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Inflation in the standard cosmological model

L'inflation dans le cadre du modèle cosmologique standard

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ABSTRACT

The inflationary paradigm is now part of the standard cosmological model as a description of its primordial phase. While its original motivation was to solve the standard problems of the hot big bang model, it was soon understood that it offers a natural theory for the origin of the large-scale structure of the universe. Most models rely on a slow-rolling scalar field and enjoy very generic predictions. Besides, all the matter of the universe is produced by the decay of the inflaton field at the end of inflation during a phase of reheating. These predictions can be (and are) tested from their imprint of the large-scale structure and in particular the cosmic microwave background. Inflation stands as a window in physics where both general relativity and quantum field theory are at work and which can be observationally studied. It connects cosmology with high-energy physics. Today most models are constructed within extensions of the standard model, such as supersymmetry or string theory. Inflation also disrupts our vision of the universe, in particular with the ideas of chaotic inflation and eternal inflation that tend to promote the image of a very inhomogeneous universe with fractal structure on a large scale. This idea is also at the heart of further speculations, such as the multiverse. This introduction summarizes the connections between inflation and the hot big bang model and details the basics of its dynamics and predictions.

RÉSUMÉ

Le paradigme de l'inflation fait aujourd'hui partie intégrante du modèle cosmologique standard en tant que modèle de la phase primordiale de son évolution. Bien que sa formulation originelle ait été motivée par la résolution de problèmes du modèle du big bang chaud, on réalisa rapidement que l'inflation offrait un méchanisme naturel pour l'origine des grandes structures de l'univers. La plupart des modèles reposent sur la dynamique d'un champ scalaire en roulement lent et fournissent des prédictions génériques et robustes. De plus, toute la matière de notre univers serait produite par la désintégration de l'inflaton à la fin de l'inflation dans une phase de réchauffement. Ces prédictions peuvent être (et sont) testées par l'étude de leurs signatures sur les grandes structures et, en particulier, sur le fond diffus cosmologique. L'inflation est une fenêtre sur un domaine où relativité générale et théorie quantique des champs sont conjointement à l'œuvre et qui peut être étudiée observationnellement. Elle connecte la cosmologie

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à la physique des hautes énergies. Aujourd'hui, la plupart des modèles sont construits dans des extensions du modèle standard, comme la supersymétrie ou la théorie des cordes. L'inflation bouleverse aussi notre représentation de l'univers, en particulier par les idées d'inflation chaotique et d'inflation éternelle, qui tendent à révéler un univers très hétérogène à grande échelle, avec une structure fractale. Cette introduction résume les liens entre inflation et modèle du big bang chaud et détaille ses propriétés dynamiques et ses prédictions.

1. Modern cosmology and the inflationary paradigm

A cosmological model is a mathematical representation of our universe that is based on the laws of nature that have been validated locally in our Solar system and on their extrapolations (see Ref. [1] for a detailed discussion). It thus seats at the crossroad between theoretical physics and astronomy. While a topic of interest for many centuries, we can safely state [2] that scientific cosmology was born with Albert Einstein's general relativity a century ago, a theory of gravitation that made the geometry of spacetime dynamical physical fields that need to be determined by solving equations known as Einstein field equations. Each solution of the theory is a spacetime, a universe and the question arose to determine the solutions [3] which are good approximations of the geometry of our universe.

This led to the construction of a standard cosmological model, mostly through four phases.

1.1. The basis of the standard cosmological model

The first era of relativistic cosmology started in 1917 with the seminal paper by Einstein [3], in which he constructed, at the expense of the introduction of a cosmological constant, a static solution to its equations in which space enjoys the topology of a three-sphere. This paved the way to the derivations of exact solutions to the Einstein equations that offer possible world models. Alexandr Friedmann and independently Georges Lemaître [4] developed the first dynamical models [5], hence discovering the cosmic expansion as a prediction of the equations of general relativity. An important step was provided by Lemaître, who connected the theoretical prediction of an expanding universe with an observations by linking it to the redshifts of electromagnetic spectra, and thus of observed galaxies. This was later confirmed by the observations by Edwin Hubble [6], whose *Hubble law*, relating the recession velocity of a galaxy to its distance from us, confirms the cosmological expansion. The law of expansion derives from the Einstein equations and thus relates the cosmic expansion rate, H, to the matter content of the universe, offering the possibility to "weight the universe". This solution to a spatially homogeneous and isotropic expanding spacetime, referred to as Friedmann–Lemaître (FL) universe, serves as the reference background spacetime for most later developments of cosmology. It relies on the so-called Copernican principle stating that we do not seat in a particular place in the universe, and introduced by Einstein [3]. While difficult to test, since we observe the universe from a special spacetime position, it has been shown in the past years that this hypothesis holds on the scale of the observable universe [7].

In a second era, starting in 1948, the properties of atomic and nuclear processes in an expanding universe were investigated (see, e.g., Ref. [8] for an early textbook). This allowed Ralph Alpher, Hans Bethe and George Gamow [9] to predict the existence of a cosmic microwave background (CMB) radiation (and to estimate its temperature), and to understand the synthesis of light nuclei, the big bang nucleosynthesis (BBN), in the early universe. Both have led to theoretical developments compared successfully to observation. It was understood that the universe is filled with a thermal bath with a black-body spectrum, the temperature of which decreases with the expansion of the universe. The universe cools down and has a thermal history, and more importantly it was concluded that it emerges from a hot and dense phase at thermal equilibrium (see, e.g., Ref. [10] for the details). This model has however several drawbacks, such as the fact that the universe is spatially extremely close to Euclidean (*flatness problem*), the fact that it has an *initial spacelike singularity* (known as big bang), and the fact that thermal equilibrium, homogeneity and isotropy are set as initial conditions and not explained (*horizon problem*). It is also too idealized, since it describes no structure, i.e. does not take into account the inhomogeneities of the matter, which is obviously distributed in galaxies, clusters and voids. The resolution of the naturalness of the initial conditions was solved by the postulate [11] of the existence of a primordial accelerated expansion phase, called *inflation*.

The third and fourth periods were triggered by an analysis of the growth of the density inhomogeneities by Lifshitz [12], opening the understanding of the evolution of the large-scale structure of the universe, that is of the distribution of the galaxies in clusters, filaments and voids. Technically, it opens the way to the theory of cosmological perturbations [13–15], in which one considers the FL spacetime as a background one, the geometry and matter content of which are perturbed. The evolution of these perturbations can be derived from the Einstein equations. For the mechanism studied by Evgeny Lifshitz to be efficient, one needed initial density fluctuations large enough so that their growth in an expanding universe could lead to a non-linear structure at least today. This motivated the question of the understanding of the origin and nature (amplitude, statistical distribution) of the initial density fluctuations, which turned out to be the second success of the inflationary theory, which can be considered as the onset as the third era, the one of *primordial cosmology*. From a theory point of view, the origin of the density fluctuation lies into the quantum properties of matter [16]. From an observational

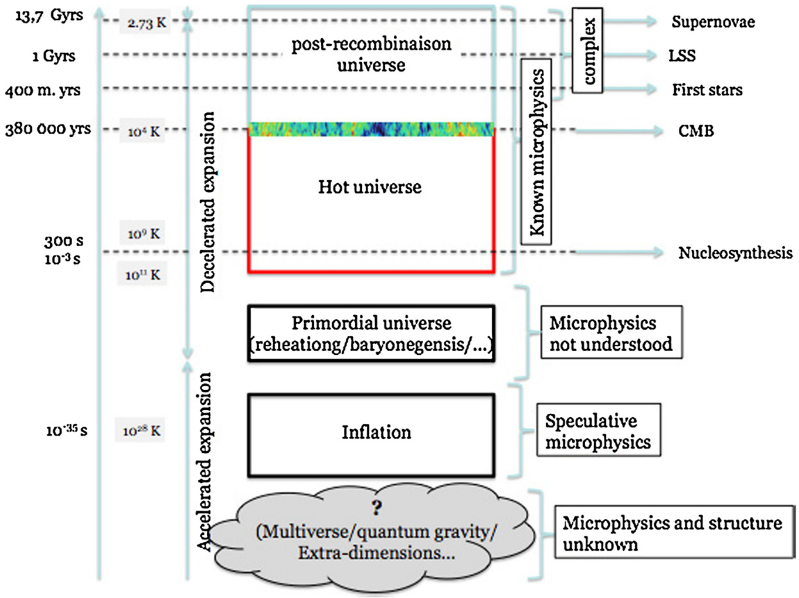

Fig. 1. (Color online.) The standard history of our universe. The local universe provides observations on phenomena from big bang nucleosynthesis to today, spanning a range between 10^{-3} s and 13.7 Gyr. One major transition is the equality that separates the universe in two eras: a matter-dominated era during which large-scale structure can grow, and a radiation-dominated era, during which the radiation pressure plays a central role, in particular in allowing for acoustic waves. Equality is followed by recombination, which can be observed through the CMB anisotropies. For temperatures larger that 10^{11} K, the microphysics is less understood and more speculative. Many phenomena such as baryogenesis and reheating still need to be understood in better details. The whole history of our universe appears as a parenthesis of decelerated expansion, during which complex structures can form, in between two periods of accelerated expansion, which do not allow for this complex structures to either appear or even survive.

point of view, the predictions of inflation can be related to the distribution of the large-scale structure of the universe, in particular in the anisotropy of the temperature of the cosmic microwave background [17]. This makes the study of inflation an extremely interesting field, since it requires to deal with both general relativity and quantum mechanics, and has some observational imprints. It could thus be a window to a better understanding of quantum gravity.

The observational developments and the progresses in the theoretical of the understanding of the growth of the large-scale structure led to the conclusion that:

- there may exist a fairly substantial amount of non-relativistic dark matter, or cold dark matter (hence the acronym CDM);
- there shall exist a non-vanishing cosmological constant (Λ).

This led to the formulation of the ΛCDM model [18] by Jeremy Ostriker and Paul Steinhardt in 1995. The community was reluctant to adopt this model until the results of the analysis of the Hubble diagram of type-Ia supernovae in 1999 [19]. This ΛCDM model is in agreement with all the existing observations of the large survey (galaxy catalogs, CMB, weak lensing, Hubble diagram, etc.),and its parameters are measured with increasing accuracy. This has opened the era of *observational cosmology* with the open question of the physical nature of the dark sector.

The main features of this model are summarized in Fig. 1 and the model gives a global understanding of the evolution of the universe and its structures from 10^{-3} s after the big bang to today (13.7 Gyr). It is important to realize that the microphysics on which this model relies is non-speculative since it span an energy range up to 100 MeV, well tested in the laboratory.

This short summary shows that inflation is now a cornerstone of the standard cosmological model and emphasizes the two roles played by inflation in the development and architecture of the standard cosmological model,

1. it was postulated in order to explain the required fine-tuning of the initial conditions of the hot big bang model,
2. it provides a mechanism for the origin of the large-scale structure,
3. it gives a new and unexpected vision of the universe on a large scale,
4. it connects, in principle [20], cosmology to high-energy physics.

1.2. The inflationary paradigm

Inflation is thus defined as a primordial phase of accelerated expansion, which has to last long enough for the standard problems of the hot big bang model to be solved.

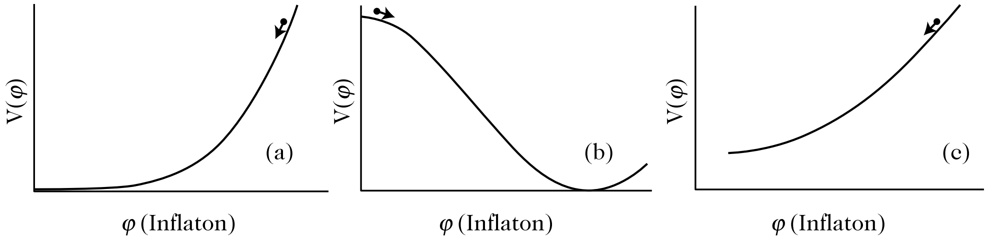

Fig. 2. The model of old inflation (left) is based on a first-order phase transition from a local to a true minimum of the potential. The false vacuum is metastable and the field can tunnel to the true vacuum. In the models of new inflation (right), a scalar field slowly relaxes towards its vacuum. From Ref. [10].

As explained above, it was initially proposed as a tentative solution to the problems of the hot big bang model. In particular, it provides a simple explanation for the homogeneity and flatness of our universe.

Before the first models of inflation were proposed, precursor works appeared as early as 1965 by Erast Gliner [21], who postulated a phase of exponential expansion. In 1978, François Englert, Robert Brout and Egard Gunzig [22], in an attempt to resolve the primordial singularity problem and to introduce the particles and the entropy contained in the universe, proposed a 'fireball' hypothesis, whereby the universe itself would appear through a quantum effect in a state of negative pressure subject to a phase of exponential expansion. Alecei Starobinsky [23], in 1979, used quantum-gravity ideas to formulate the first semi-realistic rigorous model of an inflation era, although he did not aim to solve the cosmological problems. A simpler model, with transparent physical motivations, was then proposed by Alan Guth [11] in 1981. This model, now called 'old inflation', was the first to use inflation as a means of solving cosmological problems. It was soon followed by Andrei Linde's 'new inflation' proposal [24].

From the Einstein equations, an accelerated expansion requires that the matter dominating the dynamics of the universe satisfies a relation between its energy density and pressure, $\rho + 3P < 0$. A solution to implement this condition is to use a scalar field φ. As long as it is slow rolling, it satisfies $P \sim -\rho$, hence leading to acceleration, while if it is fast rolling, it satisfies $P \sim +\rho$. As long as this scalar field is a minimally coupled canonical field, the only freedom is the choice of its potential $V(\varphi)$.

Early models, known as *old inflation*, rely on a first-order phase-transition mechanism [25] (see Fig. 2). A scalar field is trapped in a local minimum of its potential, thus imposing a constant energy density, equivalent to the contribution of a cosmological constant. As long as the field remains in this configuration, the evolution of the universe is exponential and the universe can be described by a de Sitter spacetime. This configuration is metastable and the field can tunnel to its global minimum, $V(\varphi_f) = 0$, hence creating bubbles of true vacuum, which correspond to a non-inflationary universe. In models of *new inflation* [24], the scalar field exits its false vacuum by slowly rolling towards its true vacuum (see Fig. 2). These models can only work if the potential has a very flat plateau around $\varphi = 0$, which is artificial. In most of its versions, the inflaton cannot be in thermal equilibrium with other matter fields. The theory of cosmological phase transitions then does not apply and no satisfying realization of this model has been proposed, so that it was progressively abandoned. The first models of inflation (see Ref. [26] for a review) were actually only incomplete modifications of the big bang theory as they assumed that the universe was in a state of thermal equilibrium, homogeneous on large enough scales before the period of inflation. This problem was resolved by Linde with the proposition of chaotic inflation [27]. In this model, inflation can start from a Planckian density even if the universe is not in equilibrium. A new picture of the universe then appears. The homogeneity and isotropy of our observable universe would be only local properties, while the universe is very inhomogeneous on very large scales, with a fractal-like structure.

Following the study of quantum effects near black-holes and in de Sitter spacetimes, quantum effects were investigated during inflation (see Ref. [28] for a historical prospect). Two approaches were followed. The first one, originating in the 1960s [29], uses the static form of the de Sitter metric so that an observer at the origin would detect thermal radiation from $R = 1/H$ with a temperature $T = H/2\pi$, which corresponds to vacuum polarization of the de Sitter geometry. It is often used in the superstring community, in particular in the context of holography and the thermodynamics associated with horizons and the so-called "hot tin can" picture [30]. The second approach is based on the quantization of a scalar field in a time-dependent background described by an (almost) de Sitter spacetime. This led to the formulation by Viatcheslav Mukhanov and Gennady Chibisov of the theory of cosmological perturbations during inflation [16,31], which links the origin of the large-scale structure of the universe to quantum fluctuations, quickly followed by a series of works [32,33]. The slow-roll phase is essential in this mechanism, as it is during this period that the density fluctuations that lead to the currently observed large-scale structure are generated.

At the end of inflation, all classical inhomogeneities have been exponentially washed out, and one can consider them as non-existent at this stage. The universe has become very flat so that curvature terms can be neglected. Moreover, all the entropy has been diluted. If the inflaton potential has a minimum, the scalar field will oscillate around this minimum right after the end of inflation. Due to the Hubble expansion, these oscillations are damped and the scalar field decays into a large number of particles. During this phase, the inflationary universe (of low entropy and dominated by the coherent oscillations of the inflaton) becomes a hot universe (of high entropy and dominated by radiation). This reheating phase [34] connects inflation with the hot big bang scenario and complete the picture. In principle, knowing the couplings of the inflaton to the standard matter field, one can determine the relative amount of all matter species and their distribution. This step is however very challenging theoretically and requires heavy numerical simulations.

1.3. Overview of the volume

The dynamics of inflation can be encoded in a dynamical scalar field and, to specify a model, one needs to specify the number of fields, their kinetic term and potential, as well as their couplings to other fields and to the gravity sector (minimal or not). A historical overview and the description of new models is provided by Renata Kallosh and Andrei Linde. This phase of accelerated expansion modifies the causal structure of the universe and the properties of its horizon, which is discussed by George Ellis and Jean-Philippe Uzan.

As a mechanism for the origin of the large-scale structure, inflation provides the initial power spectrum of the scalar (density) perturbations and of the tensor (gravitational waves) power spectrum. The universe evolves and structure grow under the effect of gravity. It follows that the observations of the large-scale structure encode information on both the initial conditions and the evolution of the perturbations, i.e. on the cosmological parameters. The imprint of inflation on the CMB and on the large-scale structure are respectively described by Ruth Durrer and by Francis Bernardeau. The analysis of the recent Planck data, described by François Bouchet, allows one to exclude or constrain many models. A summary of the comparison of inflationary models to cosmological data is provided by Jérôme Martin, Christophe Ringeval, and Vincent Vennin.

In most models in which there are more than one scalar fields, one expects the initial density perturbation to be non-Gaussian. This is reviewed by Sébastien Renaux-Petel, taking into account the latest Planck constraints on the deviation of the CMB from Gaussianity. Inflationary models are usually built by invoking an inflaton, the physical nature of which (i.e. the way it integrates with the architecture of the standard model of particle physics) needs to be identified. This links inflation to the physics beyond the standard model, and in particular supersymmetry [35]. A conservative approach, described by Fedor Bezrukov and Mikhail Shaposhnikov, is to assume that the inflaton is the Higgs field, the only scalar field experimentally known today, but this requires it to be non-minimally coupled with gravity. Another avenue is to construct models in the framework of a theory of quantum gravity such as string theory or loop quantum gravity, two approaches summarized by Eva Silverstein and Martin Bojowald, respectively.

Inflation also opens many questions, in particular concerning its onset. Thiago Pereira and Cyril Pitrou investigate the efficiency of the isotropization of the universe during inflation and whether it can let an observable signature. Interestingly, this imposes some limitations to the construction of the Bunch–Davies vacuum required to set initial conditions. Marc Lilley and Patrick Peter discuss the possibility of an alternative to inflation, in particular assuming a bounce, and Jun'Ichi Yokoyama summarizes the progresses in linking the origin of the cosmological magnetic field to inflation.

As an introduction, this article provides in Section 2 a description of the dynamics of the background in order to explain how inflation solves the standard problem of the hot big bang model. It provides the basics of the mechanism for the background dynamics and some of its extensions. Section 3 summarizes the origin of the large-scale structure during inflation and its connections to quantum mechanics. Section 4 summarizes the predictions of single field slow-roll inflation and some open issues.

2. Inflation as a solution to the problems of the big bang model

2.1. Hot big bang model

In the standard cosmological model, the universe on a large scale is described by a spatially homogeneous and isotropic spacetime, the metric of which is of the form:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\gamma_{ij}\,\mathrm{d}x^i x^j \tag{1}$$

where the scale factor $a(t)$ is a function of the cosmic time, x^i are comoving coordinates and γ_{ij} is the metric of the constant time hypersurfaces. Since they are maximally symmetric spaces, their geometry is either Euclidean (i.e. vanishing curvature, $K = 0$), spherical ($K > 0$) or hyperbolic ($K < 0$). The evolution of the scale factor is determined by the Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{2}$$

where ρ and P are the energy density and pressure of the cosmic matter fluid, and where $H = \dot a/a = \mathrm{d}\ln a/\mathrm{d}t$ is the Hubble function. Λ is the cosmological constant. We already read on the second equation that the expansion of the universe is decelerated unless $\rho + 3P < 0$. It has to be completed by an equation for the evolution of the cosmic fluid

$$\dot\rho + 3H(\rho + 3P) = 0 \tag{3}$$

from which we deduce that the density of a fluid with a constant equation of state $w \equiv P/\rho$ evolves $\rho \propto a^{-3w}$. This set of equations is also often written in terms of the conformal time defined by

$$\mathrm{d}t = a(\eta)\,\mathrm{d}\eta, \qquad \mathcal{H} = \frac{a'}{a} = \frac{\mathrm{d}\ln a}{\mathrm{d}\eta} \tag{4}$$

The different problems of the standard cosmological model are discussed in Chapters 3 and 5 of Ref. [10]. Let me start by the flatness and horizon problems in order to show how inflation can solve them. To that purpose, let me rewrite the equations of evolution in terms of the dimensionless parameter

$$\Omega_K = -\frac{K}{a^2 H^2}$$

as

$$\frac{\mathrm{d}\ln\Omega_K}{\mathrm{d}\ln a} = (1+3w)(1-\Omega_K)\Omega_K \tag{5}$$

It shows that Ω_K is a fixed point of the dynamics, which is not stable if $1+3w>0$. It means that with standard pressureless matter and radiation, the curvature term will tend to dominate the Friedmann equation at late time. But today, the spatial curvature of our universe is small $|\Omega_{K0}-1|<0.1$ (with the upper bound taken very generously). It implies that at the time of matter-radiation equality, $|\Omega_K-1|<3\times10^{-5}$, and at Planck time, $|\Omega_K-1|<10^{-60}$. The big bang model does not give any explanation for such a small curvature at the beginning of the universe and this fine tuning is unnatural for an old universe like ours.

The cosmological principle, which imposes space to be homogeneous and isotropic, is at the heart of the FL solutions. By construction, these models cannot explain the origin of this homogeneity and isotropy. So it would be more satisfying to find a justification of this principle, at least on observable scales. A simple way to grasp the problem is to estimate the number of initial cells, with an initial characteristic Planck size length, ℓ_P present today in the observable universe. This number is typically of order:

$$N \sim \left(\frac{1+z_p}{\ell_P H_0}\right)^3 \sim 10^{87}$$

The study of the CMB and of galaxies tends to show that their distribution is homogeneous on larger scales, so it is difficult to understand how initial conditions fixed on 10^{87} causally independent regions can appear so identical (at a 10^{-5} level!). This horizon problem is related to the state of thermodynamic equilibrium in which the universe is. The cosmological principle imposes a non-causal initial condition on spatial sections of the universe, and in particular that the temperature of the thermal bath is the same at every point. The horizon problem is thus closely related to the cosmological principle and is therefore deeply rooted in the FL solutions.

2.2. The idea of inflation

Let us see how a phase of accelerated expansion can solve the flatness problem. Since the definition of Ω_K implies that $\Omega_K = -K/\dot{a}^2$, it is clear that it will (in absolute value) decrease to zero if $\dot{a}$ increases, that is if $\ddot{a}>0$, i.e. during a phase of accelerated expansion. As discussed above, this requires $\rho+3P<0$ and cannot be achieved with ordinary matter. This can also be seen on the Friedmann equations under the form (5). They clearly show that when $-1<w<-\frac{1}{3}$, the fixed point $\Omega_K=0$ is an attractor of the dynamics.

So, the flatness problem can be resolved if the attraction toward $\Omega_K=0$ during the period of inflation is sufficient to compensate its subsequent drift away from 0 during the hot big bang, i.e. if inflation has lasted for a sufficiently long time. To quantify the duration of the inflationary period, we define the quantity

$$N \equiv \ln\left(\frac{a_\mathrm{f}}{a_\mathrm{i}}\right) \tag{6}$$

where a_i and a_f are the values of the scale factor at the beginning and at the end of inflation. This number measures the growth in the scale factor during the accelerating phase and is called the *number of "e-folds"*. To give an estimate of the required minimum number of e-folds of inflation, note that, if we assume H to be almost constant during inflation, then

$$\left|\frac{\Omega_K(t_\mathrm{f})}{\Omega_K(t_\mathrm{i})}\right| = \left(\frac{a_\mathrm{f}}{a_\mathrm{i}}\right)^{-2} = \mathrm{e}^{-2N}$$

In order to have $|\Omega_K(t_\mathrm{f})| \lesssim 10^{-60}$ and $\Omega_K(t_\mathrm{i}) \sim \mathcal{O}(1)$, we thus need

$$N \gtrsim 70 \tag{7}$$

Similarly, in a accelerated universe, the comoving Hubble radius, $\mathcal{H}^{-1}=(aH)^{-1}$, decreases in time

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 \tag{8}$$

Two points in causal contact at the beginning of inflation can thus be separated by a distance larger than the Hubble radius at the end of inflation. These points are indeed still causally connected, but can *seem* to be causally disconnected if the inflationary period is omitted. So, inflation allows for the entire *observable* universe to emerge out of the same causal region before the onset of inflation. The horizon problem can also be solved if $N \gtrsim 70$.

2.3. Dynamics of single-field inflationary models

Most models of inflation rely on the introduction of a dynamical scalar field φ, evolving in a potential, $V(\varphi)$ with an action given by

$$S = -\int \sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right]\mathrm{d}^4x \tag{9}$$

so that its energy-momentum tensor takes the form

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V\right)g_{\mu\nu} \tag{10}$$

It follows that the energy density and pressure of a homogeneous scalar field are respectively given by

$$\rho_\varphi = \frac{\dot\varphi^2}{2} + V(\varphi)\,, \qquad P_\varphi = \frac{\dot\varphi^2}{2} - V(\varphi) \tag{11}$$

These expressions show that $\rho_\varphi + 3P_\varphi = 2(\dot\varphi^2 - V)$. Since the Friedmann equations imply that the expansion is accelerated as soon as $\dot\varphi^2 < V$, it will be quasi-exponential if the scalar field is in slow-roll, i.e. if $\dot\varphi^2 \ll V$. This clearly explains why this is a natural way of implementing inflation.

The Friedmann and Klein–Gordon equations take the form

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\varphi^2 + V\right) - \frac{K}{a^2}\,, \qquad \frac{\ddot a}{a} = \frac{8\pi G}{3}\left(V - \dot\varphi^2\right)\,, \qquad \ddot\varphi + 3H\dot\varphi + V_{,\varphi} = 0 \tag{12}$$

Once the potential is chosen, the whole dynamics can be determined. As an example consider a free massive scalar

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 \tag{13}$$

The Klein–Gordon equation reduces to that of a damped harmonic oscillator. If φ is initially large, then the Friedmann equation implies that H is also very large. The friction term becomes important and dominates the dynamics so that the field must be in the slow-roll regime. The evolution equations then reduce to

$$3H\dot\varphi + m^2\varphi = 0\,, \qquad H^2 = \frac{4\pi}{3}\left(\frac{m}{M_\mathrm{p}}\right)^2\varphi^2,$$

which gives

$$\varphi(t) = \varphi_\mathrm{i} - \frac{mM_\mathrm{p}}{\sqrt{12\pi}}t\,, \qquad a(t) = a_\mathrm{i}\exp\left\{\frac{2\pi}{M_\mathrm{p}^2}\left[\varphi_\mathrm{i}^2 - \varphi^2(t)\right]\right\} \tag{14}$$

where φ_i and a_i are the values of the field and the scale factor at $t_\mathrm{i} = 0$, and M_p is the Planck mass. This can be compared to the numerical integration depicted in Fig. 3, in which we see the subsequent (non-slow-rolling) phase with the damped oscillations at the bottom of the potential and the fact that $\ddot a$ changes sign to become negative. Fig. 4 provides a phase space analysis of the dynamics, showing that the slow-roll trajectories are attractors of the dynamics. It follows that if inflation lasts long enough, the initial conditions on $(\varphi, \dot\varphi)$ become irrelevant.

2.4. Slow-roll formalism

As seen in the example of a massive scalar field, inflation occurs while the scalar field is slow rolling. This has led to the development of a perturbative formalism to describe the dynamics. If the field is slow rolling, then H is almost constant and it is convenient to define

$$\varepsilon = -\frac{\dot H}{H^2}\,, \qquad \delta = \varepsilon - \frac{\dot\varepsilon}{2H\varepsilon}\,, \qquad \xi = \frac{\dot\varepsilon - \dot\delta}{H} \tag{15}$$

These definitions depend only on the spacetime geometry. In the case of a single scalar field, they can be rewritten as

$$\varepsilon = \frac{3}{2}\dot\varphi^2\left[\frac{1}{2}\dot\varphi^2 + V(\varphi)\right]^{-1}\,, \qquad \delta = -\frac{\ddot\varphi}{H\dot\varphi} \tag{16}$$

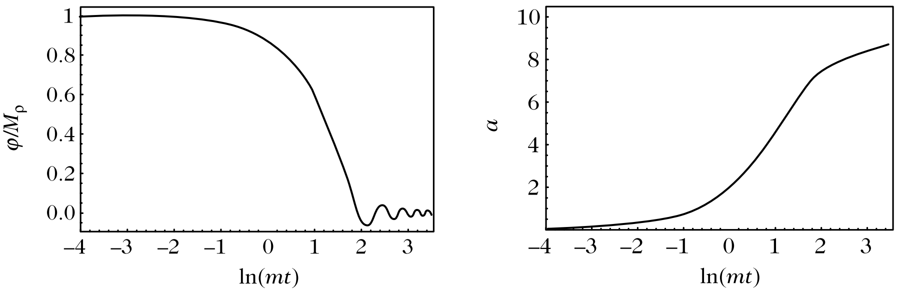

Fig. 3. Evolution of the inflaton and scale factor during inflation with a potential (13). The scalar field is initially in a slow-roll regime and the expansion of the universe is accelerated. At the end of this regime, it starts oscillating at the minimum of its potential and it is equivalent to a pressureless fluid. From Ref. [10].

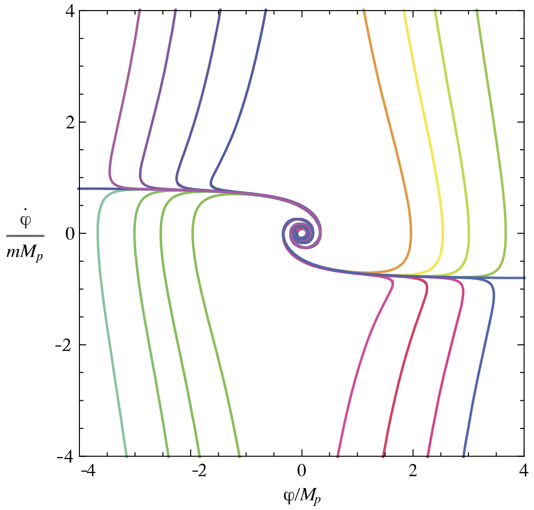

Fig. 4. (Color online.) Phase portrait in the $(\varphi, \dot{\varphi})$-plane of the dynamics of a scalar field with potential $V = m^2\varphi^2/2$, assuming $m = 10^{-6} M_{\rm p}$. This illustrates the mechanism of attraction toward the slow-roll solution. From Ref. [10].

ε can be used to rewrite the Friedmann equations as

$$H^2\left(1 - \frac{1}{3}\varepsilon\right) = \frac{8\pi G}{3} V\,, \qquad \frac{\ddot{a}}{a} = H^2(1-\varepsilon) \tag{17}$$

and the effective equation of state of the inflaton as $w_\varphi = -1 + \frac{2}{3}\varepsilon$, so that the condition for inflation reduces to:

$$\ddot{a} > 0 \Longleftrightarrow w < -1/3 \Longleftrightarrow \varepsilon < 1 \tag{18}$$

The number of e-folds between a time t, when the value of the inflaton is φ, and the end of inflation ($t = t_{\rm f}$ and $\varphi = \varphi_{\rm f}$) of the inflationary phase can be expressed as

$$N(t, t_{\rm f}) = \int_t^{t_{\rm f}} H\,{\rm d}t \tag{19}$$

since, after integration, $a(t) = a_{\rm f}\,{\rm e}^{-N}$. $N(t_{\rm i}, t_{\rm f})$ corresponds to the duration of the inflationary phase, as defined earlier. N can be expressed as

$$N(\varphi, \varphi_{\rm f}) = \int_\varphi^{\varphi_{\rm f}} \frac{H}{\dot{\varphi}}\,{\rm d}\varphi = -\sqrt{4\pi G}\int_\varphi^{\varphi_{\rm f}} \frac{{\rm d}\varphi}{\sqrt{\varepsilon}} \tag{20}$$

As long as the slow-roll parameters are small, H is almost constant. One can then develop the equations for the evolution of the background (a, H, φ) in terms of this small parameters. This allows us to derive the observational predictions in term of these parameters and, interestingly, they can be related to the derivative of the inflationary potential [10,36] as

$$\varepsilon = \frac{1}{16\pi G}\left(\frac{V_{,\varphi}}{V}\right)^2\,, \qquad \delta = \frac{1}{8\pi G}\left(\frac{V_{,\varphi\varphi}}{V}\right) - \varepsilon$$

2.5. Chaotic inflation

In order to illustrate this formalism, let us come back to the massive scalar field. It is clear that $\varepsilon = \frac{M_p^2}{4\pi\varphi^2}$ and $\delta = 0$, so that the slow-roll regime lasts until φ reaches $\varphi_f = M_p/\sqrt{4\pi}$. We infer that the total number of e-folds is:

$$N(\varphi_i) = 2\pi\left(\frac{\varphi_i}{M_p}\right)^2 - \frac{1}{2} \tag{21}$$

In order to have $N \gtrsim 70$, we need $\varphi_i \gtrsim 3M_p$. If φ takes the largest possible value compatible with the classical description adopted here, i.e. $V(\varphi_i) \lesssim M_p^4$, we find that $\varphi_i \sim M_p^2/m$. In this case, the maximal accessible number of e-folds would be $N_{max} \sim 2\pi M_p^2/m^2$. As can be deduced from the observations of the CMB impose that $m \sim 10^{-6}M_p$, so that $N_{max} \sim 10^{13}$. The maximal number of e-folds is thus very large compared to the minimum required for solving the cosmological problems.

Consequently, if the universe is initially composed of regions where the values of the scalar field are randomly distributed, then the domains where the initial value of φ is too small never inflate or only for a small number of e-folds. The main contribution to the total physical volume of the universe at the end of inflation comes from regions that have inflated for a long time and that had an initially large value of φ. These domains produce extremely flat and homogeneous zones at the end of inflation, with a very large size compared to that of the observable universe.

This is the idea of *chaotic inflation* proposed in Ref. [27] and which predicts, if we are typical observers, that we shall not be surprised to observe an extremely Euclidean, homogeneous and isotropic universe. This conclusion has to be contrasted with the early version of the big bang model and to the discussion on its problems.

2.6. End of the inflationary phase

Inflation ends when $\max(\varepsilon, \delta) \sim 1$. At the end of inflation, all classical inhomogeneities have been exponentially washed out, and one can consider them as non-existent at this stage (still the fact that we neglect them from the start of the analysis has motivated many works to carefully understand the homogenization ans isotropization of the universe, both at the background and perturbative levels; see, e.g., Ref. [37] for a discussion of the curvature and Ref. [38] for a discussion on the isotropization and their effects on the maximum number of e-folds), so that one can set $K = 0$ during all the late stages of the primordial phase.

During inflation, all the energy is concentrated in the inflaton. Shortly after the end of inflation, the universe is cold and "frozen" in a state of low entropy, where the field oscillates around the minimum of its potential. The coherent oscillations of the inflaton can be considered as a collection of independent scalar particles. If they couple to other particles, the inflaton can decay perturbatively to produce light particles. The interaction of the inflaton should therefore give rise to an effective decay rate, Γ_φ, and reheating would only occur after the expansion rate had decreased to a value $H \sim \Gamma_\varphi$. This also implies that during the first $\mathcal{O}(m/\Gamma_\varphi)$ oscillations of the inflaton, nothing happens.

To illustrate this, consider an inflaton with potential $V \propto \varphi^n$. During the oscillatory phase, $H < m$ and the inflaton undergoes several oscillations during a time H^{-1}. It is thus reasonable to use the mean value of the pressure and density over several oscillations. It follows that $\langle\dot\varphi^2/2\rangle = (n/2)\langle V(\varphi)\rangle$, so that $\langle P_\varphi\rangle = (n-2)/(n+2)\langle\rho_\varphi\rangle$. The scalar field thus behaves as a dust fluid for $n = 2$ and as a radiation fluid for $n = 4$.

In order for the inflaton to decay, it should be coupled to other fields. Including quantum corrections, the Klein–Gordon equation becomes

$$\ddot\varphi + 3H\dot\varphi + \left[m^2 + \Pi(m)\right]\varphi = 0 \tag{22}$$

where $\Pi(m)$ is the polarization operator of the inflaton. The real part of $\Pi(m)$ corresponds to the mass correction. Π has an imaginary part $\mathrm{Im}[\Pi(m)] = m\Gamma_\varphi$. Since m is much larger than both Γ_φ and H at the end of inflation, we can solve the Klein–Gordon equation by assuming that both these quantities are constant during an oscillation. It follows that

$$\varphi = \Phi(t)\sin mt\,, \qquad \Phi = \varphi_0 \exp\left[-\frac{1}{2}\int(3H + \Gamma_\varphi)\,\mathrm{d}t\right] \tag{23}$$

As long as $3H > \Gamma_\varphi$, the decrease in the inflaton energy caused by the expansion (Hubble friction) dominates over particle decay. Thus,

$$\Phi = \varphi_f\frac{t_f}{t} = \frac{M_p}{m}\frac{1}{\sqrt{3\pi}t} \tag{24}$$

where we have used $\varphi_f = M_p/\sqrt{4\pi}$, $vt_f = 2/3H_f$ and $H_f^2 = (4\pi/3)(m/M_p)^2\varphi_f^2$. Reheating occurs in the regime $\Gamma_\varphi \gtrsim 3H$ and

$$\Phi = \frac{M_p}{m}\frac{1}{\sqrt{3\pi}t}\mathrm{e}^{-\Gamma_\varphi t/2} \tag{25}$$

We define the time of reheating, $t_{\rm reh}$, by $\Gamma_\varphi = 3H$ so that the energy density at that time is $\rho_{\rm reh} = \frac{\Gamma_\varphi^2 M_{\rm p}^2}{24\pi}$. If this energy is rapidly converted into radiation, its temperature is $\rho_{\rm reh} = \frac{\pi^2}{30} g_* T_{\rm reh}^4$. This defines the reheating temperature as

$$T_{\rm reh} = \left(\frac{5}{4\pi^3 g_*}\right)^{1/4} \sqrt{\Gamma_\varphi M_{\rm p}} \simeq 0.14 \left(\frac{100}{g_*}\right)^{1/4} \sqrt{\Gamma_\varphi M_{\rm p}} \ll 10^{15}\ \text{GeV} \tag{26}$$

where the upper bound was obtained from the constraint $\Gamma_\varphi \ll m \sim 10^{-6} M_{\rm p}$, assuming $g_* \gtrsim 100$ for the effective number of relativistic particles.

This description of perturbative reheating is simple and intuitive in many aspects. However, the decay of the inflaton can start much earlier in a phase of *preheating* (parametric reheating) where particles are produced by parametric resonance. The preheating process can be decomposed into three stages: (1) non-perturbative production of particles, (2) perturbative stage, and (3) thermalization of the produced particles. We refer to Ref. [34] for details on this stage that connects inflation to the standard hot big bang model.

2.7. Eternal inflation

To finish, let us describe the process of eternal inflation that plays a central role in the discussion of the multiverse. The discovery of the self-reproduction process is a major development for inflation and cosmology. This mechanism was known for the models of old and new inflation and then extended to chaotic inflation [39]. In inflationary models with large values of the inflaton, quantum fluctuations can locally increase the value of the inflaton. The expansion of these regions is then faster and their own quantum fluctuations generate new inflationary domains. This process naturally leads to a self-reproducing universe where there are always inflating regions.

We just sketch a heuristic description here [40]. Regions separated by distances greater than H^{-1} can be considered to be evolving independently. So any region of size H^{-1} will be considered as an independent universe decoupled from other regions. Consider such a region of size H^{-1} for which the scalar field is homogeneous enough and has a value $\varphi \gg M_{\rm p}$ and consider a massive field. In a time interval $\Delta t \sim H^{-1}$, the field classically decreases by $\Delta\varphi \sim \dot\varphi \Delta t \sim \dot\varphi/H$. The Klein–Gordon equation in the slow-roll regime then implies that $\Delta\varphi \sim -M_{\rm p}^2/4\pi\varphi$. This value should be compared with the typical amplitude of quantum fluctuations, $|\delta\varphi| \sim H/2\pi \sim m\varphi/\sqrt{3\pi} M_{\rm p}$ (see next section).

Classical and quantum fluctuations have the same amplitude for

$$|\Delta\varphi| \sim |\delta\varphi| \Longleftrightarrow \varphi \sim \varphi_* \equiv \frac{M_{\rm p}}{2}\sqrt{\frac{M_{\rm p}}{m}} \tag{27}$$

We can thus distinguish between three phases in the evolution of the inflaton: a phase during which the quantum fluctuations are of the same order as (or larger than) the classical field variation, a phase in which the field is in classical slow-roll towards its minimum and a phase when the field oscillates around its minimum. Note that $V(\varphi_*) = (m/M_{\rm p}) M_{\rm p}^4/8 \lesssim 10^{-6} M_{\rm p}^4$, so that the first regime can be reached even at energies that are small compared to $M_{\rm p}$.

When $\varphi \gg \varphi_*$, $\delta\varphi \gg \Delta\varphi$. The characteristic length of the fluctuations $\delta\varphi$ generated during the time Δt is of the order of H^{-1}, so that the initial volume is divided into $(\exp H\Delta t)^3 \sim 20$ independent volumes of radius H^{-1}. Statistically, the value of the field in half of these regions is $\varphi + \Delta\varphi - \delta\varphi$ and $\varphi + \Delta\varphi + \delta\varphi$ in the other half. So the physical volume of the regions where the field has a value greater than φ_* is ten times larger:

$$V_{t+\Delta t}(\varphi > \varphi_*) \sim \frac{1}{2} (\exp H\Delta t)^3\, V_t(\varphi > \varphi_*) \sim 10 V_t(\varphi > \varphi_*) \tag{28}$$

The physical volume where the space is inflating therefore grows exponentially in time. The zones where the field becomes lower than φ_* enter a slow-roll phase. So they become inflationary universes, decoupled from the rest of the universe, with a slow-roll phase, a reheating phase and a hot big bang. These zones are *island universes* (or pocket universes) and our observable universe would be only a tiny part of such an island universe.

This scenario has important consequences for cosmology. In chaotic inflation, the universe has a very inhomogeneous structure on scales larger than H^{-1} with regions undergoing eternal inflation continuously giving rise to new zones themselves undergoing inflation. On large scales, the universe has a fractal structure with continuous production of island universes. Each one of these island universes then undergoes a phase of "classical" inflation, with a large number of e-folds and is thus composed of many regions of the size of our observable universe. This simplistic model assumes only one scalar field and a potential with a unique minimum. Realistic models in high-energy physics on the other hand involve many scalar fields. The potential of these fields can be very complex and have many flat directions and minima. So the same theory can have different vacua that correspond to different schemes of symmetry breaking. Each of these vacua can lead, at low energy, to physically different laws. Due to the exploration of this *landscape* by quantum fluctuations, the universe would find itself divided into many regions with different low-energy physical laws (for instance different values of the fundamental constants). If this vision of the primordial universe is correct, physics alone cannot provide a complete explanation for all the properties of our observable universe since the same physical theory can generate vast regions with very different low-energy properties. So our observable universe would have the properties it has not because the other possibilities are impossible or improbable, but simply because a universe with such properties allows for a life similar to ours to appear.

Eternal inflation thus offers a framework to apply the anthropic principle since the self-reproduction mechanism makes it possible to generate universes with different properties and to explore all possible vacua of a theory. This approach is used more and more to address the question of the value of the fundamental constants and to address the cosmological constant problem. This framework also allows us to address questions beyond the origin of the properties of our universe, thus defining the limitation of what we will be capable of explaining.

3. Inflation as a model for the origin of large-scale structure

The previous section has described a homogeneous scalar field. As any matter field, φ has quantum fluctuations and cannot be considered as strictly homogeneous. This drives us to study the effects of these perturbations. Any fluctuation of the scalar field will generate metric perturbations, since they are coupled by the Einstein field equations. We should thus study the coupled inflaton-gravity system to remains to understand.

In terms of the variables to consider, we start from 10 perturbations for the metric and one for the scalar field, to which we have to subtract four gauge freedoms and four constraint equations (two scalars and two vectors). We thus expect to identify three independent degrees of freedom to describe the full dynamics: one scalar mode and a tensor mode (counting for two degrees of freedom, one per polarization).

The following presentation relies on the extended description of Ref. [10] (Chapter 8).

3.1. Perturbation theory during inflation

Starting from the perturbation of the scalar field as $\varphi = \varphi(t) + \delta\varphi(\boldsymbol{x}, t)$, its stress-energy tensor takes the form

$$\delta T_{\mu\nu} = 2\partial_{(\nu}\varphi\partial_{\mu)}\delta\varphi - \left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + V\right)\delta g_{\mu\nu} - g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + g^{\alpha\beta}\partial_\alpha\delta\varphi\partial_\beta\varphi + V'\delta\varphi\right) \quad (29)$$

The most general form of the perturbed metric is

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\,\mathrm{d}\eta^2 + 2B_i\,\mathrm{d}x^i\,\mathrm{d}\eta + (\gamma_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right] \quad (30)$$

where the small quantities A, B_i and h_{ij} are unknown functions of space and time, to be determined from the Einstein equations.

There are two important technical points. We refer to Ref. [14] and to the chapters 5 and 8 of Ref. [10] or Ref. [41] for details.

- First, one can perform a Scalar–Vector–Tensor (SVT) decomposition of the perturbations and, at linear order, the modes will decouple. This decomposition is a generalization of the fact that any vector field can be decomposed as the sum of the gradient of a scalar and a divergenceless vector as

$$B^i = D^i B + \bar{B}^i, \qquad h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}\bar{E}_{j)} + 2\bar{E}_{ij} \quad (31)$$

with $D^i\bar{B}_i = 0$ and $D_i\bar{E}^{ij} = \bar{E}^i_i = 0$.
- Second, one has to carefully look at how the perturbation variables change under a gauge transformation, $x^\mu \to x^\mu - \xi^\mu$, where ξ^μ is decomposed into two scalar degrees of freedom and two vector degrees of freedom ($\bar{L}^i$, which is divergenceless $D_i\bar{L}^i = 0$) as

$$\xi^0 = T, \qquad \xi^i = L^i = D^iL + \bar{L}^i \quad (32)$$

Under this change of coordinates, the metric and the scalar field transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + \pounds_\xi g_{\mu\nu}, \qquad \varphi \to \varphi + \pounds_\xi\varphi$$

In order to construct variables that remain unchanged under a gauge transformation, we introduce gauge invariant variables that "absorb" the components of ξ^μ. We thus get the seven variables

$$\Psi \equiv -C - \mathcal{H}(B - E') \qquad \Phi \equiv A + \mathcal{H}(B - E') + (B - E')', \qquad \bar{\Phi}^i \equiv \bar{E}^{i\prime} - \bar{B}^i \quad (33)$$

and it is obvious that the tensor modes $\bar{E}^{ij}$ were already gauge invariant. For the scalar field, one can introduce the two following gauge invariant variables:

$$\chi = \delta\varphi + \varphi'(B - E'), \quad \text{or} \qquad Q = \delta\varphi - \varphi'\frac{C}{\mathcal{H}} \quad (34)$$

They are related by $Q = \chi + \varphi'\Psi/\mathcal{H}$. and χ is the perturbation of the scalar field in a Newtonian gauge, while Q, often called the Mukhanov–Sasaki variable, is the one in flat slicing gauge.

Forgetting about the technicalities of the derivation, one ends up with the following equations of evolution. We were left with $7 = 2 + 2 + 3$ degrees of freedom respectively to the T, V and S modes. For simplicity, we assume that $K = 0$.

- *Vector modes*. Since a scalar field does not contain any vector sources, there are no vector equations associated with the Klein–Gordon equation, and there is only one Einstein equation for these modes and a constraint equation,

$$\Delta\bar{\Phi}_i = 0$$

We can conclude that $\bar{\Phi}_i = 0$, independently of any model of inflation, that the vector modes are completely absent at the end of inflation. The two constraints equation kill the vector degrees of freedom.
- *Tensor modes*. There is only one tensor equation, which can be obtained from the Einstein equations:

$$\bar{E}''_{ij} + 2\mathcal{H}\bar{E}'_{ij} - \Delta\bar{E}_{ij} = 0 \quad (35)$$

It describes the evolution of the two polarizations of a gravity wave as a damped harmonic oscillator. Shifting to Fourier space and decomposing the gravity waves on a polarization tensor as $\bar{E}_{ij}(\boldsymbol{k},\eta) = \sum_\lambda \bar{E}_\lambda(\boldsymbol{k},\eta)\varepsilon^\lambda_{ij}(\boldsymbol{k})$ and defining

$$\mu_\lambda(\boldsymbol{k},\eta) = \sqrt{\frac{M_p^2}{8\pi}}a(\eta)\bar{E}_\lambda(\boldsymbol{k},\eta)$$

the equation of evolution takes the form

$$\mu''_\lambda + \left(k^2 - \frac{a''}{a}\right)\mu_\lambda = 0 \quad (36)$$

for each of the two polarizations.
- *Scalar modes*. The Einstein equations provide two constraint equations and an equation of evolution. This implies that we shall be able to derive a master equation for the only propagating degree of freedom. This can be achieved by defining [10,31]

$$v(\boldsymbol{k},\eta) = a(\eta)Q(\boldsymbol{k},\eta), \qquad z(\eta) = \frac{a\varphi'}{\mathcal{H}},$$

which can be shown to satisfy

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0 \quad (37)$$

Bote that v is a composite variable of the field and metric perturbations. As expected from general relativity, it is not possible to separate them.

The set of equations (36)–(37) for the tensor and scalar modes are second-order differential equations in Fourier space. In order for them to be predictive, one needs to fix their initial conditions. It is striking that these equations look like Schrödinger equations, with a time-dependent mass.

3.2. Setting the initial conditions

Let us first compare the evolution of a mode with comoving wave-number k in the standard big bang model and during inflation. Any equation of evolution can be shown to have two regimes, whether the mode is super-Hubble ($k < \mathcal{H}$), i.e. its wavelength is larger that the Hubble radius, or sub-Hubble ($k > \mathcal{H}$), i.e. its wavelength is smaller that the Hubble radius (see Fig. 5).

In the post-inflationary era, the expansion is decelerating so that $\mathcal{H}$ is a decreasing function of η, while k remains constant (since it is comoving). It follows that the super-Hubble modes become sub-Hubble while the universe expands. All observable modes are sub-Hubble today and one needs to set initial conditions in the early universe while they were super-Hubble. Actually, asymptotically toward the big bang all modes where super-Hubble. A difficulty arises from the observation of the CMB and of large-scale structure, which shows that there shall exist super-Hubble correlations. It seems unnatural to set correlated initial conditions on super-Hubble scales, since it would appear as acausal. Besides, there was no natural procedure to fix the initial conditions so that they were completely free and reconstructed to get an agreement with the observations.

During inflation, the picture is different because the expansion is accelerated, which means that $\mathcal{H}$ is increasing with η while k remains constant. It follows that any super-Hubble mode was sub-Hubble deep in the inflationary era. If inflation lasts long enough, then all the observable modes can be sub-Hubble, where v and μ_λ behave as harmonic oscillators. We shall thus find a mechanism to set initial conditions in this regime since we are interested only in modes that are observable today, at least as far as confronting inflation to observation is concerned.

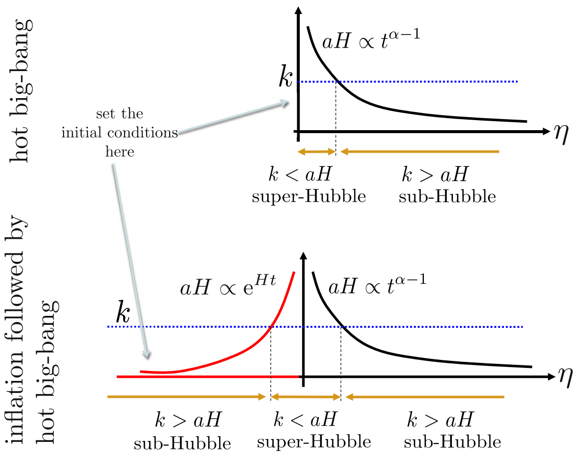

Fig. 5. (Color online.) Evolution of the comoving Hubble radius and of a comoving mode of wavenumber k with conformal time η in the standard hot big bang model (top) and with an inflationary phase (bottom). Without inflation, the mode is always super-Hubble in the past and becomes sub-Hubble as the universe expands. This implies that initial conditions have to be set on super-Hubble scales. The existence of super-Hubble correlation are thus thought to be acausal. With an inflationary phase, the mode was sub-Hubble deep in the inflationary era, which means that initial conditions have to be set on sub-Hubble scales.

The idea [16] was to take seriously the fact that Q, and thus v, enjoys quantum fluctuations. They demonstrated that when expanded at second order in perturbation the Einstein Hilbert + scalar field action reduces (see Refs. [31,38,41]) to

$$\delta^{(2)} S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\boldsymbol{x}\left[(v')^2 - \delta^{ij}\partial_i v \partial_j v + \frac{z''}{z}v^2\right] \equiv \int \mathcal{L}\, \mathrm{d}^4 x \tag{38}$$

up to terms involving a total derivative and that do not contribute to the equations of motion. The variation of this action indeed gives the equation (37), but it tells us more on the structure of the theory: we recognize the action of a canonical scalar field with a time-dependent mass, $m^2 = -z''/z$ in Minkowski spacetime. It was thus proposed that the variable to quantize is v and we shall quantize it as we quantize any canonical scalar field evolving in a time-dependent exterior field [42,44], where here the time-dependence would find its origin in the spacetime dynamics [43].

The procedure then goes as follows.

1. v is promoted to the status of quantum operator in second quantization in Heisenberg representation

$$\hat{v}(\boldsymbol{k},\eta) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\left[v_k(\eta)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}} + v_k^*(\eta)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}}^\dagger\right] \tag{39}$$

where $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^\dagger$ are creation and annihilation operators.

2. One introduces the conjugate momentum of v,

$$\pi = \frac{\delta\mathcal{L}}{\delta v'} = v' \tag{40}$$

which is also promoted to the status of operator, $\hat{\pi}$.

3. One can then get the Hamiltonian

$$H = \int (v'\pi - \mathcal{L})\, \mathrm{d}^4 x = \frac{1}{2}\int \left(\pi^2 + \delta^{ij}\partial_i v \partial_j v - \frac{z''}{z}v^2\right) \mathrm{d}^4 x \tag{41}$$

The equation of evolution for $\hat{v}$ is indeed Eq. (37) which is equivalent to the Heisenberg equations $\hat{v}' = i\left[\hat{H},\hat{v}\right]$ and $\hat{\pi}' = i\left[\hat{H},\hat{\pi}\right]$.

4. The operators $\hat{v}$ and $\hat{\pi}$ have to satisfy canonical commutation relations on constant-time hypersurfaces

$$[\hat{v}(\boldsymbol{x},\eta),\hat{v}(\boldsymbol{y},\eta)] = [\hat{\pi}(\boldsymbol{x},\eta),\hat{\pi}(\boldsymbol{y},\eta)] = 0\,, \quad [\hat{v}(\boldsymbol{x},\eta),\hat{\pi}(\boldsymbol{y},\eta)] = i\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{42}$$

5. As for quantization in Minkowski spacetime, the creation and annihilation operators appearing in the decomposition (39) satisfy the standard commutation rules

$$[\hat{a}_{\boldsymbol{k}},\hat{a}_{\boldsymbol{p}}] = [\hat{a}_{\boldsymbol{k}}^\dagger,\hat{a}_{\boldsymbol{p}}^\dagger] = 0\,, \qquad [\hat{a}_{\boldsymbol{k}},\hat{a}_{\boldsymbol{p}}^\dagger] = \delta(\boldsymbol{k}-\boldsymbol{p}) \tag{43}$$

They are consistent with the commutation rules (42) only if v_k is normalized according to

$$W(k) \equiv v_k v'^*_k - v^*_k v'_k = i \tag{44}$$

This determines the normalization of their Wronskian, W.

6. The vacuum state $|0\rangle$ is then defined by the condition that it is annihilated by all the operators $\hat{a}_{\boldsymbol{k}}$, $\forall \boldsymbol{k}$, $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$. The sub-Hubble modes, i.e. the high-frequency modes compared to the expansion of the universe, must behave as in a flat spacetime. Thus, one decides to pick up the solution that corresponds adiabatically to the usual Minkowski vacuum

$$v_k(\eta) \to \frac{1}{\sqrt{2k}} \mathrm{e}^{-\mathrm{i}k\eta}, \qquad k\eta \to -\infty \tag{45}$$

This choice is called the *Bunch–Davies vacuum*. Let us note that in an expanding spacetime, the notion of time is fixed by the background evolution, which provides a preferred direction. The notion of positive and negative frequency is not time invariant, which implies that during the evolution positive frequencies will be generated.

Note that this quantization procedure amounts to treating gravity in a quantum way at linear order since the gravitational potential is promoted to the status of operator.

It can be seen that it completely fixes the initial conditions, i.e. the two free functions of integration that appear when solving Eq. (37). This can be seen on the simple example of a de Sitter phase ($H = \text{constant}$, $a(\eta) = -1/H\eta$ with a test scalar field. Then $z''/z = 2/\eta$ and the equation (37) simplifies to $v''_k + \left(k^2 - \frac{2}{\eta^2}\right) v_k = 0$, for which the solutions are given by

$$v_k(\eta) = A(k)\mathrm{e}^{-\mathrm{i}k\eta}\left(1 + \frac{1}{ik\eta}\right) + B(k)\mathrm{e}^{\mathrm{i}k\eta}\left(1 - \frac{1}{ik\eta}\right) \tag{46}$$

The condition (45) sets $B(k) = 0$ and fixes the solution to be (for $Q = v/a = H\eta v$)

$$Q_k = \frac{H\eta}{\sqrt{2k}}\left(1 + \frac{1}{ik\eta}\right)\mathrm{e}^{-\mathrm{i}k\eta} \tag{47}$$

Notice that on sub-Hubble, $Q_k \propto 1/\sqrt{k}$, as imposed by quantum mechanics, but when the mode becomes super-Hubble ($k\eta \ll 1$), it shifts to $Q_k \propto 1/\sqrt{k^3}$, which corresponds to a scale-invariant power spectrum. Hence the scale invariance on super-Hubble scales is inherited from the small scale properties fixed by quantum mechanics and no tuning is at work.

On super-Hubble scales, the field Q acquires a constant amplitude $|Q_k|(k\eta \ll 1) = \frac{H}{\sqrt{2k^3}}$. In this limit

$$\hat{Q} \to \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \hat{\chi}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \frac{H}{\sqrt{2k^3}}\left(\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

All the modes being proportional to $(\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}})$, $\hat{Q}$ has the same statistical properties as a Gaussian classical stochastic field and its power spectrum defined by

$$\langle Q_{\boldsymbol{k}} Q^*_{\boldsymbol{k}'}\rangle = P_Q(k)\,\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \tag{48}$$

from which one easily deduces that

$$P_Q(k) = \frac{2\pi^2}{k^3}\mathcal{P}_Q(k) = |Q_k|^2 = \frac{|v_k|^2}{a^2} \tag{49}$$

It is thus a scale-invariant power spectrum since $\mathcal{P}_Q(k) = \left(\frac{H}{2\pi}\right)^2$.

4. Conclusion: generic predictions

The analysis described in the previous section applies in any model of inflation and the conclusions can be computed analytically in the slow-roll regime. The departure from a pure de Sitter phase reflects itself in a spectral index in Eq. (49). They also apply identically for the gravity waves, the only difference being that v and μ_λ have different "mass terms" and that they are related to Q and $\bar{E}_\lambda$ respectively by z and a. It implies that they will have different spectral indices and amplitudes.

In conclusion, single-field inflationary models have robust predictions that are independent of their specific implementation.

- The observable universe must be homogeneous and isotropic. Inflation erases any classical inhomogeneities. The observable universe can thus be described by a FL spacetime.
- The universe must be spatially Euclidean. During inflation, the curvature of the universe is exponentially suppressed. Inflation therefore predicts that $\Omega_K = 0$ up to the amplitude of the super-Hubble density perturbations, i.e. up to about 10^{-5}.

- Scalar perturbations are generated. Inflation finds the origin of the density perturbations in the quantum fluctuations of the inflaton, which are amplified and redshifted to macroscopic scales. All the modes corresponding to observable scales today are super-Hubble at the end of inflation. Inflation predicts that these perturbations are adiabatic, have a Gaussian statistics and have an almost scale-invariant power spectrum. The spectral index can vary slightly with wavelength. These perturbations are coherent, which results in a structure of acoustic peaks in the angular power spectrum of the CMB's temperature anisotropies.
- There are no vector perturbations.
- Gravitational waves are generated. In the same way as for scalar modes, the gravitational waves have a quantum origin and are produced through parametric amplification. They also have Gaussian statistics and an almost scale invariant power spectrum.
- There is a consistency relation between their spectral index and the relative amplitude of the scalar and tensor modes.
- The quantum fluctuations of any light field ($M < H$) are amplified. This field develops super-Hubble fluctuations of amplitude $H/2\pi$.

More generally, inflationary models radically change our vision of cosmology in at least three respects. These predictions are in full agreement with observations, and CMB in particular.

- Predictions are by essence probabilist. One can only predict spectra, correlations so that only the statistical properties of the galaxy distribution can be inferred, not their exact position.
- Particles are produced by the preheating mechanism. The inflaton decay allows us to explain the production of particles at the end of inflation. It may affect the standard predictions [45].
- Inflation is eternal. Placing inflationary models in the context of chaotic initial conditions, we obtain a very different picture of the universe. The latter should be in eternal inflation and gives rise to island universes.

Many questions still need to be answered. To which extent can we reconstruct the potential of the inflaton from cosmological observations? How is the inflaton connected to the standard model of particle physics (can it be the Higgs)? Can one construct models without scalar field? How robust are the predictions and can one construct radically different models (as the now-ruled out opponent theory of topological defects)? Can one relate inflation to supersymmetry or string theory in a satisfying way? What happens to other fields? Should we be worried by trans-Planckian modes? Can one detect features (non-Gaussianity, correlations) of the initial conditions that would reveal the physics at work? Can one constrain the number of *e*-folds? Can one detect the primordial gravity waves? What is the structure of spacetime in eternal inflation?...

All these questions are at the crossroad between theoretical physics, high-energy physics and cosmology, and are discussed in this volume.

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