{pmatrix} \tag{3}$$

Here $(\eta_{\mu\nu})$ is the flat Minkowski metric, $\Psi(t,\mathbf{x})$ and $\Phi(t,\mathbf{x})$ are the so called Bardeen potentials of scalar perturbations and $H_{ij}(t,\mathbf{x})$ with $H^i_i = \partial_i H^i_j = 0$ describes a gravitational wave. It can be shown that one can always bring the perturbed metric into this form, the longitudinal gauge as long as vector type (vorticity) perturbations can be neglected, see, e.g., [6]. The first-order perturbed Einstein equations relate the metric perturbations to the perturbations of the energy momentum tensor. They are given by

$$\frac{\rho a^2}{2M_P^2}\left(D_s + 3(1+w)\mathcal{H}V\right) = \Delta\Phi \tag{4}$$

$$\frac{\rho a^2}{2M_P^2}(1+w)V = \mathcal{H}\Psi + \dot\Phi \tag{5}$$

$$\frac{\rho a^2}{M_P^2}w\Pi^{(S)} = \Psi + \Phi \tag{6}$$

[1] Actually the resolution of the horizon problem does not *explain* the homogeneity and isotropy of the Universe. It just renders is causally possible. How to generate it from an arbitrarily fluctuating initial spacetime is still an unsolved problem. Also for chaotic inflation, we require an initially homogeneous and isotropic patch of some small size.

$$\frac{\rho a^2}{M_P^2} w \Pi^{(T)} = \ddot{H} + 2\mathcal{H}\dot{H} - \Delta H \tag{7}$$

Here ρ is the mean density of the Universe and the pressure is given by $P = w\rho$. D_s is the matter density perturbations, V is the peculiar velocity potential, $\mathbf{V} = -\nabla V$ and $\Pi^{(S)}$ is the scalar anisotropic stress potential. The traceless part of the stress tensor is given by

$$T_i^j - P\delta_i^j = P\left[\partial_i \partial^j \Pi^{(S)} - \frac{1}{3}\Delta \Pi^{(S)} \delta_i^j + \Pi_i^{(T)j}\right]$$

Eq. (7) is the evolution equation for an arbitrary tensor mode and the transverse traceless part of the stress tensor, $\Pi^{(T)}$ is its source. Eqs. (4)–(6) are not independent. They are related via the energy momentum conservation equations. Using this one can derive, e.g., one single second-order equation for Φ, the so-called Bardeen equation. There D_s and V are eliminated by the constraints (4) and (5). $\Pi^{(S)}$ as well as a combination of matter perturbations that describes the divergence of the entropy flux, i.e. the deviation of the perturbations from adiabaticity,

$$\Gamma \equiv \delta P/P - \frac{c_s^2}{w} D_s$$

source the Bardeen potential Φ. Here we shall not write the Bardeen equation, which can be found, e.g., in [6], but an equation for the so-called Mukhanov–Sasaki variable, see [4,7], which represents the spatial curvature in the comoving gauge and is given by:

$$\zeta = \frac{2}{3(1+w)}\left[\Psi + \mathcal{H}^{-1}\dot{\Phi}\right] \tag{8}$$

The Bardeen equation is equivalent to the following equation for ζ,

$$\dot{\zeta} = \frac{w}{w+1}\mathcal{H}\Gamma - \frac{2c_s^2}{2(1+w)}\mathcal{H}^{-1}k^2\Psi \tag{9}$$

This shows that for adiabatic perturbations, i.e., if $\Gamma = 0$, the variable ζ is conserved on super Hubble scales, i.e. scales with $k/\mathcal{H} \ll 1$.

2.2. Generation of scalar perturbations during inflation

We briefly discuss the generation of scalar perturbations during inflation. More details can be found in the contribution by A. Starobinsky.

An inflationary phase is a period during which the cosmic expansion is accelerated, $\mathrm{d}^2 a/\mathrm{d}\tau^2 > 0$ and the physical Hubble parameter changes only very slowly,

$$\epsilon_1 = -\frac{\mathrm{d}H/\mathrm{d}\tau}{H^2} = \frac{\mathcal{H}^2 - \dot{\mathcal{H}}}{\mathcal{H}^2} \ll 1 \tag{10}$$

One also introduces

$$\epsilon_2 = \frac{\mathrm{d}^2 H/\mathrm{d}^2\tau}{2H\mathrm{d}H/\mathrm{d}\tau} \tag{11}$$

and requires that $|\epsilon_2| \ll 1$. With this, one finds that ϵ_1 changes slowly,

$$\frac{\dot{\epsilon}_1}{\epsilon_1} = 2\mathcal{H}(\epsilon_1 + \epsilon_2) \tag{12}$$

If matter is given by a canonical scalar field with Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

inflation happens when the energy density is dominated by the potential,

$$\rho_\phi = \frac{1}{2a^2}\dot{\phi}^2 + V(\phi) \simeq V(\phi) \tag{13}$$

The slow roll parameters then simply become $\epsilon_1 = \frac{3}{2}(\mathrm{d}\phi/\mathrm{d}\tau)^2/V$ and $\epsilon_2 = -M_P^2 V_{\phi\phi}/V$.

Introducing

$$z = \sqrt{2\epsilon_1} a M_\mathrm{p} \quad \text{and} \quad v = z\zeta \tag{14}$$

the second-order perturbation of the action can be written as [8,6]

$$S_2 = \frac{-1}{2}\int d^4x \left[\partial_\mu v \partial^\mu v + m^2(t)v^2\right] \tag{15}$$

where the indices are raised with the flat metric and $m^2 = -\ddot{z}/z$. Now v is a canonically normalised scalar degree of freedom with a time-dependent mass. The best-known example of quantum particle creation in an external classical field. Requiring standard quantum initial conditions for v,

$$v_k(t) = \frac{1}{\sqrt{2k}} \exp(-\mathrm{i}kt) \quad \text{for } k \gg m \simeq \mathcal{H} \tag{16}$$

we can solve the classical mode equation and obtain v and ζ at late time. For the power spectrum of ζ, one finds the following result within the slow roll approximation,

$$k^3\langle 0|\zeta(\mathbf{k})\zeta(\mathbf{k}')^*|0\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')P_s(k) \tag{17}$$

with

$$P_s(k) = \frac{H^2}{4\epsilon_1 M_P^2}\left(\frac{k}{\mathcal{H}}\right)^{-2(3\epsilon_1+\epsilon_2)} = A_s\left(\frac{k}{k_*}\right)^{n_s-1}, \quad \text{for } k/\mathcal{H} \ll 1 \tag{18}$$

Here n_s is the scalar spectral index and A_s the scalar amplitude, which in general, i.e. if $n_s - 1 \neq 0$, depends on the 'pivot scale' k_*. Since ϵ_1 and ϵ_2 are small, slow roll inflation predicts a nearly scale-invariant spectrum, $|n_s - 1| \ll 1$. More details on the derivation of this result are found, e.g., in [8,6].

2.3. Generation of tensor perturbations during inflation

We also briefly discuss the generation of tensor perturbations during inflation. More details can be found in the contribution by A. Starobinsky.

Like for the scalar case, we can write the second-order perturbed action for gravitational wave perturbations in the form (15). The only difference is that now v and m^2 are defined differently. The canonically normalised variable v is now given by:

$$H_{ij} = \frac{v}{M_p a} e_{ij} \tag{19}$$

where e_{ij} is an arbitrary but normalised polarisation tensor, and $m^2 = -\ddot{a}/a$. Choosing again the quantum initial condition (16), one obtains the gravitational wave spectrum,

$$4k^3\langle 0|H^{ij}(\mathbf{k})H^*_{ij}(\mathbf{k}')|0\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')P_t(k) \tag{20}$$

$$P_t(k) = 4\frac{H^2}{M_P^2}\left(\frac{k}{\mathcal{H}}\right)^{-2\epsilon_1} = A_t\left(\frac{k}{k_*}\right)^{n_t}, \quad k/\mathcal{H} \ll 1 \tag{21}$$

Here we have taken into account the fact that the tensor perturbations are $2H_{ij}$ and another factor of 2 comes from the sum over the two polarisation states of H_{ij}. Details of the derivation are found in [6,8]. A_t and n_t denote the tensor amplitude and spectral index.

Since during inflation the slow roll parameters are small, both spectra are nearly scale invariant, with spectral indices $n_s - 1 = -6\epsilon_1 - 2\epsilon_2$ and $n_t = -2\epsilon_1$. For historical reasons, the spectral index of ζ is not simply denoted n_s, but $n_s - 1$. The tensor spectrum is suppressed with respect to the scalar one by a factor

$$r = \frac{P_t}{P_s} \simeq 16\epsilon_1 = -8n_t \simeq \frac{A_t}{A_s} \tag{22}$$

This is the so-called slow-roll consistency relation. If we could ever measure P_t, we could in principle test this relation. The $\simeq$ signs are strict equal signs only for scale invariant spectra. Otherwise the expressions depend on the scale.

The fluctuations generated in this way during slow-roll inflation are typically Gaussian. The non-Gaussianities that come from interactions of the inflation field are suppressed. The parameter f_{NL}, defined as the ratio between the 3-point function and the square of the 2-point function, is of the order of the slow-roll parameters [9].

3. Inflationary fluctuations in the CMB

In this section, which is the heart of this contribution, we discuss how perturbations generated during inflation are imprinted on the CMB.

3.1. Transfer functions

Let us first note that we can relate the curvature spectrum P_s on super-Hubble scales to the spectrum of the Bardeen potential via Eq. (8). In the radiation and matter dominated regimes this yields

$$P_\Psi = \frac{4}{9} P_s \qquad \text{(radiation dom.)} \tag{23}$$

$$P_\Psi = \frac{9}{25} P_s \qquad \text{(matter dom.)} \tag{24}$$

Here we have used that in a radiation and in a matter dominated Universe Ψ is constant on super Hubble scales. Using also that both, Γ and Π vanish in a pure radiation and in a pure matter Universe one can use the Bardeen equation to compute Φ, D_s or V at any time such that

$$\begin{aligned} \Psi(\mathbf{k},t) &= T_\Psi(k,t)\zeta(\mathbf{k}) \\ D_s(\mathbf{k},t) &= T_D(k,t)\zeta(\mathbf{k}) \\ V(\mathbf{k},t) &= T_V(k,t)k^{-1}\zeta(\mathbf{k}) \end{aligned} \tag{25}$$

Here $T_X(k,t)$ is the transfer function of the variable X. It relates the fluctuations in X at scale k^{-1} at time t to the initial fluctuations of ζ after inflation at the same scale. We have introduced a factor k^{-1} in the equation for V, since the potential V has a dimension of length and we want to keep the transfer functions dimensionless. Since the perturbation equations are linear, different scales do not mix and, in simple inflationary models, there is just one non-trivial initial condition that is usually cast in the variable ζ. If the matter content of the Universe is more complicated, for example during the radiation–matter transition or allowing for collisionless neutrinos which have anisotropic stress, $\Pi^{(S)} \neq 0$, we just have to introduce more transfer functions and the corresponding linear perturbation equations which determine their evolution. In order to compute the transfer functions T_X, we need to know the coefficients of the corresponding linear differential equation, which depend on the background Universe, i.e. on the cosmological parameters. Therefore, in cosmology we cannot isolate cosmological parameters from initial conditions. We always estimate them together. In simple inflationary models, the initial spectra from inflation together with the matter content of the Universe yield a set of at least six parameters that we can estimate from the observed CMB anisotropies and polarisation.

3.2. Photon propagation in a perturbed Universe

Let us first consider photons coming from the last scattering surface into our antenna and compute the temperature (energy) fluctuations of such photons. We assume here that decoupling happens instantaneous and at a fixed temperature T_*.

The photons propagate in the perturbed metric,

$$ds^2 = a^2(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

But photon propagation is conformally invariant, only the affine parameter λ of the photons depends on the scale factor. We can therefore forget about the factor a^2 and consider photon propagation the a perturbed Minkowski metric. We define the photon 4-velocity by

$$\left(n^\mu\right) = \begin{pmatrix} 1 + \delta n^0 \\ \mathbf{n} + \delta\mathbf{n} \end{pmatrix} \tag{26}$$

where $\mathbf{n}$ is a unit vector, $\mathbf{n}^2 = 1$. The geodesic equation, $\ddot{n}^\mu + \Gamma^\mu_{\alpha\beta} n^\alpha n^\beta = 0$ leads to the following first-order perturbation equation for δn^0:

$$\frac{d}{d\lambda}\delta n^0 = h_{\alpha 0,\beta} n^\alpha n^b - \frac{1}{2}\dot{h}_{\alpha\beta} n^\alpha n^b$$

To first order, we can replace n^μ by its unperturbed value on the r.h.s. Integrating we obtain:

$$\delta n^0 \Big|_i^f = \left[h_{00} + h_{0j} n^j\right]_i^f - \frac{1}{2}\int_i^f \dot{h}_{\alpha\beta} n^\alpha n^b d\lambda \tag{27}$$

The energy of a photon with 4-momentum p as seen by an observer moving with 4-velocity u is given by $(p \cdot u)$. The observer 4-velocity is $u^\mu \partial_\mu = a^{-1}[(1-\Psi)\partial_t + V^i\partial_i]$, where V^i denotes the peculiar velocity. The photon 4-momentum in the expanding universe is proportional to $p^\mu \propto a^{-2} n^\mu$, since the affine parameters are related by a factor a^{-2} (see [6] for details). To first order in the perturbation, we therefore have:

$$\frac{E_f}{E_i} = \frac{a_i}{a_f}\left(1 + [\Psi - \delta n^0 + V_i^b n^i]_i^f\right) \tag{28}$$

But we must be careful, the last scattering surface is not at constant scale factor a_i, but at constant temperature $T_{\rm dec} = T_i + \delta T_{\rm dec}$, hence a_i is also perturbed,

$$\frac{a_i}{a_f} = \frac{T_0 - \delta T_0}{T_{\rm dec} - \delta T_{\rm dec}} = \frac{T_0}{T_{\rm dec}}\left(1 - \frac{1}{4}\, D_{\rm s}^{(r)}\Big|_i^f\right) \tag{29}$$

Here T_0 and $T_{\rm dec}$ are the true physical temperatures today and at decoupling, which are related to the unperturbed background temperatures via $T_f = T_0 - \delta T_0$ and $T_i = T_{\rm dec} - \delta T_{\rm dec}$. For the second equal sign, in (29) we made use of the fact that

$$\frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho^{(r)}}{\rho^{(r)}} = \frac{1}{4} D_{\rm s}^{(r)}$$

Inserting our scalar and tensor perturbations, we obtain the following result:

$$\frac{E_f}{E_i} = \frac{T_0}{T_{\rm dec}}\left\{1 - \left[\frac{1}{4}D_{\rm s}^{(r)} - V_i^b n^i + \Psi\right]_i^f + \int_i^f (\dot\Psi + \dot\Phi + \dot H_{ij} n^i n^i)\,{\rm d}\lambda\right\} \tag{30}$$

In an experiment we consider photons coming in from different directions, $\mathbf{n}_1$ and $\mathbf{n}_2$, and we measure the difference in their energy. Neglecting the terms at $t_{\rm f} = t_0$, which apart from the dipole due to our motion with respect to the last scattering surface, $V_j(t_0)n^i$, are simple monopole terms that drop out in a difference measurement, we find:

$$\frac{\Delta T}{T} \equiv \frac{\Delta T(\mathbf{n}_1)}{T} - \frac{\Delta T(\mathbf{n}_2)}{T} = \frac{T_{\rm dec}}{T_0}\left(\frac{E_{\rm f}}{E_i}(\mathbf{n}_1) - \frac{E_{\rm f}}{E_i}(\mathbf{n}_2)\right)$$

with

$$\frac{\Delta T(\mathbf{n}_1)}{T} = \left[\frac{1}{4}D_{\rm s}^{(r)} - V_i^b n^i + \Psi\right](t_{\rm dec}, \mathbf{x}_{\rm dec}) + \int_{t_{\rm dec}}^{t_0} (\dot\Psi + \dot\Phi + \dot H_{ij} n^i n^i)(t, \mathbf{x}(t))\,{\rm d}t \tag{31}$$

Here $\mathbf{x}(t) = \mathbf{x}_0 - \mathbf{n}(t - t_0)$ denotes the background trajectory of the photon. This first-order perturbative expression corresponds to the 'Born approximation', i.e., we neglect the perturbation of the photon trajectory.

This is the CMB temperature fluctuation in the instant decoupling approximation first derived in [10]. It contains the Sachs–Wolfe term [11], the integrated Sachs–Wolfe term and the acoustic oscillations, but not Silk damping. We here discuss the former and leave Silk damping for the next section.

Let us determine the CMB power spectrum from this expression. We first define the correlation function,

$$\xi(\mathbf{n}\cdot\mathbf{n}') = \left\langle \frac{\Delta T(\mathbf{n})}{T}\frac{\Delta T(\mathbf{n}')}{T}\right\rangle \tag{32}$$

where $\langle\cdots\rangle$ denotes a statistical expectation value. The quantum perturbations generated during inflation become classical statistical fluctuations at the end of inflation, which are usually Gaussian. For an explanation of this 'cosmological squeezing' process, see, e.g., [12]. Because of statistical isotropy, this function can only depend on the scalar product, $\mathbf{n}\cdot\mathbf{n}'$. Since ξ is a function on the interval $[-1, 1]$ we can expand it in Legendre polynomials. Denoting $\mathbf{n}\cdot\mathbf{n}' = \mu$ we set:

$$\xi(\mu) = \frac{1}{4\pi}\sum_\ell (2\ell + 1) C_\ell P_\ell(\mu) \tag{33}$$

Expanding the temperature fluctuations in spherical harmonics, one finds that for

$$\frac{\Delta T(\mathbf{n})}{T} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\mathbf{n}) \tag{34}$$

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell \tag{35}$$

Again, the Kronecker deltas are a consequence of statistical isotropy. The C_ℓs are the CMB power spectrum, i.e., the expectation values of the square amplitude of its multipole coefficients. The quantity $\mathcal{D}_\ell = (2\pi)^{-1}\ell(\ell+1)C_\ell$ is the square of the fluctuation amplitude on the angular scale $\theta_\ell \simeq \pi/\ell$.

Let us compute the CMB power spectrum from the expression (31) for the temperature fluctuation. Since scalar and tensor fluctuations are uncorrelated (also this is simply a consequence of statistical isotropy), we can calculate their contribution separately. We start with the scalars. For given initial fluctuations $\zeta(\mathbf{k})$ after inflation, the expressions for each one of the required perturbation variables in Fourier space is given by the transfer function, see Eq. (25).

We first insert its Fourier representation for each variable appearing in (31). For example,

$$V(\mathbf{x}(t),t)=\int\frac{\mathrm{d}^3k}{k}T_V(k,t)\zeta(\mathbf{k})\exp(\mathrm{i}\mathbf{kn}(t_0-t))$$

where we have set $\mathbf{x}_0=0$. We then use that

$$e^{\mathrm{i}\mathbf{kn}r}=\sum_\ell(2\ell+1)i^\ell j_\ell(kr)P_\ell(\hat{\mathbf{kn}}) \tag{36}$$

where j_ℓ denotes the spherical Bessel function [13] or order ℓ. Using also the addition theorem of spherical harmonics, we find

$$C_\ell^{(\mathrm{s})}=\frac{2}{\pi}\int\frac{\mathrm{d}k}{k}P_{\mathrm{s}}(k)|\Delta_\ell^{(\mathrm{s})}(k)|^2,\quad\text{where} \tag{37}$$

$$\Delta_\ell^{(\mathrm{s})}(k)=\left(\frac{T_D^{(\mathrm{r})}}{4}+T_\Psi\right)j_\ell(k\Delta t_{\mathrm{dec}})+iT_V j'_\ell(k\Delta t_{\mathrm{dec}})+\int_{t_{\mathrm{dec}}}^{t_0}\mathrm{d}t(\dot T_\Phi+\dot T_\Psi)j_\ell(k\Delta t) \tag{38}$$

Here $\Delta t=t_0-t$ and in the first and second lines, the transfer functions have to be evaluated at t_{dec}, while in the integral they are evaluated at t.

When we take into account the finite thickness of the decoupling surface as well as polarisation; this will simply change the expression for $\Delta_\ell^{(\mathrm{s})}(k)$, but not Eq. (37). The $\Delta_\ell^{(\mathrm{s})}(k)$ are called the transfer functions for the scalar CMB power spectrum.

Note that the transfer functions depend only on the background cosmology and all the physics related to inflation is in the curvature power spectrum P_{s}.

In the same manner, we find the tensor contribution to the power spectrum, see [6] for a detailed derivation,

$$C_\ell^{(t)}=\frac{(\ell+2)!}{(\ell-2)!}\frac{1}{2\pi}\int\frac{\mathrm{d}k}{k}P_t(k)|\Delta_\ell^{(t)}(k)|^2 \tag{39}$$

where

$$\Delta_\ell^{(t)}(k)=\int_{t_{\mathrm{dec}}}^{t_0}\mathrm{d}t\dot T_H\frac{j_\ell(k\Delta t)}{(k\Delta t)^2} \tag{40}$$

Let us estimate the result on large scales, $kt_{\mathrm{dec}}\ll 1$, where Silk damping and polarisation on the last scattering surface are irrelevant. For this, we use the fact that $j_\ell(x)$ peaks at $x\simeq\ell$, so that the multipole ℓ is dominated by the fluctuations at scale $k\simeq\ell/\Delta t_{\mathrm{dec}}\simeq\ell/t_0$. On super-Hubble scales, we can approximate $T_\Psi\simeq 3/5$ and $T_D/T_\Psi=-2$. Using also the fact that, at decoupling, the Universe is already matter dominated (this is not a very good approximation...) and the fact that, for adiabatic perturbations, $D_{\mathrm{s}}^{(r)}=4D_{\mathrm{s}}^{(m)}/3$, we obtain $T_D^{(r)}/(4T_\Psi)=-2/3$. The transfer function T_V is suppressed by a factor $k/\mathcal{H}$. Neglecting also the integrated term, we obtain the Sachs–Wolfe result [11],

$$\Delta_\ell^{(\mathrm{s})}(k)\simeq\frac{1}{3}T_\Psi(k)j_\ell(kt_0) \tag{41}$$

If P_{s} is a simple power law, we can integrate (37) in this approximation and find

$$C_\ell^{(\mathrm{SW})}=\frac{A_{\mathrm{s}}}{25}\frac{\Gamma(3-n_{\mathrm{s}})\Gamma(\ell-\frac{1}{2}+\frac{n_{\mathrm{s}}}{2})}{2^{3-n_{\mathrm{s}}}\Gamma^2(2-\frac{n_{\mathrm{s}}}{2})\Gamma(\ell+\frac{5}{2}-\frac{n_{\mathrm{s}}}{2})} \tag{42}$$

where Γ denotes the Gamma-function. For a scale-invariant spectrum, $n_{\mathrm{s}}=1$ this yields:

$$\ell(\ell+1)C_\ell^{(\mathrm{SW})}=\frac{A_{\mathrm{s}}}{25\pi} \tag{43}$$

To obtain an analytic estimate for the tensor spectrum, we neglect the time dependence of Δt and approximate $\Delta t\sim t_0$. With this the integral, (40) becomes simply:

$$-T_H(k,t_{\mathrm{dec}})j_\ell(kt_0)/(kt_0)^2$$

On super Hubble scales, the transfer function for a gravitational wave mode is $1/(2\sqrt{2})$ (the factor $1/2$ coming from our normalization of H and the factor $1/\sqrt{2}$ from the fact that T_H denotes the transfer function for one mode). Inserting this in (39), we can perform the integral and obtain:

$$C_\ell^{(t)}=\frac{A_t}{16}\frac{(\ell+2)!}{(\ell-2)!}\frac{\Gamma(6-n_t)\Gamma(\ell-2+\frac{n_t}{2})}{2^{7-n_{\mathrm{s}}}\Gamma^2(\frac{7}{2}-\frac{n_t}{2})\Gamma(\ell+4-\frac{n_t}{2})} \tag{44}$$

For a scale-invariant spectrum, $n_t = 1$, this yields:

$$\ell(\ell+1)C_\ell^{(t)} = \frac{A_t}{60\pi}\frac{\ell(\ell+1)}{(\ell+3)(\ell-2)} \tag{45}$$

For the ratio, we obtain:

$$\frac{C_\ell^{(t)}}{C_\ell^{(\mathrm{SW})}} \simeq \frac{5A_t}{12A_\mathrm{s}} \simeq \frac{5}{12}r \tag{46}$$

This result is of course very rough, but it captures the fact that for $r = 1$ the tensor signal is about a factor 2–3 times smaller than the scalar signal on large scales.

Once a tensor fluctuation of wavenumber k 'enters the horizon', i.e. $k/\mathcal{H} = 1$, it starts decaying. Therefore the tensor CMB anisotropy spectrum starts decaying around $\ell \sim 60$, from the loss of power in modes with $kt_\mathrm{dec} \simeq k/\mathcal{H}_\mathrm{dec} > 1$.

During the radiation, dominated Universe scalar perturbations obey an acoustic wave equation. Hence $D_\mathrm{s}^{(r)}$ and V oscillate at constant amplitude. The initial conditions from inflation are such that each mode starts out as $D_\mathrm{s}^{(r)}(\mathbf{k}, t) = A(\mathbf{k})\cos(c_\mathrm{s}kt)$, where $c_\mathrm{s} = 1/\sqrt{3}$ is the sound speed of radiation. At the time of decoupling, $D_\mathrm{s}^{(r)}$ therefore exhibits a series of acoustic peaks, the first at $k_1 = \pi/c_\mathrm{s}t_\mathrm{dec} \simeq \sqrt{3}\pi\mathcal{H}_\mathrm{dec}/2$. Since for $k/\mathcal{H}_\mathrm{dec} > 1$ the transfer function $\Delta_\ell(k)$ is dominated by $D_\mathrm{s}^{(r)}$ and V; these peaks are imprinted in the CMB, see Fig. 2. The first peak is at $\ell_1 \simeq \sqrt{3}\pi\mathcal{H}_\mathrm{dec}/\mathcal{H}_0 \sim 200$.

Note that for the existence of these peaks, it is very important that all modes with wavenumber k are in phase, i.e. are generated with the same amplitude and the same time derivative during inflation, so that they are all really $\propto \cos(c_\mathrm{s}kt_\mathrm{dec})$, and not $\cos(c_\mathrm{s}kt_\mathrm{dec} + \alpha)$, i.e. that these acoustic fluctuations are coherent. This is not so if perturbations are generated by seeds, e.g., cosmic strings or other causal scaling seeds. In this case, the fluctuations of a given wavenumber are typically generated when this wavenumber enters the horizon, but for different wave vectors $\mathbf{k}$ in general with different phases. In this case one does not see a clear patterns of acoustic peaks, but the peaks are smeared out into one broad hump by decoherence [14].

The presence of the acoustic peaks in the CMB power spectrum proves that these fluctuations are not generated by some stochastic seeds at horizon crossing, but that they have been laid down early in the Universe when they were still super Hubble. Hence it proves that some mechanism like inflation, which generates coherent fluctuations on super-Hubble scales, was at work.

3.3. The Boltzmann hierarchy: temperature anisotropies and polarisation

In this section, we want to consider the decoupling process in somewhat more detail and present more precise results for the CMB power spectra. We discuss the decoupling process, but shall not derive the Boltzmann hierarchy. A detailed derivation can be found in [6].

In the previous section, we approximated the last scattering surface as infinitely thin. This is of course not realistic. During recombination, the mean free path of the photons gradually grows until it becomes larger than the Hubble scale and the photons become free. During this process, the photons go through a phase of imperfect coupling where on small scales photons can move out of overdensities into underdensities, hence small fluctuations are damped by photon diffusion. This process is called Silk damping [15]. It seriously affects scales smaller than a few h^{-1}Mpc (comoving), which corresponds roughly to the third peak in the CMB spectrum. But already the first peak would be about 10% higher without Silk damping.

The only scattering process relevant before decoupling is non-relativistic, elastic Thomson scattering. The photon energies are not affected by Thomson scattering, photons are simply deflected. But Thomson scattering depends on polarisation: the polarisation amplitude for photons with linear polarisation in the scattering plane is suppressed by a factor $\cos\theta$, where θ denotes the scattering angle [16]. For an electron density given by n_e, depending on whether the polarisation is parallel or orthogonal to the scattering plane, one finds the following scattering rates:

$$\tau_{sp}^{-1} = \frac{3}{8\pi}n_\mathrm{e}\sigma_T\cos^2\theta \quad \text{parallel}$$
$$\tau_{so}^{-1} = \frac{3}{8\pi}n_\mathrm{e}\sigma_T \qquad\quad \text{orthogonal}$$

If there is a quadrupole anisotropy of incoming photons onto an electron, this leads to a net polarisation of outgoing photons. The most extreme case, with a scattering angle of $\pi/2$, is shown in Fig. 1. On average, the final polarisation is on the order of a few percent of the anisotropy.

To take these effects correctly into account, one has to solve the coupled Boltzmann equation for the temperature fluctuation, $\Delta T/T = \Delta I/4I$ and the polarisation that can be cast in terms of the dimensionless Stokes parameters $\mathcal{Q} = Q/4I$ and $\mathcal{U} = U/4I$. Thomson scattering does not generate circular polarisation, so that it is consistent to set the Stokes parameter $V = 0$. $\mathcal{Q}$ and $\mathcal{U}$ depend on the chosen basis and it is more convenient to exploit the fact that $\mathcal{Q} \pm i\mathcal{U}$ are helicity ± 2 objects in the sky, which we can expand in spin-weighted spherical harmonics,

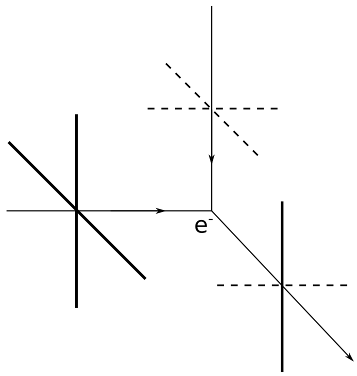

Fig. 1. Incoming photons scattered at $\pi/2$ are fully polarized. If the photon density in the vertical direction is smaller that the one in the horizontal direction; this leads to a net polarisation of the outgoing photons here in forward direction.

$$(\mathcal{Q} \pm i\mathcal{U})(\mathbf{n}) = \sum (e_{\ell m} \pm i b_{\ell m})_{\pm 2} Y_{\ell m}(\mathbf{n}) \tag{47}$$

Here ${}_{\pm 2}Y_{\ell m}(\mathbf{n})$ are the spin-weighted spherical harmonics of helicity ± 2, see [6]. Note that under parity, $\mathbf{n} \to -\mathbf{n}$, ${}_{\pm 2}Y_{\ell m}(\mathbf{n}) \to {}_{\mp 2}Y_{\ell m}(\mathbf{n})$, so that the coefficients $e_{\ell m}$ are parity even while the $b_{\ell m}$ are parity odd. One can actually show that E-type polarisation (i.e. polarisation with $b_{\ell m} \equiv 0$) is described by a gradient field, while B-type polarisation (i.e. polarisation with $e_{\ell m} \equiv 0$) is a pure curl. This makes B-polarisation so interesting. It cannot come from linear scalar perturbation, but is a smoking gun of gravitational waves.

Like for the temperature we can now define the polarisation spectra

$$C_\ell^{(EE)} = \langle |e_{\ell m}|^2 \rangle \tag{48}$$

$$C_\ell^{(BB)} = \langle |b_{\ell m}|^2 \rangle \tag{49}$$

$$C_\ell^{(TE)} = \langle a^*_{\ell m} e_{\ell m} \rangle \tag{50}$$

$C_\ell^{(TE)}$ is the correlation between E-type polarisation and the temperature anisotropy. For a parity invariant Universe, there are no E–B or T–B correlations.

The Boltzmann hierarchy yields coupled linear differential equations for the evolution of the transfer functions $X_\ell(k)$ such that

$$C_\ell^{(XY)} = \frac{2}{\pi} \int \frac{\mathrm{d}k}{k} P_{\mathrm{in}}(k) X^*_\ell(k) Y_\ell(k) \tag{51}$$

Here P_{in} is P_t for tensor perturbations and P_s for scalar perturbations and X, Y denotes T, E or B. This method to compute the CMB perturbation and polarisation spectra has been first derived in [17,18] and is explained in detail in [6]. Here we do not write down the somewhat cumbersome Boltzmann hierarchy equations and their integral solutions.

The Boltzmann hierarchy actually allows for integral solutions similar to Eqs. (38) and (40) in terms of combinations of spherical Bessel functions, which contain only the lowest moments of the hierarchy, the baryon perturbation variables and the gravitational field. One can therefore solve the hierarchy up to $\ell \sim 10$ coupled with the Einstein equations and the fluid equations for baryons and dark matter, and then use these solutions to determine all the higher multipoles via the integral solutions. This method has been introduced in [19] and has become the standard for the present fast CMB codes like CAMB [20] and CLASS [21].

3.4. Observations

It was clear for a long time that the CMB should have fluctuations if the gravitational instability picture of structure formation is to be correct. Therefore, workers in the field, like David Wilkinson started searching for these fluctuations right after the discovery [22] of the CMB in 1965. It took however until 1992 for the first positive fluctuations to be discovered by the COBE satellite team. At this time there was a clear indication of a Sachs–Wolfe plateau at large scale, $\ell < 20$ in good agreement with inflation [5]. However, such a plateau was also predicted by fluctuations from topological defects and so it took a couple more years to detect the very coherent peak structure of the CMB, which is in perfect agreement with inflation, but at odds with topological defects and other causal scaling seed models, to convince also the ‘heretics’ (like the author of this paper) that a process similar to inflation must have generated the initial fluctuations. The latest CMB temperature and polarisation spectra from the Planck satellite are shown in Figs. 2 to 4.

The quantity $\mathcal{D}_\ell$ plotted in the figures is

$$\mathcal{D}_\ell = T_0^2 \ell(\ell+1) C_\ell/(2\pi)$$

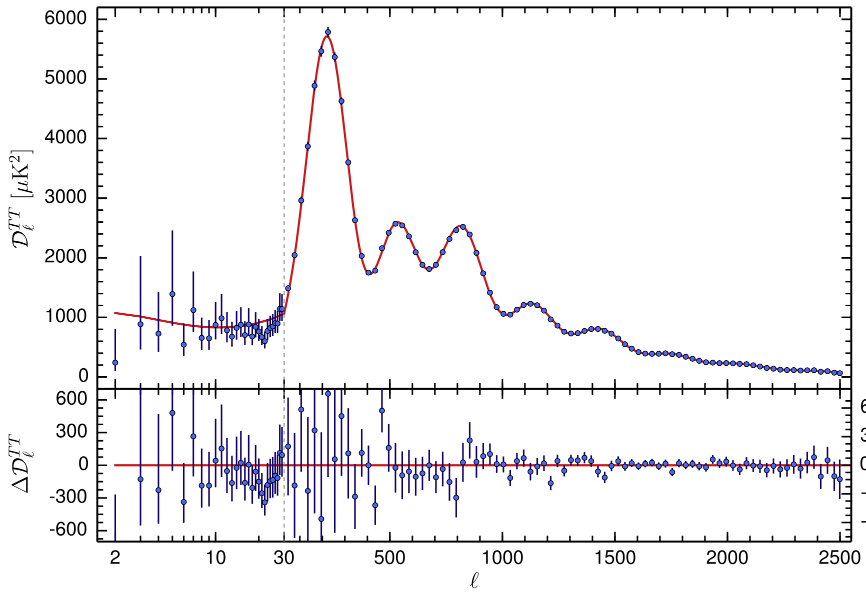

Fig. 2. (Color online.) The temperature anisotropy spectrum measured by the Planck satellite. Note the different scaling of the ℓ-axis from 2 to 29 and from 30 to 2500. This renders the Sachs–Wolfe plateau at low ℓ visible. Figure from [23].

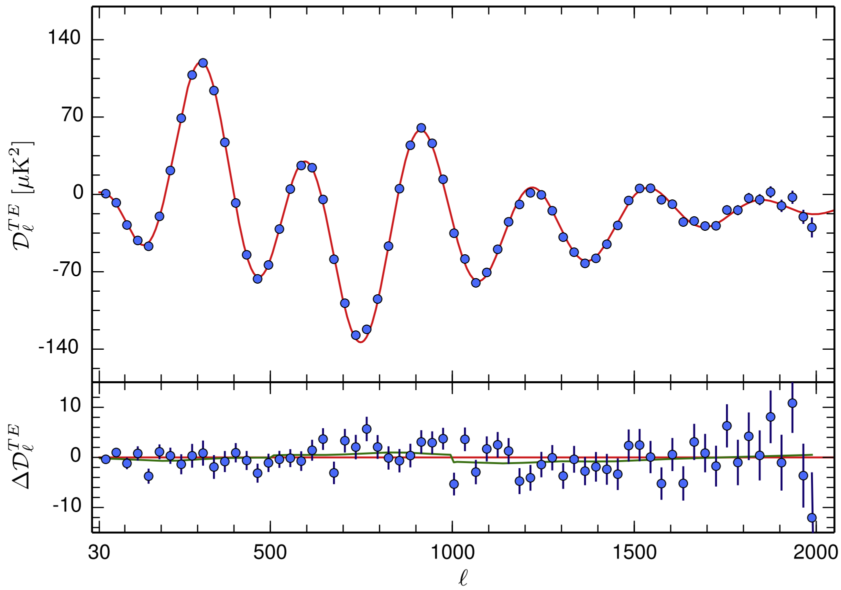

Fig. 3. (Color online.) The temperature–polarisation cross-correlation spectrum measured by the Planck satellite. Figure from [23].

The small strip at the bottom of the figure indicates the difference to the old 2013 results.

The acoustic peaks are very well visible. Interestingly, the T–E correlation has a first negative peak at $\ell \simeq 100$. There is therefore T–E anti-correlation already at $\ell < 100$, which comes from scales that are larger than the Hubble scale at decoupling. Since polarisation has been laid down at the time of decoupling, this proves the existence of correlations on super-Hubble scales. It has been postulated in [24] and it has been shown numerically in [25] that such correlations cannot be generated by causal scaling seeds like, e.g., topological defects.

The observed fluctuations are consistent with Gaussianity; for example, a 3-point function of the local type is constrained by $f_{\rm NL} = 0.8 \pm 5$ at the 68% confidence level and is well compatible with zero. Also other types of non-Gaussianities that have been investigated are significantly constrained and are compatible with zero [26].

The CMB measurements can be used to estimate the cosmological parameters and the inflationary parameters by searching for a best-fit model. This is usually done with a Markov chain Monte Carlo routine, see [27]. For the inflationary parameters the findings of the Planck Collaboration [23,26] are given in Table 1.

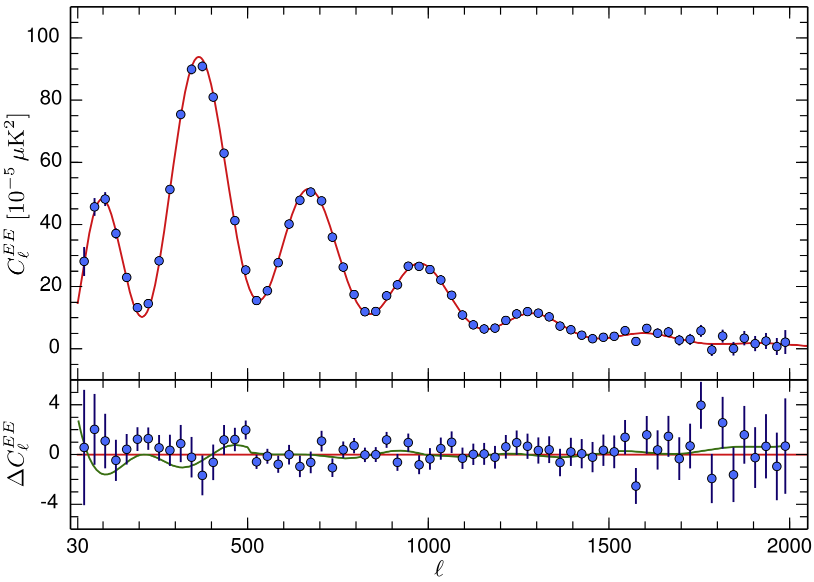

Fig. 4. (Color online.) The polarisation spectrum measured by the Planck satellite. (The vertical axis should indicate $\mathcal{D}_\ell$.) Figure from [23].

Table 1
The inflationary parameters from present CMB data. The values of the amplitude and the spectral index of scalar perturbations have been estimated by setting $r = 0$. Allowing for a positive r somewhat increases their error bars.

$\ln(10^{10} A_s)$	n_s	r	$\lvert f_{NL}\rvert$
3.094 ± 0.034	0.9645 ± 0.0049	< 0.11	< 6

The fluctuations of the CMB are also affected by lensing from foreground density perturbations. This second-order effect is relevant at the present accuracy. It widens and damps somewhat the acoustic peaks, as described in detail in [28,6]. The analysis of the Planck data has even allowed us to determine the lensing potential [29] and to use it to lift degeneracies in the determination of cosmological parameters [23].

The next exciting discovery we are waiting for is primordial B-polarisation. Unfortunately, even if there are no gravitational waves, the lensing of foreground structure affects polarisation and rotates some of the scalar E-polarisation into a B-mode [28]. This makes it more difficult, but not impossible to disentangle the gravitational wave signal. The B-modes from lensing of scalar perturbations have recently been detected [30]. Last spring, a collaboration had also announced a positive value of $r \simeq 0.2$, which however turned out to be most probably due to an underestimation of the contribution from dust to B-polarisation [31].

A positive value of r in the regime $0.001 < r < 0.1$ would be extremely interesting. It would point towards and inflationary energy scale of about 10^{16} GeV, a scale where effects of quantum gravity, e.g., from string theory, might be relevant. Therefore, the CMB is extremely interesting for the connection of cosmology and very high-energy physics.

4. Conclusion

In this paper I have argued that present observations of CMB anisotropies and polarisation are in very good agreement with slow-roll inflation with

$$1 - n_s = 2(3\epsilon_1 + \epsilon_2) \simeq 0.04, \quad \text{and} \tag{52}$$

$$r = 16\epsilon_1 < 0.1 \tag{53}$$

The main features in favor of inflation are

– the nearly scale-invariant spectrum,
– Gaussian fluctuations,
– the coherence of the acoustic peaks,
– the anti-correlation of E-polarisation and temperature anisotropy at $\ell < 100$,
– and the fact that there is no evidence of vector perturbation.

Causal scaling seeds typically only reproduce the first of these features. We have not discussed vector perturbations at all in this paper but they are typically very relevant, e.g. for topological defects. There exist also 'exotic' inflationary models that generate vector perturbations, but in a radiation-dominated Universe, they then decay.

Are there alternatives to inflation? The problem of this question is that it is not well posed. If we define inflation to be a phase in the early Universe where correlations are generated on scales which are larger than the Hubble scale at decoupling, the answer is definitively no, since we have observed such correlations. However, if we define inflation somewhat more narrowly as a phase where the expansion of the Universe is driven by the potential energy of a scalar field, the inflation, there the answer is positive. There exist other possibilities, as, e.g., bouncing Universes, see the contribution by Lilley and Peter to this volume [32] or pre-big bang cosmology, which is inspired by string theory [33–35]. The main difference of these models of the early Universe, which typically also solve the horizon and flatness problem, and which also lay down coherent fluctuations on large scales, with scalar field inflation lies in their gravitational wave spectrum. In these models, the Hubble scale is not nearly constant during the pre-big bang phase, but it is actually growing. Therefore, they typically lead to a blue spectrum of gravitational waves that peaks at very small scales and is negligible on cosmological scales.

Discovering gravitational waves in the CMB would rule out all the alternative models presently on the market. But we have to keep in mind that data or observations cannot 'prove' a physical theory. It is the theoretical consistency, stringency and elegance that finally convince us. On this level, we must admit that present inflationary models are either simple toy models not connected to the standard model, or contrived constructions which are far from elegant.

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