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Geometric aspects of ordering phenomena

Aspects géométriques des phénomènes d'ordre

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ABSTRACT

A macroscopic system prepared in a disordered phase and quenched across a second-order phase transition into an ordered phase undergoes a coarsening process whereby it orders locally in one of the equilibrium states. The study of the evolution of the morphology of the ordered structures in two dimensions has recently unveiled two interesting and generic features. On the one hand, the dynamics first approach a critical percolating state via the growth of a new lengthscale and satisfying scaling properties with respect to it. The time needed to reach the critical percolating state diverges with the system size, though more weakly than the equilibration time. On the other hand, once the critical percolating structures established, the geometrical and statistical properties at larger scales than the one established by the usual dynamic growing length remain the ones of critical percolation. These observations are common to different microscopic dynamics (single spin flip, local and non-local spin exchange, voter) in pure or weakly disordered systems. We discuss these results and we refer to the relevant publications for details.

RÉSUMÉ

Si un système macroscopique préparé dans une phase désordonnée est refroidi brusquement à une température inférieure à celle où, à l'équilibre, il y a une transition du second ordre, il subit alors un processus de coarsening au cours duquel il prend localement l'une des structures ordonnées stables à l'équilibre. L'étude de l'évolution de la morphologie des structures ordonnées en deux dimensions a récemment révélé deux propriétés génériques intéressantes. D'une part, la dynamique approche d'abord un état critique de percolation grâce à la croissance d'une nouvelle échelle de longueur, et vérifie des relations d'échelle vis-à-vis de celle-ci. Le temps nécessaire pour rejoindre l'état critique de percolation diverge avec la taille du système, moins faiblement que le temps nécessaire pour atteindre l'équilibre. D'autre part, après avoir atteint l'état critique de percolation, les propriétés géométriques et statistiques aux échelles plus longues que la longueur dynamique de croissance habituelle demeurent celles de la percolation critique. Ces observations sont communes aux différents types microscopiques de dynamique (retournement de spin simple, échange de spin local ou non, électeur) dans les systèmes purs ou faiblement

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désordonnés. On discute ces résultats et on renvoie aux publications originales pour davantage de détails.

1. Introduction

Understanding the out-of-equilibrium evolution of collective phenomena in complex systems is a hard task. In many interesting cases of current interest, such as glassy materials, there is no clear comprehension of the mechanisms whereby these systems progress and, concomitantly, which are the microscopic rearrangements that lead to the slow but steady approach to equilibrium, when this asymptotic state is possible. There are, however, some lucky out of equilibrium relaxing systems in which the time-dependent microscopic configurations can be followed with numerical simulations or experimental techniques and simple mechanisms can thus be identified. Consequently, a better understanding of their dynamics can be reached. Among these are the systems that coarsen [1–6].

Coarsening or phase ordering kinetics is the process whereby an open system orders locally in its equilibrium states until the maximal order compatible with thermal fluctuations, conservation laws and boundary conditions is achieved. This phenomenon occurs when a macroscopic system is taken across a continuous phase transition by changing (abruptly, or slowly though not quasi-statically) one of its parameters or the environmental conditions. In this note, we focus on classical systems though, in principle, similar questions could be asked in quantum problems. Typical examples of coarsening systems are magnets taken from their paramagnetic to their ordered phase or, say, water and oil mixtures taken into demixing conditions.

The characterisation of coarsening has been largely circumscribed to the one of the space–time correlation [1–4,6] and linear response [4,7,8] functions, while the direct analysis of the morphology of the spatial structures has remained, in comparison, much less developed. However, this is a problem of fundamental but also practical interest since, from a materials science point of view, not only the average grain size but also its stochastic counterpart, the size distribution function, influences many material properties.

The morphology of critical *equilibrium* systems has been well characterised by a plethora of studies of the statistical and fractal properties of different kinds of geometrical structures [9–13]. In particular, the distributions of domain sizes, Fortuin–Kasteleyn cluster areas, interfaces lengths, winding angles, etc. are known. In contrast, a similar characterisation of systems evolving *out of equilibrium* after quenches across and to a critical point has not been performed yet, with the exception of two prominent cases. One is the distribution of droplets in Ostwald ripening, the process whereby the droplets of a minority phase diluted in a majority one organise, and has been the focus of much attention since the publication of the celebrated Lifshitz–Slyozov–Wagner theory [14,15]. The other one is the characterisation of small domains in polycrystalline formation [16], magnetic grains [17] soap froths [18,19] and biological tissues [20,21] usually done using kinetic Potts models.

In recent years, an impressive theoretical, numerical and experimental effort has been devoted to the measurement of the *density of defects* left in a system after its slow quench through a second-order phase transition [22] (the defects could be domain walls, vortices, or others depending on the system). However, little is known about the *size distribution, geometric properties and spatial organisation of the defects* inherited from such slow quenches.

A series of works aim at start filling this hole. In this article, I will shortly summarise recent advances in this line of research. I will focus on two-dimensional spin models, where most of the studies have been performed. I will describe results for the ferromagnetic 2d Ising and Potts spin models with microscopic dynamics satisfying detailed balance, and the planar voter model that goes beyond the physically constrained dynamic rules. The effects of quenched disorder will be shortly mentioned. A few words on 3d systems, including spin models and continuous field theories, will also be written. A final discussion section closes the paper with ideas for future research.

1.1. Models

The classical *Ising model* is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j \tag{1}$$

with $J > 0$. The bimodal spin variables take values $s_i = \pm 1$. The energy function of the classical *Potts model* is given by

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{s_i s_j} \tag{2}$$

with $s_i = 0, \ldots, q-1$. In both cases, the sum runs over nearest-neighbours on a finite-dimensional lattice. Both models undergo a second-order phase transition at a finite temperature as long as $2 \leq q \leq 4$ and $d \geq 2$. The Ising model is the

paradigm for ferromagnetic ordering. The Potts model is used to describe soap froths [18,19], polycrystalline formation [16], magnetic grains [17], and cellular tissues [20,21]. The main difference between the two models is that while the former has two degenerate ground states, the latter has q of them.

The term *weak quenched disorder* is used to classify random interactions or fields that do not induce frustration and do not drastically change the nature of the equilibrium states (although they can sometimes kill an ordered phase in sufficiently low dimensions, see the example below). More precisely, weak disorder typically reduces the extent of the ordered phase by reducing the value of the critical temperature. Moreover, it can smoothen a first-order phase transition into a second-order one, or modify the values of the critical exponents in continuous phase transitions. However, if the ordered phase is, say, ferromagnetic and it survives under the quenched randomness, it remains ferromagnetic. Weak randomness is realised, for example, by choosing the exchanges from a probability distribution with positive support, or by using random fields with symmetric distribution around zero. For bimodal variables, these choices correspond to the *random-bond Ising model* and the *random-field Ising model*, respectively. Disordered extensions of the Potts model also exist. (An extreme effect of the random fields is to destroy the ferromagnetic phase of the 2d Ising model.)

Stochastic dynamics mimics the spin flips due to coupling with a thermal bath. They can or cannot conserve the local order parameter. At a microscopic scale, the individual spins are updated. In the former case, only pairs of neighbouring anti-aligned spins can be updated. In the latter case, any spin in the sample can be turned round. The moves are accepted with stochastic rules that satisfy detailed balance ensuring that the dynamics can (but not necessarily do, as we will show) take a finite sample to thermal equilibrium at any temperature. In the numerical simulations, a single spin flip is attempted at each microscopic time step. The flip is accepted with probability

$$1 \text{ for } \Delta E < 0\,, \qquad \mathrm{e}^{-\beta \Delta E} \text{ for } \Delta E > 0\,, \qquad \text{and} \qquad p \text{ for } \Delta E = 0 \tag{3}$$

and, at $T = 0$, no update with $\Delta E > 0$ is allowed. p is a parameter. For $p = 0$, the system can get easily trapped. The behaviour for all $p > 0$ is essentially the same and typical choices are $p = 1/2$ for the heat-bath and Glauber rules and $p = 1$ for the Monte Carlo rule. The unit of Monte Carlo time is defined as N attempts to perform a spin-flip, with N the number of spins in the sample.

At a larger lengthscale, one follows the Ginzburg–Landau paradigm and proposes a stochastic differential equation to describe the time evolution of a local field, that is to say, a coarse-grained description of the spin configuration. This is the stochastic time-dependent Ginzburg–Landau equation which can incorporate, or not, the condition on the conservation of the order parameter. At zero temperature, the equation is deterministic. The analysis of the scalar equation for an Ising model with non-conserved order parameter, around a rather flat interface, leads to the Allen–Cahn result that the motion of any point on a domain wall is driven by the local curvature. The conserved order parameter dynamics of Ising-like systems [23,24] and the Potts model can also be treated with this formalism [25].

Another interesting case is the *voter model* [26], which has no energy function attached to it and whose dynamic rules mimic the spreading of opinions in a society with finite-range interactions: at each microscopic time step, a spin with two possible orientations is chosen at random and its configuration is set to be equal to the one of one of its nearest neighbours, also chosen at random. If the spins were already in agreement, no change would be induced in this time step. If they were in disagreement the chosen spin would flip. This model is essentially out of equilibrium but, in two dimensions and for a finite system size, the evolution approaches an absorbing state with full consensus of one of the two kinds. The dynamics are not driven by curvature in this model, but by interface noise. Another particular feature is that there is no bulk noise, as no spin surrounded by neighbours of the same kind can fluctuate. Generalisations with multi-opinion voters are also possible.

As already said, we focus here on 2d models. Still, we will mention a few 3d analyses. Three dimensional studies are of two kinds. On the one hand, scalar-order-parameter cases (Ising models and phase separating systems) were considered. On the other hand, 3d complex field theories with continuous symmetry breaking, used to describe phase transitions in a host of physical systems including superfluid ^{4}He, type-II superconductors, nematic liquid crystals, magnetic materials, weakly interacting bosons, and the early universe, were also studied. These theories lead to the formation of linear topological defects that can be vortices, disclinations, or cosmic strings, depending on the interpretation of the field and the theory. Indeed, we very recently studied the dynamics engendered by the stochastic Gross–Pitaevskii equation for weakly interacting bosons [27,28] from the point of view of the vortex tangle.

Special care will be given to the study of the effect of the *initial conditions*.

The density of spins of one and another type (only two in Ising systems or more in Potts' models) is an important parameter with strong effects on the dynamics. For rules that do not conserve the order parameter, a finite imbalance in this parameter, say the density of up spins being larger than the one of down spins, drives the system to equilibrium of the positively magnetised equilibrium state very quickly. These initial states are not interesting to study the competition between the growth of both types of order. For rules that do conserve the density of all species, the dynamics of slightly unbalanced initial states is still interesting and we will discuss it below. One can also impose the same number of up and down spins to ensure strict zero initial magnetisation.

Having said this, there remain, basically, two types of initial states:

- with only short-range correlations, as obtained, for example, by sampling equilibrium configurations in the disordered phase;
- with long-range correlations, as, for example, in equilibrium configurations at a critical point.

Table 1
Connectivities, n, critical thresholds, p_c, and exponents for the approach to critical percolation, z_p, after a zero-temperature quench of the Ising model on several two-dimensional lattices evolved with non-conserved order parameter dynamics (see Eq. (12) for its definition). In the case of the bow-tie lattice, $n = 5$ is the average between the connectivities of the kinds of sites with connectivities 4 and 6. (Regularly odd-coordinated lattices are not suited for our study because the evolution rapidly freezes due to metastable droplets. The honeycomb lattice is one such case [34] and this is the reason why an exponential in L time scale is found [35].)

Lattice	n	p_c	z_p
Honeycomb	3	0.70	∞
Kagomé	4	0.65	1/2
Square	4	0.59	1/2
Bowtie-a	5	0.55	0.38(5)
Triangular	6	1/2	1/3

1.2. Geometry

The geometric analysis of critical structures finds its origin in Flory's study of *percolation theory* with the goal of describing the vulcanisation of rubber [29]. It was later very successfully used in the context of *polymer science* [30], and many details are given in the book [31] and in a more recent review article [32].

The *random site percolation problem* is very simply defined. A lattice with typically free (FBC) or periodic (PBC) boundary conditions is chosen. (It is also possible to consider regular or random graphs, but we will not discuss these in this note.) The sites of the lattice are occupied or not with probability p and $1-p$, respectively. A *cluster* is then defined as the ensemble of neighbouring sites occupied by particles. p is a parameter that controls the sizes of the clusters and the possible presence of a spanning or wrapping cluster. It is quite clear that for $p \to 0$, all configurations are void and there is no cluster in any of them, while for $p \to 1$ the full lattice is always occupied by a single cluster. In the limit $L \to \infty$, a sharp threshold at a finite p_c separates a phase with no spanning cluster (or no wrapping cluster for PBC) from one with one spanning cluster (or a wrapping cluster for PBC) of occupied sites for FBC. For finite L, there is only a rounded changeover around p_c. The critical parameter p_c depends on the geometry of the lattice and values for several finite dimensional lattices are given in Table 1. The interest is set in the full characterisation of the number and structure of these clusters, as a function of their size, that is to say, the number of sites belonging to it, as a function of the control parameter p.

The percolation problem is completely geometric. Still, the transition at p_c is a critical phenomenon with features similar to the ones of a physical phase transition. The critical behaviour is characterised by a set of universal critical exponents, which describe, for example, the fractal behaviour of the percolating cluster and the size distribution of the non-percolating ones, independently of the microscopic details of the system.

The value of the percolation threshold decreases for increasing spatial dimensionality. For cubic lattices, $p_c \simeq 0.59$ in 2d and $p_c \simeq 0.32$ in $d = 3$. This implies that the case $p = 1/2$ is below the threshold in 2d for all the lattices in Table 1 apart from the triangular lattice, which is exactly at the percolation threshold, but above the threshold in the cubic lattice in $d = 3$. The case $p = 1/2$ is of special interest in the coarsening context, since it corresponds to a fully random, infinite temperature, initial condition.

An extension of the simple occupied-empty, or black and white, percolation problem generalises to state variables with more than two values. This problem is called *polychromatic percolation* [33]. More precisely, one takes q-valued variables, $i = 0, \ldots, q-1$, and assigns a probability of presence of each of them $p_0,\ p_1, \ldots, p_{q-1}$ with $p_0 + p_1 + \ldots p_{q-1} = 1$. The simplest case is the equiprobable one, $p_0 = p_1 = \ldots = p_{q-1} = 1/q$. This problem has received attention as for lattices with sufficiently low p_c two or more species can percolate simultaneously. For our purposes, we work with equiprobable cases, $p = 1/q$, and in low space dimensionality, $d = 2$, so that $p < p_c$.

Geometric clusters can also be defined in spins models by joining together neighbouring spins with the same orientation. It is known since the 1970s that these are not, however, the relevant structures that describe the equilibrium critical phenomenon of the spin models [36,37]. Instead, in order to describe the thermodynamic phase transition with geometric arguments, the so-called *Fortuin–Kasteleyn* (FK) *clusters* should be constructed from the geometric ones by erasing the bonds linking two occupied sites with probability $p_{FK} = e^{-2\beta J}$, thus reducing their size. In the 2d Ising model, the geometric and FK clusters percolate at the same temperature $T_p = T_c$ but the ones that lead to the correct critical exponents are only the latter. In the 3d Ising model the geometric clusters percolate at $T_p < T_c$ [38] and it is therefore even clearer that they cannot be linked to the thermodynamic critical properties of the model. In 3d, the FK clusters are also the ones that fully capture the thermodynamic critical point.

Another interesting geometric property of a patchwork structure are the interfaces themselves and their lengths. There are several possible definitions of the boundary of a cluster. In various papers, we focused on the hull and the area enclosed by it. The hull is the external border of a domain, constructed by joining sites on the dual lattice with links that cut broken bonds on the original lattice.

Hull-enclosed areas do not have holes while domains can have. By simple visual inspection of the instantaneous configurations, one can see that in the 2d Ising model hull-enclosed areas and domains differ significantly, while for increasing values of q the presence of holes diminishes, and already for $q \geq 3$, there is very little difference between the two observables.

For any reasonable definition of volume, surface and length of a geometric compact object embedded in 3d, with regular areas and (closed) interfaces, these should scale as

$$V \simeq R^3 \qquad A \simeq R^2 \qquad \ell \simeq R \tag{4}$$

with R the radius of the smallest sphere that contains the loop and the length ℓ being larger than the microscopic lengthscale l_0. These relations are easily extended to objects embedded in d dimensions. For example, a hull-enclosed area in 2d is compact, therefore it scales as $A \simeq R^2$.

At a critical point, be it thermodynamic or geometric, the critical objects are fractal and their volume, surface and length will be characterised by fractal exponents,

$$V \simeq R^{D_V} \qquad A \simeq R^{D_A} \qquad \ell \simeq R^{D_\ell} \tag{5}$$

As an example, a domain and its interface at the critical 2d Ising point are not compact. Its area and lengthscale as above and

$$A \simeq \ell^{D_A/D_\ell} \tag{6}$$

In a dynamic situation, the reference linear scale R is a growing function of time, $\xi_d(t)$. The observed length and areas should then be scaled with the growing length and growing areas (including, possibly, some fractal exponents), to describe the area–length relations and also the time-dependent number densities that we discuss below.

Take a geometric object with a given property (area, length, etc.) generically called x, which takes values X. For large but not necessarily infinite L, close to criticality, its probability distribution per lattice site or number density, $n_x(X)$, takes the form:

$$n_x(X) \simeq X^{-\alpha_x} f_x(X^{\sigma_x}(u - u_c)) + n_{\rm fs}(X/L^{D_x}) \tag{7}$$

We called u the control parameter that could be p for percolation or T for a statistical physics problem. The quantity $m_x = (u - u_c)^{1/\sigma_x}$ is sometimes called the 'mass' and α_x the 'Fisher exponent'. The fractal dimension D_x and α_x are linked by $D_x = d/(\alpha_x - 1)$, with d the space dimension. The scaling function, f_x, at criticality, takes a constant value $f_x(0) = c_x$, which depends on the quantity x. The first contribution is the only one that survives in the infinite size limit while the second one controls the finite-size corrections.

In the infinite system size limit, the number density of hull-enclosed areas at critical percolation is

$$n_{\rm h}(A) \simeq \frac{2c_{\rm h}^{(2)}}{A^2} \tag{8}$$

The value of $c_{\rm h}^{(2)}$ is known exactly: $c_{\rm h}^{(2)} = 1/(16\sqrt{3}\pi)$ [39].

At the critical point of the Potts model with $2 \leq q \leq 4$, which includes the $q = 2$ Ising case, $n_{\rm h}(A)$ is [39]

$$n_{\rm h}(A) \simeq \frac{(q-1)c_{\rm h}^{(q)}}{A^2} \tag{9}$$

with the same $c_{\rm h}^{(2)} = 1/(16\sqrt{3}\pi)$, while the values of $c_{\rm h}^{(q)}$ for $2 < q \leq 4$ are known only numerically and are very close to the one for $q = 2$. Note the factor 1/2 with respect to the critical percolation point, and the fact that the power is the same.

The geometric domains at the critical Ising and Potts points are distributed in a similar way,

$$n_{\rm d}(A) \simeq \frac{(q-1)c_{\rm d}^{(q)}}{A^{\alpha_{\rm d}}} \tag{10}$$

The values of the exponent $\alpha_{\rm d}$, and consequently the fractal dimensions, have been elucidated in the 1980s and are summarised in Table 2 for a number of interesting cases. The values of $c_{\rm d}^{(q)}$ are known only numerically, and some approximate relations to $c_{\rm h}^{(q)}$ have been discussed in [40].

The interfaces are the domain walls between clusters of different kinds. In models with bimodal variables, they must be closed loops for PBCs. For FBCs, they are closed loops in the bulk or they may start and end at the system's border. In the Potts model, the interfaces can have a beginning and an end even with PBCs. For $q = 3$, such interfaces necessarily terminate at T-junctions, while three or more interfaces can meet for $q \geq 4$. T-junctions pin the dynamics at zero temperature, since each of the spins at the interface has at least two neighbours of the same species and cannot flip.

At the percolation threshold, the probability of there being (i) a cluster that percolates in both the two Cartesian directions, (ii) one or more clusters that percolate in the horizontal or the vertical direction, (iii) one or more clusters that percolate diagonally are known exactly (for different boundary conditions) [41–43]. These observables are very useful to decide whether a dynamic configuration has come close to critical percolation or not.

Table 2
Fisher exponent of a number of critical structures for the Potts model with $q = 1$ (percolation), $q = 2$ (Ising).

	$q \to 1$	$q = 2$
Geometric cluster (GC) areas	187/91	379/187
Fortuin–Kasteleyn areas		31/15
Hull-enclosed areas	2	2
GC hulls	15/7	27/11
GC external perimeter	5/2	27/11

2. Dynamics

After a quench below T_c, the systems tend to order, the number of domains steadily decreases and the average linear size of the remaining ones steadily increases. Indeed, from the simple visual inspection of the time-evolving configurations, one observes that there is a growing length $\xi_d(t)$, which, *grosso modo*, corresponds to the average linear size of the *finite-size ordered patches*. The *dynamic scaling hypothesis* states that this is the unique growing length that controls the scaling of all correlation functions and observables in these systems.

The time dependence of $\xi_d(t)$ serves to classify different dynamic universality classes determined by the dimension of the order parameter and the conservation laws. In clean systems (i.e. with no quench disorder)

$$\xi_d(t) \simeq t^{1/z_d} \tag{11}$$

with $z_d = 2$ for curvature-driven grain growth and interface-controlled Ostwald ripening, and $z_d = 3$ for phase separation and diffusion-controlled Ostwald ripening. These values have been derived analytically for the continuous description of coarsening with non-conserved [44] and conserved [23,24] scalar-order parameter dynamics, respectively. The relaxation dynamics of the Potts model with non-conserved order parameter is also expected to satisfy dynamic scaling with respect to a growing length $\xi_d(t) \simeq \lambda_q t^{1/2}$ [18]. However, this system gets easily trapped in very long-lived configurations at low temperatures. Therefore, although a fully ordered state will eventually be reached at finite temperature, the process to actually achieve it is quite complex [45]. For the voter model in 2d, $\xi_d(t) \simeq t^{1/z_d}$ with $z_d = 2$ [26,46], but there are logarithmic violations of dynamic scaling. In cases with weak disorder, $\xi_d(t)$ is expected to grow logarithmically [47] although this regime is attained at very long times only.

In systems with an energy function, the growing length is usually measured from the time-dependent excess energy with respect to the ground-state one, E_0. In a simulation, the extracted function $\ell_G(t) = E_0/[E_0 - E(t)]$ does not quickly reach the expected stable power law t^{1/z_d} for a clean system or $(\ln t)^{1/\psi}$ in a weakly disordered one, but instead it is described by a time- and/or a temperature-dependent and/or disorder-dependent effective exponent $t^{1/z_{\rm eff}}$ [35,48–51]. This is a fact to be taken into account in the dynamic scaling analysis.

This discussion assumes that there is a single growing length $\xi_d(t)$, controlling the full dynamics of the system. However, as we will show, after the quench another process takes place, one of aggregation in which these systems approach critical percolation, with a different time-dependent growing length $\xi_p(t)$ that has to be taken into account in all scaling arguments.

2.1. Disordered initial states

An initial condition taken from equilibrium in the disorder phase has a vanishing order parameter and short-range correlations with a microscopic correlation length unless T_0 is very close to T_c. There is no spanning cluster of parallelly aligned spins in it. The fate of this initial state under the short-time evolution is the subject of this section.

2.1.1. Approach to critical percolation

Evidence for standard critical percolation influencing the scaling regime and asymptotic states reached by the 2d square-lattice Ising model after a quench from high to low temperature was first given in [52,40]. These papers show that after a very short time span, a few Monte Carlo (MC) steps for the simulated cells, the dynamic configurations look like the ones at critical percolation. More precisely, the morphological and statistical properties of structures such as the areas of the domains or the lengths of the interfaces, which are larger than the average ones, $\xi_d^2(t)$ and $\xi_d(t)$, respectively, are the ones of random percolation at its threshold. As the occupation probability for up and down spins in a high-temperature equilibrium configuration on a square lattice is smaller than the one for critical uncorrelated site percolation, this fact suggests that the system must have reached critical percolation at some point of its evolution while it was building correlations. It is worth stressing that this critically is not the one of the thermodynamic critical point of the model.

A detailed analysis of the time needed to reach the critical percolation state after quenches from infinite to zero temperature single-spin flip stochastic dynamics (non-conserved order parameter) in the 2d Ising model on different lattices was performed in [53,35]. Several criteria lead to the predictions in the third column of Table 1 for the exponent characterising the algebraic divergence of this time scale with the linear size of the system

$$t_p \simeq L^{z_p} \tag{12}$$

independently of the boundary conditions, and with a prefactor of the order of one. We do not know whether the fractions are exact results; we only measure numerical values that are very close to them. This law was found from: (1) the study of the overlap between two copies of the system, $Q(t, t_w)$, made at a time t_w scaled with L in such a way to find a non-zero asymptotic value; (2) the scaling of the size of the largest geometric cluster area, its hull-enclosed area and their interfaces; (3) the distribution of the largest geometric cluster and hull-enclosed areas; (4) the behaviour of the spanning or wrapping probabilities, depending on the boundary conditions; (5) the average of the square-winding angle of various interfaces. In [53], we conjectured, on the basis of the study of the four last lattices in Table 1, that

$$z_p = \frac{z_d}{n} \tag{13}$$

or at least that the exponent $1/z_p$ should increase for faster dynamics (smaller z_d) and for increasing coordination of the lattice. (The honeycomb lattice is special as it allows for the existence of blocked domains with finite size at zero temperature [34].)

The fact that the conserved order parameter dynamics of an Ising model, with spin exchange stochastic updates à la Kawasaki, also approaches critical percolation was already observed in [54]. A similar study on segregating binary Bose–Einstein condensates arrived at the same conclusion [55]. We recently presented the analysis of t_p in the scalar conserved order parameter universality class [56]. This study suggests that in systems with such slow dynamics, the effective exponent in the dynamic growing length should be used to characterise the growing length towards critical percolation $\xi_p(t)$. More precisely, if the usual dynamic growing length measured, for example, from the excess energy with respect to the equilibrium one, is $\xi_d(t)$, then

$$\xi_p(t) = \xi_d^n(t) \tag{14}$$

The study of n on different lattices [56] suggests that it is a lattice-dependent parameter, but a clear link with the lattice connectivity is harder to assure for these dynamics.

We have not seen any important influence of temperature on the time scale t_p of the Ising model [35]. At the other end of the evolution, for very long times, temperature fluctuations quite naturally shorten the life-time of striped metastable states allowing the system to eventually reach equilibrium with the environment in a time scale t_{eq} that also diverges with the system size and $t_p \ll t_{eq}$.

The approach to critical percolation is not exclusive of physical dynamics respecting detailed balance. In the voter model this phenomenon also exists but it is achieved in a much longer time scale than in the Ising model. We found with numerical simulations on the square lattice [57]

$$t_p \simeq L^{1.66} \tag{15}$$

One can be tempted to propose that in this case the exponent is $z_p = 3/5$. Note that the connectivity of the lattice is not a relevant parameter for this problem, since the update of a spin is decided by only one of its neighbours, independently of the geometry of the lattice, and there is no energy function. In this sense, one could expect $n = 1$ for all lattices, and an exponent close to 2 in (15) seems then natural. We have checked the independence of (15) on the lattice geometry for the voter model [35]. (Note that the growing length in the voter model is $t^{1/2}$ as in the Ising model, but it also violates dynamic scaling with logarithmic factors that may render the numerical evaluation of ξ_p difficult.)

As regards the role of critical percolation in the Potts model, a completely random initial condition in which each spin takes one of the q possible states with probability $1/q$ is quite far away from the percolation threshold. The size of the largest cluster steadily increases in time but, at least within the time window explored in [58,59], no cluster spans the system area. We plan to study this problem in more detail in the future and establish whether there is a t_p and which is its scaling with the system size.

The effects of weak quenched disorder are under study as well [60].

2.1.2. Dynamic scaling

In the asymptotic time domain, when $\xi_d(t)$ has grown much larger than any microscopic length and remains smaller than the system size, $\ell_0 \ll \xi_d(t) \ll L$, a *dynamic scaling symmetry* sets in, similarly to the usual scaling symmetry observed in equilibrium critical phenomena. According to this hypothesis, the growth of $\xi_d(t)$ is the only relevant process and the whole time dependence enters only through $\xi_d(t)$. This implies that the statistical properties should not depend on time provided that distances are measured in units of the dominant length $\xi_d(t)$. Due to this fact, correlations such as $C(r, t) = \langle s_i(t) s_j(t) \rangle$, where $r = |\vec{r}_i - \vec{r}_j|$, take the scaling form

$$C(r, t) \simeq f\left(\frac{r}{\xi_d(t)}\right) \tag{16}$$

where $f(x)$ is a scaling function (different for $T_0 > T_c$ and $T_0 = T_c$ [1]), expressing the fact that there is a unique relevant length in the system.

We have revisited the scaling form (16) to take into account the existence of the length that characterises the approach to percolation [53]. The existence of a different growing correlation, $\xi_p(t) \simeq t^{1/z_p}$, suggests that the usual dynamic scaling

hypothesis needs to be revised to take it into account. More precisely, we proposed to extend the scaling of the space–time correlation function as [53]

$$C(r,t;L) \simeq f\left(\frac{r}{\xi_{\rm d}(t)}, \frac{\xi_{\rm p}(t)}{L}\right) \tag{17}$$

The numerical data for the 2d Ising model with non-conserved order parameter are much better scaled when this extended form is used. The relevance of $\xi_{\rm p}(t)$ in the scaling of the linear response functions should still be evaluated.

The analysis of the scaling properties of the correlations in other systems (conserved order parameter, Potts model, weak-disordered systems) is under way.

Fisher and Huse enunciated another hypothesis, the one of *super-universality*, that states that the scaling functions of the correlation functions after sub-critical quenches should be independent of weak disorder (its strength, distribution, etc.) as long as it does not change the nature of the equilibrium phases [47,1], once the correct growing length is used in the scaling variables. The idea behind this proposal is that the length at which the effects of weak disorder are important should be shorter than the growing length $\xi_{\rm d}(t)$ and, therefore, the mechanism for the dynamics of the large structures should remain the one of the pure system. Favourable numerical evidence for this feature appeared in [61–63,48] although some recent studies tend to falsify the claim [64]. In our view, a more detailed analysis is needed to conclude, taking into account the new growing length $\xi_{\rm p}(t)$ that will be affected by weak quench disorder. This study is also under way [60].

2.2. Critical initial states

Taking a critical initial condition one can then quench the system into an ordered phase or to another critical point. We distinguish these two cases here.

2.2.1. From critical percolation to the ordered phase

The triangular lattice is right at critical percolation for equal concentration of up and down spins. One could then naively suppose that the percolating cluster in the initial configuration will be stable under the zero temperature evolution. This, however, is not the case. Under the dynamics the initial percolating cluster breaks in pieces, to next reform in different places, acquire a thicker structure by building correlations and eventually remains for all subsequent times. A time scale $t_{\rm p} \simeq L^{z_{\rm p}}$ with $z_{\rm p} \simeq 1/3$ is needed to achieve stability under non-conserved order parameter dynamics [53].

2.2.2. From the critical temperature to the ordered phase

If the initial condition is chosen to be one in equilibrium at the critical point, the initial configuration already has long-range correlations and, moreover, spanning or wrapping clusters are the ones of the thermodynamic critical point and exist with known probabilities. There is no $t_{\rm p}$ in this case. The statistical and morphological properties of the time-evolving configurations inherit properties from the initially correlated state and these influence the scaling functions of the correlation functions and other macroscopic observables. They do not, however, change the growing length $\xi_{\rm d}(t)$ that still behaves as $t^{1/z_{\rm d}}$ in clean systems [65].

2.2.3. Quenches between critical points

The equilibrium and out of equilibrium dynamics at a second order critical point are usually characterised in terms of a dynamic critical exponent $z_{\rm c}$ that characterises the relation between the equilibrium correlation length and time, $\xi_{\rm eq} \simeq \tau_{\rm eq}^{1/z_{\rm c}}$ in the former case, and the time dependence of the growing length after critical quenches

$$\xi_{\rm c}(t) \simeq t^{1/z_{\rm c}} \tag{18}$$

in the latter case. The identity of the dynamic critical exponent in equilibrium and after a quench to the critical point is a non-trivial result shown in [66,67]. For the 2d Ising model with non-conserved order parameter dynamics, $z_{\rm c} \simeq 2.17$ [68].

In [69], we studied quenches from one critical point to another one from the geometric point of view. We achieved this process by quenching the 2d Ising model on the triangular lattice from $T_0 \to \infty$, where it is at critical percolation, to the thermodynamic instability temperature, $T_{\rm c} = 4J/\ln 3$. In this way, we demonstrated that the structures that are smaller than the growing length $\xi_{\rm c}(t)$, Eq. (18), acquire the geometric properties of the target $T_{\rm c}$ equilibrium state, while the ones that are larger than $\xi_{\rm c}(t)$ keep the ones dictated by the initial $T_0 \to \infty$ state, i.e. the ones of random critical percolation. In this case, both structures are fractal, though with different fractal dimensions, as the initial and final temperatures are critical.

2.3. Statistics of finite-size geometric objects

Many statistical and geometric properties of finite-size hull-enclosed areas and domain areas in the two-dimensional coarsening of Ising [52,40,54] and Potts [58,59] models with non-conserved and conserved order parameter dynamics have been elucidated with analytic and numerical studies. The effects of weak quenched disorder were considered in [70], and experimental data for liquid crystals were analysed along these lines in [71]. In all these cases, times $t \gg t_{\rm p}$ were considered for quenches from $T_0 \to \infty$.

The scaling properties at times shorter than $t_{\rm p}$ have been recently addressed.

We summarise these results here.

2.3.1. Number density of finite areas and lengths

The distributions of domain lengths in single-spin flip kinetic Ising and Potts chains have been derived analytically in several papers [72–75]. However, the methods used in these works are not adapted to higher dimensions. A mixture of numerical and analytical methods can be used instead.

– $t \gg t_{\rm p}$, percolation established.

Thanks to the dynamic scaling hypothesis, in 2d coarsening the distribution of domain areas relative to their average is expected to reach a *universal function* independent of the initial state, as long as it is one with short-range correlations, since the long-distance properties are controlled by the infinite-temperature 'fixed point'. Other geometric objects (hull-enclosed areas, interface lengths, etc.) should also reach a scaling behaviour with universal scaling functions when measured with respect to their averages. A critical initial state, with long-range correlations, shares the same growing length as the disordered ones, but leads to different scaling functions [1].

From the zero-temperature time-dependent Ginzburg–Landau equation for a scalar field, with no conservation law, Allen and Cahn proved that the local interface velocity is proportional to its local mean curvature [44]. All dynamic processes with this property form the *curvature-driven* dynamic class. Accordingly, in 2d at $T = 0$ the domains cannot break nor merge because the velocities of any points on two nearby interfaces point in the direction that pushes them apart, while in 3d these processes are not prohibited. The Gauss–Bonnet theorem that involves the Gaussian curvature can be used in 2d where an interface has only one independent curvature to derive a series of expressions for the scaling functions of the distribution of various planar and linear geometric objects with finite size, if one knows their initial distribution. This is not possible in 3d where the domain boundaries have two independent curvatures. For the conserved order parameter universality class, the Potts model or the voter model, such an exact approach is not possible but approximate calculations are.

What is then the initial distribution of finite-size structures that should be used as an input in these calculations? From what we discussed in the previous sections it follows that it should be the one of critical percolation after quenches from infinite temperature if the distributions were studied at times $t \geq t_{\rm p}$, and the one of the thermodynamic critical point if the initial state were drawn from equilibrium at $T_{\rm c}$.

Summarising, the number density of *finite-size hull-enclosed areas* after a zero-temperature quench in the 2d non-conserved order scalar parameter universality class has been derived using two previously known results: (1) the exact distribution of these objects at the critical percolation and critical Ising points [39], see Eq. (9), (2) the Allen–Cahn evolution [44] of each area:

$$\frac{{\rm d}A}{{\rm d}t} = -\lambda \tag{19}$$

with λ a non-universal parameter that depends on the model and/or material (and temperature). The functional form is [52, 40]

$$n_{\rm h}(A,t) \simeq \frac{(2)c_{\rm h}}{(A+\xi_{\rm d}^2(t))^2} \tag{20}$$

with the factor 2 present for $T_0 > T_{\rm c}$ and absent for $T_0 = T_{\rm c}$. Indeed, Eq. (20) with the factor 2 is obtained assuming that at $t_{\rm p}$ the system reached critical percolation and using the corresponding distribution in Eq. (9) as the initial one for the evolution. This equation is one of the very few proofs of dynamic scaling for a nontrivial quantity and, as far as I know, the only one for a finite (not uni-) dimensional model with short-range interactions. It shows that short-time dynamics has a long-lasting effect on the morphology of the system, with small and large structures having geometric and statistical properties proper of the target and short-time state, respectively. We later confirmed this observation on a liquid crystal [71], and recent experiments using phase separating glasses (though in 3d) observe a similar phenomenology [76].

A morphological analysis along these lines was later performed on the same 2d model with conserved-order parameter modelling phase separation [54], see also [77,55]. The effect of weak quenched disorder was considered in [70]. We are currently revisiting these two problems [56,35,60] with the aim of computing the time scale $t_{\rm p}$ in these cases.

Concerning the distribution of hull-enclosed areas in the Potts model, one can distinguish the cases $T_0 \to \infty$ and $T_0 = T_{\rm c}$. For $T_0 \to \infty$, the initial state is one of random polychromatic percolation with $p = 1/q$ and it is very far from critical percolation. During the evolution in time scales such that no evidence of percolation exists, the distribution of hull-enclosed cluster areas do not become critical and, as a consequence, do not develop a power-law tail. However, the curves for different times develop an envelope, with slope A^{-2} [58]. This fact was confirmed in [78], where other grain growth models were also considered.

For $T_0 = T_{\rm c}$, some intriguing results can also be derived. The von Neumann law [79] for the individual area evolution depends on the number of sides of the boundary, n_s,

$$\frac{{\rm d}A}{{\rm d}t} = (n_s - 6)\ \lambda \tag{21}$$

but not on their area. Therefore, some areas can increase while other ones decrease in the course of time. Moreover, for $q > 2$ the interfaces can be made of pieces separating the interior from clusters of different colour.

An approximate form for $n_{\rm h}$ can be obtained with the replacement $n_s \mapsto \langle n_s \rangle$, an average number and therefore the same for all cells. Under this assumption, the number density of hull-enclosed areas conserves its initial algebraic critical form and gets advected in time as for the Ising case [58,59]:

$$n_{\rm h}(A,t) \simeq \frac{(q-1)c_{\rm h}^{(q)}}{(A+\xi_{\rm d}^2(t))^2} \tag{22}$$

These results should be relevant to evolutionary game theory on lattices [80], such as models of many species population dynamics of the rock-paper-scissor type [81] and of May–Leonard kind [82].

A specially interesting case is the one of a quench between critical points. The number density of hull-enclosed finite areas scales as [69]

$$n_{\rm h}(A,t) \simeq A^{-\alpha_A}\; g\left(\frac{A}{\xi_{\rm c}^{D_A}(t)}\right) \tag{23}$$

with $\alpha_A = 1 + d/D_A$ and D_A the Fisher exponent and fractal dimension of the target state, i.e. the critical working temperature, and $g(y \to 0) = c_{\rm h}$ and $g(y \to \infty) = 2c_{\rm h}$.

In Ising models, the areas and hulls or external interfaces are in one-to-one relation; from the number density of areas, we have derived the number densities of interface lengths in the various cases discussed above.

– $t < t_{\rm p}$, approach to percolation.

An educated guess based on the results of the regime $t \gg t_{\rm p}$ evaluate $n_{\rm h}(A,t)$ in the form $A^{\alpha_A} n_{\rm h}(A,t)$ against $A/\xi_{\rm p}^{D_A}(t)$ in a variety of models. At short time scales, the good data scaling confirms that $\xi_{\rm p}(t)$ is the dominant growing length until $t_{\rm p}$ is reached. The scaling function is constant at small arguments and it has a weak variation at large arguments.

2.3.2. Area-to-length relations

Finally, in these works we analysed the evolution of the area against perimeters, a relation that generalises (6) out of equilibrium. For instance, for sub-critical quenches, after the time scale $t_{\rm p}$, this relation also scales in the sense that $y = A/\xi_{\rm d}^2(t)$ against $x = \ell/\xi_{\rm d}(t)$ yields a master curve. This master curve has double algebraic form, with a crossover from $y = x^2$ for $x \ll 1$ to $y = x^{D_A^{(p)}/D_\ell^{(p)}}$ for $y \gg 1$ with $D_A^{(p)}$ and $D_\ell^{(p)}$ the fractal dimensions of the areas and length involved at critical percolation.

Special care needs to be taken in the scaling analysis of double critical quenches as the initial and final structures are fractal though with different fractal dimensions. More precisely, if the geometric objects studied, X, are fractal with fractal dimension D_X, their scaling analysis has to be done using $X/\xi_{\rm c}^{D_X}(t)$ as the scaling variable. The clearest use of these ideas is in the analysis of the area-to-length relations [69]

$$\frac{A}{\xi_{\rm c}^{D_A^{(I)}}(t)} \simeq \left(\frac{\ell}{\xi_{\rm c}^{D_\ell^{(I)}}(t)}\right)^{\zeta} \qquad \text{with} \quad \zeta = \begin{cases} D_A^{(I)}/D_\ell^{(I)} & \text{for} \quad \ell/\xi_{\rm c}^{D_\ell^{(I)}}(t) \ll 1 \\ D_A^{(p)}/D_\ell^{(p)} & \text{for} \quad \ell/\xi_{\rm c}^{D_\ell^{(I)}}(t) \gg 1 \end{cases} \tag{24}$$

where $D_\ell^{(I)}$ and $D_\ell^{(p)}$ are the fractal dimensions of the interfaces at critical Ising and critical percolation conditions, and $D_A^{(I)}$ and $D_A^{(p)}$ are the fractal dimensions of the growing areas at $T_{\rm c}$ and the areas at critical percolation, respectively.

2.4. Metastability

The one-dimensional Ising model with non-conserved order parameter dynamics at zero temperature always converges to complete order in one of the two ferromagnetic states. In contrast, the $d \geq 2$ Ising model does not necessarily order at zero temperature. This can be easily understood. For instance, on a square lattice, any configuration with a stripe of one phase, separated by straight interfaces from the other, is stable at zero temperature.

It was realised in [83,84] that the finite-size kinetic 2d Ising model with non-conserved order parameter does not reach its ground state after an instantaneous quench from infinite to zero temperature in, approximately, one third of the occasions. In these cases, states with an even number of straight stripes are infinitely long-lived for periodic boundary conditions. The same phenomenon is observed at low but finite temperature, although the metastable states have extremely long but finite relaxation time under thermal fluctuations, and equilibrium is reached in all runs after an even longer time scale. After our work [52,40], the probability of appearance of each kind of blocked state after zero-temperature quenches from the disordered phase was put in quantitative contact with the ones of the different kinds of spanning or wrapping clusters in critical continuum percolation [85,43]. In particular, for a square lattice with PBC, the probability of reaching a stripe state is 0.356, and for FBC it is 0.339 [86]. Quenches from the critical Ising point to zero temperature were considered in [87] using different lattice geometries (rectangular lattices with varying aspect ratio; FBC, PBC and fixed boundary conditions with

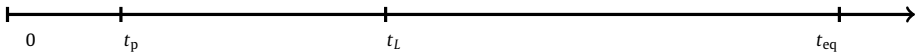

Fig. 1. A sketch of the time scales involved after a quench from the disordered phase into the ordered one, with $t_{\rm p} \simeq L^{z_{\rm p}}$, $t_L \simeq L^{z_{\rm d}}$ and $t_{\rm eq} \simeq L^{z_{\rm eq}}$, and $z_{\rm p} < z_{\rm d} \leq z_{\rm eq}$.

$s = +1$ on the horizontal borders and $s = -1$ on the vertical ones, imposing two domain walls in the sample; several connectivities). The topology of the asymptotic state was shown to be decided by the critical percolation configuration reached at $t_{\rm p}$ by taking advantage of the analytic results for various crossing probabilities (Cardy formula for percolation [88] and extensions for critical Ising [89]), and comparing them to the long-time limit of the zero temperature evolved data.

The metastable states in the 2d Potts model are geometrically much richer than the stripe states in the Ising model. In the asymptotic zero-temperature dynamics on a square lattice, the ground state, a variety of blocked metastable states, and states with *blinker spins* that flip *ad infinitum* with no energy cost are reached with finite probability [25]. This is similar to what is seen in the 3d Ising model.

In three dimensions [90,86], the metastable states are geometrically complex, high-genus 'gyroid' states, that are discrete versions of minimal zero-curvature surfaces. Because of the discreteness of the spin system, these gyroid states are not static, but contain a small (but finite) fraction of blinker spins that can flip *ad infinitum* at no energy cost and can wander around a considerable part of the system (10% of the system's volume). The states in this iso-energy manifold are composed of two inter-twinned domains, and the average genus of the domains scales as L^{γ} with $\gamma \approx 1.7$ [90,86], suggesting a *sponge-like* structure. The continuum version of this state has zero mean curvature, there are many such configurations, and this problem is named the 'plumber's nightmare'.

3. Conclusions

We finish with a short summary of the scenario suggested by the results presented in the main text and a discussion of lines for future research.

3.1. Time and lengthscales

In short, on the basis of a series of works described in the text, this note states that different stochastic dynamics of various 2d statistical physics models with bimodal variables initially distributed at random with equal densities, and quenched below their critical point, evolve in the three time regimes sketched in Fig. 1. These features are shared by the voter model.

Initially, the systems evolve *via* a kind of aggregation process in which critical percolation is approached. One can associate a growing length $\xi_{\rm p}(t) \simeq t^{1/z_{\rm p}}$ with this regime that ends at a time of the order of $t_{\rm p} \simeq L^{z_{\rm p}}$, when this growing length reaches the system's size. Interestingly enough, the models 'self-organise' into a critical situation, the one of random critical percolation.

This 'approach-to-percolation' dynamics is accompanied by the usual dynamic mechanism. The latter could be the curvature-driven dynamics set in action on long and flat enough interfaces in the case of non-conserved-order-parameter dynamics. Or it could be another dynamic mechanism determined by the conservation laws. The local growth of order of the competing equilibrium states is characterised by the usual growing length $\xi_{\rm d}(t) \simeq t^{1/z_{\rm d}}$.

This ordering process finishes when $\xi_{\rm d}(t)$ reaches the system size, that is to say, at $t_L \simeq L^{z_{\rm d}}$. This time scale is usually associated with the equilibration one. However, at low temperatures the approach-to-percolation process leaves configurations with stripes that span the systems horizontally, vertically or diagonally, with probabilities dictated by the ones of critical percolation (and the choice of boundary conditions and lattice geometry). At zero temperature, these striped states are stable (in several lattices) and inhibit the system from reaching a fully ordered state. At low but finite temperatures, a new time scale $t_{\rm eq}$ exists such that these metastable states decay and the system completely orders in one of the equilibrium or absorbing states.

The existence of two growing lengths over a certain period of time forces the use of both in the dynamic scaling of space–time correlation functions and possibly also linear response functions.

The current challenge is to give an analytical proof of the approach to critical percolation after quenches from the disordered to the ordered phase. Concomitantly, it would be very nice to compute the characteristic time $t_{\rm p}$ and its associated growing length $\xi_{\rm p}(t)$ analytically in at least one model.

3.2. Discussion

From a theoretical perspective, similar issues can be addressed in different settings with their own physical relevance. We mention a few here.

Much attention is currently being paid to the analysis of the density of topological defects left in a system after a slow quench across its phase transition [22]. How does a slow cooling affect the approach to critical percolation and the mesoscopic morphology is a question that needs a detailed study. We are currently working on this problem [91].

Other systems in which similar effects should arise are 2d kinetically constrained particle systems with a transition and coarsening dynamics such as the spiral model [92–94], or frustrated magnets of different kinds modelled with microscopic descriptions [95] or with the simpler sixteen-vertex model [96].

Phase ordering kinetics in stochastic dynamic models that are not necessarily motivated by physical situations attract the attention of many researchers at the frontier between physics and other disciplines. We studied the morphology of clusters in the voter model, that does not have an energy function [57]. Cases with an energy function and breaking of detailed balance are also relevant. For example, Godrèche & Pleimling [97] studied the asymmetric dynamics of an Ising model in a recent paper. The analysis of the possible approach to percolation and the morphology of clusters is being performed by this group [98].

Let me close the theoretical discussion with an interesting connection between our results and the problem of aggregation and gelation [99–101], sometimes put in contact with the glassy one. Aggregation is the process whereby particles (or objects) randomly distributed in space attach together upon encounter (with probability one or smaller). In some cases, the links thus formed cannot be broken. In others, they can, at some rate. The gelation phenomenon consists in the formation of a network that percolates across the system. The gelation time, $t_{\rm G}$, is identified as the time needed to construct the gel. It has been claimed that this time depends on the density of material, the dynamic rules, and other details of the model but, as far as I know, no study of the possible system size dependence of $t_{\rm G}$ has been performed. Moreover, the fractal properties of the percolation cluster and the size distribution of the infinite and finite clusters were also studied in many cases. There is some consensus in the literature that for certain models, at concentration $\phi = 0.5$, the percolating network has the properties of the percolating cluster in random critical percolation [102]. Connections between the Smoluchowski mean-field model of gelation and critical percolation on the Cayley tree were also reckoned [103]. How general these features are, and how could they be put in contact with our results on simpler lattice spin models, will definitely constitute an interesting theme for future research.

Our results are amenable to experimental checks *via* the visualisation of 2d coarsening. This has been partially carried out in the study of domain size distributions in electric-field-driven deracemisation in achiral bent-core liquid crystals through the formation and coarsening of chiral domains [104,71,105]. But other two-dimensional systems, with different dynamic mechanisms, should also be relatively simple to observe with various techniques. For instance, very recently, Castro et al. [106] used atomic force microscopy to observe the surface of silicon targets eroded by an ion plasma, a cellular system.

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