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The Riemann tensor and the Bianchi identity in 5D space–time

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A B S T R A C T

The initial assumption of theories with extra dimension is based on the efforts to yield a geometrical interpretation of the gravitation field. In this paper, using an infinitesimal parallel transportation of a vector, we generalize the obtained results in four dimensions to five-dimensional space–time. For this purpose, we first consider the effect of the geometrical structure of 4D space–time on a vector in a round trip of a closed path, which is basically quoted from chapter three of Ref. [5]. If the vector field is a gravitational field, then the required round trip will lead us to an equation which is dynamically governed by the Riemann tensor. We extend this idea to five-dimensional space–time and derive an improved version of Bianchi's identity. By doing tensor contraction on this identity, we obtain field equations in 5D space–time that are compatible with Einstein's field equations in 4D space–time. As an interesting result, we find that when one generalizes the results to 5D space–time, the new field equations imply a constraint on Ricci scalar equations, which might be containing a new physical insight.

1. Introduction

The idea of the existence of higher-dimensional space–times has a long history. It has been founded by Kaluza [1] and Klein [2,3] in the 1920s, as they were trying to unify the gravitational and electromagnetic interactions by assuming a five-dimensional space–time (5D). A new version of the idea of Kaluza and Klein appeared later, around the 1970s, which is still considered by the majority of the physics community to be the best hope for a completed unified theory of all fundamental interactions [4], despite growing opposition against extra-dimensional theories.

All methods about unification are based on the fact that the Riemannian geometry in general relativity can be generalized to higher dimensions without implying any constraint on the gravitational field. In other word, as we will discuss in this paper, the antisymmetric nature of Riemann's tensor and the Bianchi identity as a constraint on Riemann's tensor are implied when we consider two and three spatial dimensions of space–time. But no one is concerned by any constraint on gravitational field equations when four or more spatial dimensions of space–time are considered!

In this paper, by following the geometrical approach described in Ref. [5] and using the concept of parallel transport [6] in five dimensional space–time, we get a new second-order differential identity for the Riemann tensor. As we will see by doing contractions on the new identity, we can define the gravitational field equations in 5D space–time, which is

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compatible with Einstein's field equations in 4D space–time. Also we will prove that in order to convert the new 5D field equations to Einsteins's field equations that are supposed to be correct in all extra-dimensional spaces, the d'Alembertian of 5D Ricci scalar R should be zero ($\Box R = 0$).

The organization of this paper is as follows. In section 2, the geometry of curved space is reviewed. Bianchi identities in five dimensional space are obtained in Section 3. In section 4, we will generalize the parallel transportation of a vector in a four-dimensional space–time to a five-dimensional one. We will see that this leads us to a new identity. By using this identity, the new 5D gravitational equation is derived. We give finally our conclusion in section 5.

2. An overview of the geometrical features of a curved space

Riemannian geometry rests essentially on the consequences derived from the displacement element ds in which the related metric tensor is appeared as it follows

$$ds^2 = \mathbf{g}_{\mu\nu} dx^\mu dx^\nu, \;\; \mu, \nu = 0, 1, 2, 3 \tag{1}$$

Riemannian geometry requires a symmetric linear connection coefficient, Γ, in which the infinitesimal parallel transport of a vector always preserves the length of the vector. These coefficients have been called Christoffel symbols and defined by [7]:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\mathbf{g}^{\lambda\sigma}\left(\partial_\mu \mathbf{g}_{\nu\sigma} + \partial_\nu \mathbf{g}_{\sigma\mu} - \partial_\sigma \mathbf{g}_{\mu\nu}\right) \tag{2}$$

The Christoffel symbols are used to determine the parallel transport of vector A_μ:

$$A_\mu \to A_\mu + \Gamma^\lambda_{\mu\nu} dx_\lambda A^\nu \tag{3}$$

where dx is related to the vector displacement of A_μ. The Christoffel symbols measure the curvature of the coordinate axes. The covariant derivative is then defined as:

$$D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda \tag{4}$$

This derivative results from the comparison of a vector at coordinate x with its parallel transformed one at $x + dx$. The equations that are written in terms of the covariant derivatives preserve their physical properties under gauge transformations.

The curvature of a manifold manifests itself when a vector is transported parallel around a closed path. By displacement around an infinitesimal square loop with elements dx and dy which is plotted in Fig. 1, the difference between the initial gauge field vector A^μ and the transported vector A'^μ, using the Taylor expansion, is given by [5,8]:

$$\Delta A^\mu = A'^\mu - A^\mu = -[D_\sigma, D_\lambda]^\mu_\rho \, A^\rho dx^\sigma dy^\lambda = R^\mu_{\rho\sigma\lambda} A^\rho \Delta S^{\sigma\lambda} \tag{5}$$

where $\Delta S^{\sigma\lambda}$ represents the area of the performed rectangle, and the curvature tensor $R^\mu_{\rho\sigma\lambda}$ is given by

$$R^\mu_{\rho\sigma\lambda} = \partial_\sigma \Gamma^\mu_{\lambda\rho} - \partial_\lambda \Gamma^\mu_{\sigma\rho} + \Gamma^\mu_{\sigma\nu}\Gamma^\nu_{\lambda\rho} - \Gamma^\mu_{\lambda\nu}\Gamma^\nu_{\sigma\rho} \tag{6}$$

So ΔA^μ is non-zero if and only if the space is intrinsically curved. Now if A'^μ moves in the opposite direction along the path to reach the point where A^μ exists, we are expecting the difference between A^μ and A''^μ to be zero, i.e.

$$\left(R^\mu_{\rho\sigma\lambda} + R^\mu_{\rho\lambda\sigma}\right) A^\rho \Delta S^{\sigma\lambda} = 0 \tag{7}$$

This leads to an anti-symmetric nature of the curvature tensor, which is confirmed by Eq. (6) [7]:

$$R^\mu_{\rho\sigma\lambda} = -R^\mu_{\rho\lambda\sigma} \tag{8}$$

This is the first property of the curvature tensor. This tensor is called the Riemann tensor.

We now consider a 3-dimensional closed path and transport the vector A^μ through the path. The path consists of circuits round the top and bottom face of the box with elements dx and dy, which are connected by another element of the cube with length dz (see the right panel of Fig. 1). By moving the vector from the starting point and transport it around the closed path, the vector will be changed by [8]:

$$\Delta A^\mu = \left(D_\rho R^\mu_{\nu\sigma\lambda}\right) A^\nu \Delta V^{\sigma\lambda\rho} \tag{9}$$

where $\Delta V^{\sigma\lambda\nu} = dx^\sigma dy^\lambda dz^\nu$ is the volume of the box. The concerned cube contains clearly another two similar circuits enclosing the two other pairs of faces. If we transport the vector along these two paths, the following result for the new transported vector is obtained [5]:

$$\Delta A'^\mu = \left(D_\rho R^\mu_{\nu\sigma\lambda} + D_\sigma R^\mu_{\nu\lambda\rho} + D_\lambda R^\mu_{\nu\rho\sigma}\right) A^\nu \Delta V^{\sigma\lambda\rho} \tag{10}$$

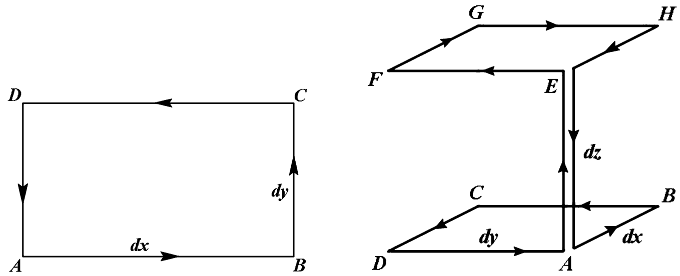

Fig. 1. Two- and three-dimensional closed paths. These two figures have been taken from Ref. [5].

Obviously, the total path traverses each side of the box as many times in one direction as the opposite one, so the path is equivalent to its own reverse, and consequently the vector A^{μ} would not be changed, and so $\Delta A'^{\mu} = 0$, and hence [5]

$$D_{\lambda} R_{\rho\sigma\mu\nu} + D_{\rho} R_{\sigma\lambda\mu\nu} + D_{\sigma} R_{\lambda\rho\mu\nu} = 0 \tag{11}$$

This is known as the Bianchi identity. This constraint is related to the features of the Riemann tensor, which was obtained just by considering the geometrical feature of space–time.

3. Bianchi identity in five-dimensional space–time

We are now going to generalize the method employed in the previous section to five-dimensional (5D) space–time. For simplicity, we make the new extra-dimension non-compact, like the other three spatial dimensions. Since we are concerned about the very small hyper-volume of space, the result can be generalized to a space with higher dimensions. Therefore, the new metric tensor in 5D metric is given by:

$$\mathrm{d}s^2 = \mathbf{g}_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}, \ \mu, \nu = 0, 1, 2, 3, 4 \tag{12}$$

The functional form of the covariant derivative will not be changed, and is:

$$D_{\mu} A^{\nu} = \partial_{\mu} A^{\nu} + \Gamma^{\nu}_{\mu\lambda} A^{\lambda}, \ \mu, \nu, \lambda = 0, 1, 2, 3, 4 \tag{13}$$

The four-dimensional analog of the cube in space coordinates is a tesseract, which consists of 8 cubical cells. So for our closed path in 4D, we consider two connected 3D cubical cells, whose extra-dimension is denoted by w. According to what we have already done with a three-dimensional cube, for the four-dimensional tesseract we should choose cube pair paths, while a transported vector is considered along them. By moving the vector field from the starting point and transport it parallel around the closed path, it will be changed into

$$A'^{\mu}_{A} = \left(1 - \mathrm{d}w^{\lambda} D_{\lambda}\right) \left(1 - \Delta V^{\eta\nu\rho} D_{\eta} R^{\mu}_{\sigma\nu\rho}\right) \left(1 + \mathrm{d}w^{\eta} D_{\eta}\right) \left(1 - \Delta V^{\eta\nu\rho} D_{\eta} R^{\mu}_{\sigma\nu\rho}\right) A^{\sigma}_{A} \tag{14}$$

After doing some algebra manipulations, we will arrive at

$$A'^{\mu}_{A} = \left(1 - \Delta T^{\lambda\eta\nu\rho} D_{\lambda} D_{\eta} R^{\mu}_{\sigma\nu\rho}\right) A^{\sigma}_{A} \tag{15}$$

or

$$\Delta A^{\mu} = \left(D_{\eta} D_{\rho} R^{\mu}_{\nu\sigma\lambda}\right) A^{\nu} \Delta T^{\sigma\lambda\rho\eta} \tag{16}$$

where $\Delta T^{\sigma\lambda\rho\eta}$ is the hyper-volume of the performed box in 5D space–time. A tesseract is comprised of four cube pairs paths, so the whole path leads to

$$D_{\eta} D_{\lambda} R_{\rho\sigma\mu\nu} + D_{\lambda} D_{\rho} R_{\sigma\eta\mu\nu} + D_{\rho} D_{\sigma} R_{\eta\lambda\mu\nu} + D_{\sigma} D_{\eta} R_{\lambda\rho\mu\nu} = 0 \tag{17}$$

This new identity can be considered as the improved Bianchi identity in 5D space–time. Substituting the Riemann tensor into the left-hand side of Eq. (17) automatically confirms its right-hand side (see Appendix A). This confirms the validity of the obtained Eq. (17). We try to employ this equation in the next section to achieve the gravitational field equations in 5D space–time.

4. Gravitational field equations in five-dimensional space–time

Einstein's tensor as an outstanding equation in general relativity can be obtained by twice contracting the Bianchi identity, Eq. (11). So we can write:

$$\begin{aligned} & \mathbf{g}^{i\sigma} \mathbf{g}^{j\lambda} \left(D_{\lambda} R_{\rho\sigma ij} + D_{\rho} R_{\sigma\lambda ij} + D_{\sigma} R_{\lambda\rho ij}\right) = 0 \\ & \Rightarrow D^{i} R_{\rho i} - D_{\rho} R + D^{j} R_{\rho j} = 0 \\ & \Rightarrow D^{i} \left(R_{\rho i} - \tfrac{1}{2} R \mathbf{g}_{i\rho}\right) = 0 \end{aligned} \tag{18}$$

where $\sigma, \lambda, \rho, i, j = 0, 1, 2, 3$. The Einstein tensor is then defined by

$$G_{ij} = R_{ij} - \frac{1}{2}\mathbf{g}_{ij}R \tag{19}$$

The Einstein tensor G_{ij}, which is constructed from the Riemann metric and the Ricci tensor, does not have any divergence:

$$D^i G_{ij} = 0 \tag{20}$$

By considering the conservation law of energy and momentum, i.e.

$$D^i T_{ij} = 0 \tag{21}$$

we can write

$$G_{ij} = \kappa_{4\text{D}} T_{ij} \tag{22}$$

or

$$R_{ij} - \mathbf{g}_{ij}\frac{1}{2}R = \kappa_{4\text{D}} T_{ij} \tag{23}$$

where $\kappa_{4\text{D}}$, as the proportional coefficient, is Einstein's gravitational constant in four-dimensional space–time. Eq. (23) is known as Einstein's field equation, which is the basic equation in theory of general relativity [7].

The general form of Einstein's field equation can be written as follows:

$$R_{ij} - \mathbf{g}_{ij}\frac{1}{2}R + \Lambda \mathbf{g}_{ij} = \kappa_{4\text{D}} T_{ij} \tag{24}$$

where Λ is the cosmological constant that is used to justify the expansion of the universe [7].

Let us now apply the contraction to the improved Bianchi identity, Eq. (17), which leads us to (see details of the calculation in Appendix B):

$$D^\mu D^\nu \left(R_{\mu\nu} - \frac{1}{3}R\mathbf{g}_{\mu\nu}\right) = 0, \quad \mu, \nu = 0, 1, 2, 3, 4 \tag{25}$$

Now, according to the procedure that leads us to the Einstein equation above, we can choose a second-rank tensor that is proportional to the bracket term in Eq. (25). The second covariant derivative of this stress tensor should be zero, and we call it $\Lambda_{\mu\nu}$. Also, in a way similar to the one in Eq. (24), we can add this stress tensor to the concerned second-rank tensor in Eq. (25). Therefore, we can write the field equations derived from Eq. (25) as

$$R_{\mu\nu} - \frac{1}{3}R\mathbf{g}_{\mu\nu} + \Lambda_{\mu\nu} = \kappa_{5\text{D}} T_{\mu\nu} \tag{26}$$

where

$$D^\nu D^\mu \Lambda_{\mu\nu} = 0 \tag{27}$$

and $\kappa_{5\text{D}}$ is proportional coefficient that suppose to be a constant. By choosing

$$\Lambda_{\mu\nu} = -\frac{1}{6}R\mathbf{g}_{\mu\nu} + \Lambda \mathbf{g}_{\mu\nu} \tag{28}$$

Eq. (26) will turn into Einstein's equation in 5D, i.e.

$$R_{\mu\nu} - \frac{1}{2}R\mathbf{g}_{\mu\nu} + \Lambda \mathbf{g}_{\mu\nu} = \kappa_{5\text{D}} T_{\mu\nu} \tag{29}$$

But, according to Eq. (27), considering the fact that the covariant derivative of the metric tensor is always zero, the Ricci scalar in five-dimensional space–time should now satisfy the following relation:

$$\Box_{5\text{D}} R = 0 \tag{30}$$

where $\Box_{5\text{D}}$ stands for the 5D d'Alembertian operator that contains the second covariant derivative ($\Box = D^\mu D_\mu$).

We now say that Eq. (30) can be considered as a constraint on the curvature of 5D space–time when we try to generalize the 4D Einstein equation to higher dimensions, i.e. Eq. (26). The obtained equation, Eq. (26), with the supposed tensor $\Lambda_{\mu\nu}$ as in Eq. (28), is completely similar to the 4D Einstein field equation. Investigating the general form of the assumed second-rank tensor, whose second-order covariant derivative is zero and also satisfies Eq. (26), can be considered as an open research area.

In summary, using the improved Bianchi identity, we derive a gravitational field equation that contains the second covariant derivatives as in Eq. (25). This equation satisfies automatically the Einstein field equation in 4D space–time. But it additionally contains an extra feature, which is related to the constraint, imposed on Ricci's scalar by Eq. (30). The Ricci scalar in 4D space–time is in fact a quantity that provides us with a specified curvature for the universe, but appears as a scalar field in 5D space–time, which can describe a universe with a changing curvature. Other features of Eq. (29), which obviously contain more information than the Einstein's equation in 4D space–time, can be considered as a new research area in future.

5. Conclusion

A curved space–time may be defined by measuring the change of a vector when a parallel transport is occurring around an infinitesimal closed loop. We began with three-dimensional space–time, where, by transporting a vector around an infinitesimal square loop, we could extract the antisymmetric nature of the Riemann tensor. In 4D space–time, the same method led us to the Bianchi identity.

A second-order differential identity for the Riemann tensor was obtained as the improved Bianchi identity, when we applied parallel transportation to five-dimensional space–time. By doing contractions on the new identity, we could define the gravitational field equations in 5D space–time. As we saw, the new 5D field equations that arise from the geometrical feature of space–time imply a constraint on Ricci's scalar. On the other hand, by a proper choice of the stress tensor, we arrived at a field equation in 5D space–time whose apparent form is similar to Einstein's field equation 4D space–time. Investigating the features of the new gravitational field, Eq. (29), can be considered as a research task in future.

Appendix A

We intend here to evidence that the improved Bianchi identity, which was obtained by a geometrical interpretation of 5D space–time, is satisfied by the Riemann tensor. For this purpose, by considering the second-order covariant derivative on the Riemann tensor, evaluated in locally inertial coordinates, we can write:

$$D_\eta D_\lambda R_{\rho\sigma\mu\nu} = \partial_\eta \partial_\lambda R_{\rho\sigma\mu\nu} = \frac{1}{2}\partial_\eta \partial_\lambda \left(\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\nu \partial_\rho g_{\mu\sigma}\right) \tag{31}$$

One has to notice that using the inertial coordinates does not impose any limitation on our calculations because the terms that we are neglecting are all proportional to $\partial_\sigma g_{\mu\nu}$, and therefore automatically vanish [7].

So the four terms in Eq. (17) appear as:

$$\begin{aligned}
\partial_\eta \partial_\lambda R_{\rho\sigma\mu\nu} &= \tfrac{1}{2}\left(\partial_\eta \partial_\lambda \partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\eta \partial_\lambda \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\eta \partial_\lambda \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\eta \partial_\lambda \partial_\nu \partial_\rho g_{\mu\sigma}\right)\\
\partial_\lambda \partial_\rho R_{\sigma\eta\mu\nu} &= \tfrac{1}{2}\left(\partial_\lambda \partial_\rho \partial_\mu \partial_\eta g_{\sigma\nu} - \partial_\lambda \partial_\rho \partial_\mu \partial_\sigma g_{\nu\eta} - \partial_\lambda \partial_\rho \partial_\nu \partial_\eta g_{\sigma\mu} + \partial_\lambda \partial_\rho \partial_\nu \partial_\sigma g_{\mu\eta}\right)\\
\partial_\rho \partial_\sigma R_{\eta\lambda\mu\nu} &= \tfrac{1}{2}\left(\partial_\rho \partial_\sigma \partial_\mu \partial_\lambda g_{\eta\nu} - \partial_\rho \partial_\sigma \partial_\mu \partial_\eta g_{\nu\lambda} - \partial_\rho \partial_\sigma \partial_\nu \partial_\lambda g_{\eta\mu} + \partial_\rho \partial_\sigma \partial_\nu \partial_\eta g_{\mu\lambda}\right)\\
\partial_\sigma \partial_\eta R_{\lambda\rho\mu\nu} &= \tfrac{1}{2}\left(\partial_\sigma \partial_\eta \partial_\mu \partial_\rho g_{\lambda\nu} - \partial_\sigma \partial_\eta \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\sigma \partial_\eta \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\eta \partial_\nu \partial_\lambda g_{\mu\rho}\right)
\end{aligned} \tag{32}$$

Therefore, the sum of cyclic permutations of the first four indices in the Riemann tensor lead us to:

$$\begin{aligned}
&D_\eta D_\lambda R_{\rho\sigma\mu\nu} + D_\lambda D_\rho R_{\sigma\eta\mu\nu} + D_\rho D_\sigma R_{\eta\lambda\mu\nu} + D_\sigma D_\eta R_{\lambda\rho\mu\nu} =\\
&\tfrac{1}{2}(\partial_\eta \partial_\lambda \partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\eta \partial_\lambda \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\eta \partial_\lambda \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\eta \partial_\lambda \partial_\nu \partial_\rho g_{\mu\sigma}\\
&+ \partial_\lambda \partial_\rho \partial_\mu \partial_\eta g_{\sigma\nu} - \partial_\lambda \partial_\rho \partial_\mu \partial_\sigma g_{\nu\eta} - \partial_\lambda \partial_\rho \partial_\nu \partial_\eta g_{\sigma\mu} + \partial_\lambda \partial_\rho \partial_\nu \partial_\sigma g_{\mu\eta}\\
&+ \partial_\rho \partial_\sigma \partial_\mu \partial_\lambda g_{\eta\nu} - \partial_\rho \partial_\sigma \partial_\mu \partial_\eta g_{\nu\lambda} - \partial_\rho \partial_\sigma \partial_\nu \partial_\lambda g_{\eta\mu} + \partial_\rho \partial_\sigma \partial_\nu \partial_\eta g_{\mu\lambda}\\
&+ \partial_\sigma \partial_\eta \partial_\mu \partial_\rho g_{\lambda\nu} - \partial_\sigma \partial_\eta \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\sigma \partial_\eta \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\eta \partial_\nu \partial_\lambda g_{\mu\rho}\\
&+ \partial_\sigma \partial_\eta \partial_\mu \partial_\rho g_{\lambda\nu} - \partial_\sigma \partial_\eta \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\sigma \partial_\eta \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\eta \partial_\nu \partial_\lambda g_{\mu\rho}) = 0
\end{aligned} \tag{33}$$

This result, as we were expecting, confirms the validity of the improved Bianchi identity, i.e. Eq. (17).

Appendix B

Here we are going to evidence how we can extract the second derivative in Eq. (25). The improved Bianchi identity in 5D space–time is:

$$D_\eta D_\lambda R_{\rho\sigma\mu\nu} + D_\lambda D_\rho R_{\sigma\eta\mu\nu} + D_\rho D_\sigma R_{\eta\lambda\mu\nu} + D_\sigma D_\eta R_{\lambda\rho\mu\nu} = 0 \tag{34}$$

Applying the contraction to this equation will lead us to:

$$\begin{aligned}
&g^{\rho\mu} g^{\sigma\nu}\left(D_\eta D_\lambda R_{\rho\sigma\mu\nu} + D_\lambda D_\rho R_{\sigma\eta\mu\nu} + D_\rho D_\sigma R_{\eta\lambda\mu\nu} + D_\sigma D_\eta R_{\lambda\rho\mu\nu}\right) = 0\\
&\Rightarrow -D_\eta D_\lambda R + D_\lambda D^\mu R_{\eta\mu} + D^\mu D^\nu R_{\eta\lambda\mu\nu} + D^\nu D_\eta R_{\lambda\nu} = 0\\
&\Rightarrow g^{\eta\lambda}\left(-D_\eta D_\lambda R + D_\lambda D^\mu R_{\eta\mu} + D^\mu D^\nu R_{\eta\lambda\mu\nu} + D^\nu D_\eta R_{\lambda\nu}\right) = 0\\
&\Rightarrow D^\eta D^\mu R_{\eta\mu} + D^\mu D^\nu R_{\mu\nu} + D^\nu D^\lambda R_{\nu\lambda} - D^\lambda D_\lambda R = 0
\end{aligned} \tag{35}$$

By using the fact that all the indices are dummy, it is obvious that the first three terms in the last row of Eq. (35) are equal to each other; thus we arrive at:

$$D^\mu D^\nu \left(R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R\right) = 0 \tag{36}$$

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