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Spatial networks / Réseaux spatiaux

Robustness of spatial networks and networks of networks

Robustesse des réseaux spatiaux et des réseaux de réseaux

Louis M. Shekhtman [a,*], Michael M. Danziger [b], Dana Vaknin [a], Shlomo Havlin [a,c]

[a] *Department of Physics, Bar-Ilan University, Ramat Gan, Israel*
[b] *Network Science Institute and Department of Physics, Northeastern University, Boston, USA*
[c] *Tokyo Institute of Technology, Tokyo, Japan*

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A B S T R A C T

Many complex networks have recently been recognized to involve significant interdependence between different systems. Motivation comes primarily from infrastructures like power grids and communications networks, but also includes areas such as the human brain and finance. Interdependence implies that when components in one system fail, they lead to failures in the same system or other systems. This can then lead to additional failures finally resulting in a long cascade that can cripple the entire system. Furthermore, many of these networks, in particular infrastructure networks, are embedded in space and thus have unique spatial properties that significantly decrease their resilience to failures. Here we present a review of novel results on interdependent spatial networks and how cascading processes are affected by spatial embedding. We include various aspects of spatial embedding such as cases where dependencies are spatially restricted and localized attacks on nodes contained in some spatial region of the network. In general, we find that spatial networks are more vulnerable when they are interdependent and that they are more likely to undergo abrupt failure transitions than interdependent non-embedded networks. We also present results on recovery in spatial networks, the nature of cascades due to overload failures in these networks, and some examples of percolation features found in real-world traffic networks. Finally, we conclude with an outlook on future possible research directions in this area.

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R É S U M É

Récemment, il a été montré que de nombreux réseaux complexes font intervenir une interdépendence fondamentale entre différents systèmes. La motivation provient principalement des infrastructures telles que les réseaux électriques et les réseaux de communication, mais comprend également des domaines tels que le cerveau humain et la finance. L'interdépendance implique que, lorsque des composants d'un système tombent en panne, ils entraînent des défaillances dans le même système ou dans d'autres. Cela peut conduire à des défaillances supplémentaires, aboutissant finalement à une longue cascade susceptible de paralyser l'ensemble du système. En outre, nombre de ces réseaux, en particulier

* Corresponding author.
E-mail address: louis.shekhtman@biu.ac.il (L.M. Shekhtman).

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certaines infrastructures, sont intégrés dans l'espace et possèdent des propriétés spatiales uniques qui ont pour effet de réduire considérablement leur résilience aux pannes. Nous présentons également des résultats sur la guérison des réseaux spatiaux, la nature des cascades dues à des défaillances de surcharge dans ces réseaux, ainsi que quelques exemples choisis dans les réseaux de trafic réel et qui présentent des caractéristiques similaires à celles de la percolation. Enfin, nous concluons sur une discussion des futures directions de recherche possibles dans ce domaine.

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1. Introduction

Much work has been devoted to modeling and analyzing properties of real complex networks [1–3]. The tools from network science have been successfully applied to many systems in social sciences [4], biology [5,6], traffic [7,8], economics [9,10], climate [11,12], and others. In many cases the robustness of the network is of significant interest [13–16]. Given a set of nodes and edges, we consider what happens when a fraction $1-p$ of the nodes are removed or fail for some reason. One of the most common measures of robustness is the size of the largest connected component left with respect to the original size of the overall network. Percolation theory, which originated from physics [17], is used to determine this largest component and to understand the evolution of the system as $1-p$ is increased [18,19]. Moreover, many of the networks whose robustness is of interest, such as infrastructure networks, are embedded in space and thus have unique spatial constraints where links are usually of relatively short length. [20,21].

While the study of individual networks began around 2000 [2,3,14], it was only recently recognized that many systems can be more accurately modeled as multilayer networks [22–26]. These multilayer networks consist of various component networks that are part of a larger system. For example, if one considers transportation in a city, there is a layer representing commuter trains, a layer representing buses, a layer representing private cars, etc. Each of these layers is a network in its own right, but is also part of the larger multilayer network system. In the above example, one can easily transfer from one layer to another (e.g., take a bus to a train) and thus the links between the networks are the same type of connectivity links as those within each network. In contrast, another system involving multiple networks is a network of interdependent networks (NON) [22,23,26]. Here connections between networks do not represent the usual idea of flow or transfer from one network to another, but rather dependency relationships between systems. The primary examples for such interdependence come from infrastructure systems such as power grids, communications networks, water supply, etc. [27] but also include brain networks [28], ecological systems [29], and financial networks [30] (see Fig. 1). For example, the antennas that form the nodes of a communications network require power from nodes in the power grid. Conversely, the nodes of the power grid need communications in order to inform one another about their incoming and outgoing loads. Initial work on networks of interdependent networks with random structure i.e. not spatially embedded, found that the interdependence between the networks leads to large cascades of failures and abrupt collapses [31–33].

In a NON there are multiple networks within each of which there exist the usual connectivity links of single networks. The interdependence is incorporated into the network via dependency links that are assigned between nodes in the same network or in different networks. When one node at the end of a dependency link fails, it causes the failure of the node at the other end of the dependency link. The failure of the dependent node may then cause other nodes in its own network to become disconnected from the largest component causing them to fail and thus creating a cascade as more dependent nodes fail, etc. For a single network that includes both connectivity and dependency links see Parshani et al. [35]. In our above example of a power grid, one could think about power stations supplying current to one another as one network with connectivity links between nodes and antennas of a communications network transferring information as another network. Dependency links would then exist between antennas and the power stations that supply them with power, and likewise the power stations would have dependency links to the antennas that they use for communication.

Once the NON includes more than two networks, the nature of the dependency relations can take multiple shapes such as treelike structures and looplike structures [23]. Furthermore, there is the potential to include a no-feedback condition, meaning that dependency links are undirected and each node is assigned at most one dependency link. If feedback is allowed (directed dependency links) then the failure of a single node could in principle lead to an endless cascade of failures [36]. Aside from the question of whether dependency links will be directed, a model can also be designed where only some fraction q of nodes are assigned a dependency link with the remaining $1-q$ nodes functioning independently [37]. This could be the case for example when a communication tower has its own generator and is not dependent on any power station in the grid for electricity.

Having presented the basic model choices we will now briefly review the results that were previously obtained for random networks. For spatial networks many of these results turn out to be surprisingly different. As previously mentioned, the transition for a pair of Erdős–Rényi (random) networks was found to be first-order and abrupt in p for the case of no-feedback and full interdependence ($q=1$) [22]. For this case, the equation describing P_∞, the fractional size of the giant component, is

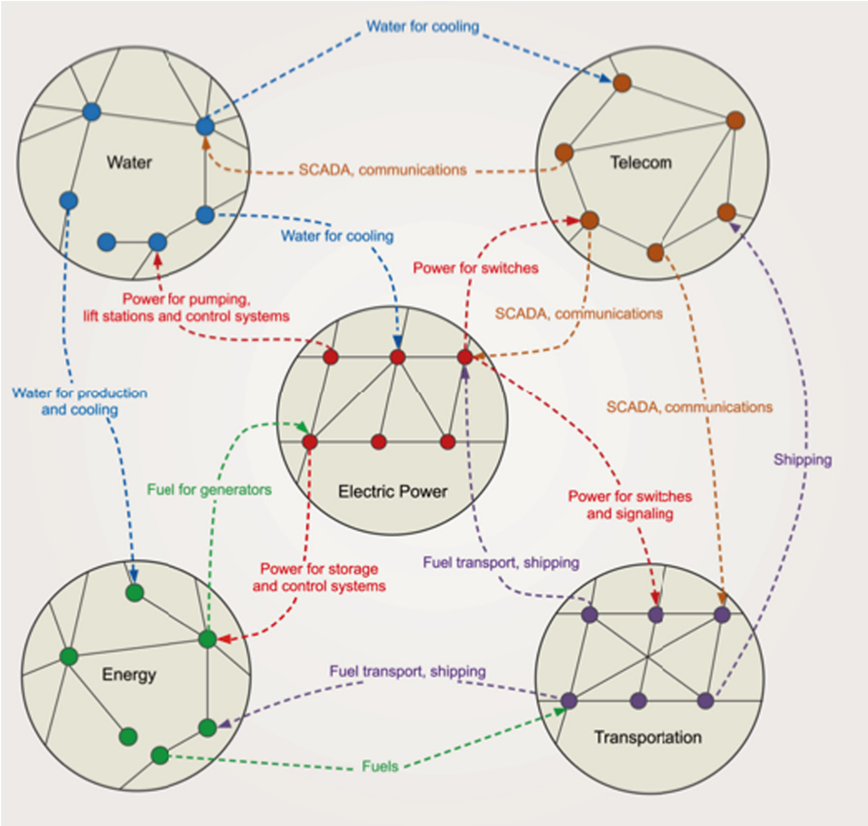

Fig. 1. Here we show the interdependence between different infrastructure systems. Each infrastructure such as water, telecom, energy, electric power, and transportation can be represented as a network. Nodes in these networks are in turn dependent on particular nodes in other networks for various reasons such as needing water for cooling, telecom for SCADA (Supervisory Control And Data Acquisition), etc. The connections within and between the networks thus form a network of interdependent networks. After [34].

$$P_\infty = p\,(1 - \mathrm{e}^{-kP_\infty})^2 \tag{1}$$

where k is the average degree of each of the networks and $1 - p$ is the fraction of nodes removed from one of the networks. When there are any number of networks, but the structure of their dependency links forms a tree, the equation governing the size of the giant component is

$$P_\infty = p\,(1 - \mathrm{e}^{-kP_\infty})^n \tag{2}$$

where n is the number of networks in the tree [38–40]. If we substitute $n = 1$ into Eq. (2) we obtain the well-known equation for P_∞ in an Erdős–Rényi network as was discovered over 50 years earlier [41]. In this way we can recognize the previous result as simply a special case of the more general theory governing interdependent networks.

Additional works considered fractions of dependent nodes between networks, i.e. $q \neq 1$. The first work to consider such a situation analyzed a pair of Erdős–Rényi networks with a fraction q nodes between them having dependency links [37]. There it was found that as q decreases from $q = 1$ to $q = 0$ there is a critical value, q_c below which the percolation transition is no longer abrupt and first-order, but instead continuous and second-order. Further work [36,39,40] considered more than two networks where only a fraction of nodes q are interdependent. Among the structures considered were a loop with directed dependency links (i.e. feedback allowed) and a random-regular (RR) NON where each network depends on exactly m other networks. The RR NON was analyzed both with cases where feedback is allowed and when feedback is not allowed [36].

Later work incorporated additional realistic features. These include restrictions on the nature of the dependency links such as coupling between nodes of similar degree [42–44], coupling based on similarity of connections [45–50], or redundancy of dependencies [35,51]. An important recent study showed that when some nodes are reinforced such that they are able to support the entire cluster of nodes connected to them, the transition can become continuous as opposed to abrupt [52]. In addition, different percolation schemes, such as k-core percolation have also been considered on interdependent networks [53–55].

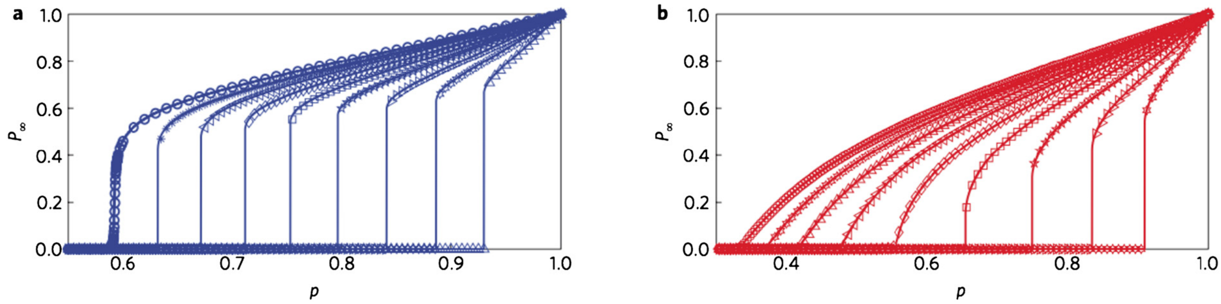

Fig. 2. Here we compare the different transitions in **(a)** spatial networks and **(b)** random networks for different fractions of coupling, q. We begin with no coupling ($q = 0$) for the curves farthest on the left in each plot and move up to $q = 0.8$ for the curve farthest to the right by increments of 0.1. In spatial networks, the transition is always abrupt for any value of $q > 0$, whereas for the random networks the transition is abrupt only above a critical q_c. After [56].

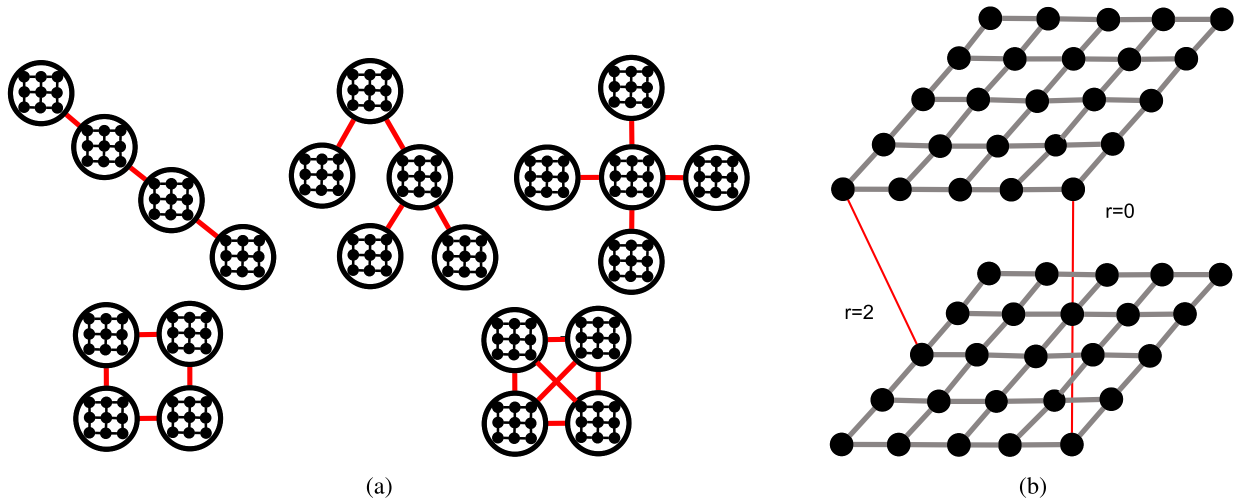

Fig. 3. A model of spatial interdependent networks. (**a**) Different possible structures of NONs. Each large circle contains a spatial network and the red links represent dependency relations between the different networks. The top 3 examples show treelike structures where the dependency relations form a tree and the bottom 2 show random-regular connections allowing for loops. (**b**) When assigning dependency links between nodes we can fix a maximum r distance between the two nodes on each end of the dependency link. Here we show dependency links of length $r = 0$ and $r = 2$. After [60].

2. Robustness of interdependent spatial networks

Having introduced previous work on interdependent networks, we will now review results on the robustness of interdependent networks composed of spatially embedded networks. When researchers first began studying spatial networks they typically used square lattices as their model and noted that any other network embedded in 2D space will be in the same universality class [19,57,58]. This provided a way to focus on the specific fundamental properties of spatial networks without getting bogged down in the unique details of specific networks.

Some of the first and most significant studies on interdependent networks which are spatially embedded modeled the system as a pair of interdependent lattices [59]. In one model [56] only a fraction q of nodes were interdependent, with dependency links being assigned randomly between pairs of nodes in different networks. In contrast to the results of random networks, where there was a critical value of q_c below which the transition was continuous, for spatially embedded networks the transition was found to always be abrupt and first-order for any values of $q > 0$ (see Fig. 2). This result was shown both numerically and analytically [56]. It is a consequence of the fact that at the point of transition, p_c, in a single spatial network, the derivative of the size of the giant component, P'_∞, diverges thus leading to the result that $q_c = 0$. For this reason interdependent spatial networks have been referred to as being 'extremely vulnerable' since they always collapse abruptly for any level of interdependence [56].

Another early work considered spatial networks with full dependency $q = 1$, but where the dependency links are restricted such that nodes can only be dependent on other nodes that are up to a certain distance r apart from them (see Fig. 3b) [59]. In this case, it is rather obvious that, for $r = 0$, we must recover the case of percolation for a single lattice network since, in this case, the dependency links have no effect and the two networks are identical. However, the goal was to find if there is a critical r_c only above which the transition is abrupt. Indeed, in that work it was found that only for $r_c > 8$, i.e. if a node can be dependent on another node up to 8 lattice spaces away then the transition is abrupt. It was also found that the value of p_c, the critical point of transition, is at a maximum at $r_c = 8$ and decreases as r increases beyond $r_c = 8$. This is because at $r = r_c$ a hole with a radius of 8 can form in the network and will then propagate throughout the entire system (as in a nucleation transition), whereas as $r \to \infty$ the random damage actually leads to the abrupt transition (similar to the case of random networks) as opposed to the growth of a hole.

Further work, considered varying both r and q and found that as q decreases, r_c increases or conversely as r decreases, q_c increases [61]. As $q \to 0$ it was found that $r_c \to \infty$. Similarly, p_c also reaches a maximum at the value of r_c for the various values of q yet the peak is less and less sharp as q decreases.

Later work considered a system of interdependent spatially embedded networks where the number of networks, n, is larger than 2 [60]. In this case, as for random NONs, the dependencies can form different structures such as trees, loops, a fixed number of dependencies for each network, etc. Several of these structures are shown in Fig. 3a. For the case of treelike spatially embedded NONs it was found that as n increases, r_c decreases, until eventually for $n = 11$, it reaches $r_c = 1$, meaning that even if nodes are only dependent on their nearest neighbors, the transition is abrupt. Also, while 11 interdependent systems may seem large, it is not unreasonable for real-world large-scale infrastructure systems [27]. The work in [60] also considered loops of interdependent spatial networks and RR spatial NONs finding relationships between r and q that were similar to those obtained in [61].

Finally, it should be noted that many of the works on spatially embedded interdependent networks used a novel methodology to obtain analytic results, which are generally very difficult to obtain for robustness of spatial networks. Specifically, it was recognized that if one could assume that the graph of P_∞ vs. p for a single spatial network is known, then one could use that result to determine the robustness of a network of such networks [56,59,60]. While in those works this methodology was applied to spatial networks, the same equations are applicable to any NON where the resilience of the single networks is known [60].

3. Spatial multiplexes

Though the initial models for interdependent networks were based on lattices with dependency links of variable length r (as described above), in many cases a *spatial multiplex* model would more realistically describe the system. A multiplex is a specific type of multilayer network where the nodes in one layer play a role in other layer(s) too. Essentially a multiplex involves different sets of links that connect the same nodes in different layers. One example of such a multiplex network could be a social multiplex formed by considering relationships of individuals on various social networking sites like Facebook, Twitter, etc. The same individual may have an account on both Facebook and Twitter yet each account may have different connections even though they represent the same person. Viewing these accounts as a single node in a multiplex can often simplify the analysis and provide insight into behavior across all the layers [47,50,63]. For our multiplex model each site in space has representations in two layers and functionality (as determined by connectivity to the giant component) in both layers is required. This reflects the assumption that, for example, each communications network asset is dependent on its nearest power grid component even as the links within the power grid and communications network have some variability. In [62], following empirical observations of infrastructure networks, the authors adopted a variation of the Waxman topology [64] with an exponential distribution of link-lengths in space for each layer of the spatial multiplex. In this manner the topology can be continuously varied from lattice-like to random by varying a single parameter: ζ, the average link length (see Fig. 4).

In the spatial multiplex the connectivity links are of variable length while the dependency links are of minimal length. This is the opposite of the interdependent lattice models where the connectivity links were the standard square grid and the dependency links could be longer. For finite r, the dependency links were found to carry damage from a hole to a concentrated region around it and trigger a spreading cascade. Since the dependency links in a spatial multiplex are of length zero while the connectivity links — which can only strengthen the robustness of the network — are longer, it would seem that the damage may not spread in a cascade as before. Surprisingly, spatial multiplexes also produce spreading cascades, in much the same way as the interdependent lattices. This happens since the connectivity link length can bring the damage further similar to long dependency links. The cascading process is explained in Fig. 4b and [62].

Under random attack it was shown that changing the average connectivity link length ζ (and thereby changing the "strength" of the embedding), causes opposite effects in single networks and in the multiplex. In single networks, increasing ζ (decreasing the spatiality) causes the percolation threshold p_c to decrease smoothly and rapidly approach its mean-field value of $1/\langle k\rangle$. It was also shown that in single networks this spatial model displays a unique property of bi-universality where at short scales the network is random whereas at longer scales the network reveals its spatial embedding [65].

In contrast to the resilience for single networks, in a spatial multiplex increasing ζ initially increases p_c, until a certain point and beyond that point p_c decreases. This is similar to the results for interdependent lattices with dependency lengths of varying lengths [66,67] (see Fig. 5).

In addition to highlighting new effects of modulating the strength of the spatiality, the fact that spatial multiplexes–with their more realistic structure–largely recapitulate the results of interdependent lattices, demonstrates the universal features of interdependent spatial networks.

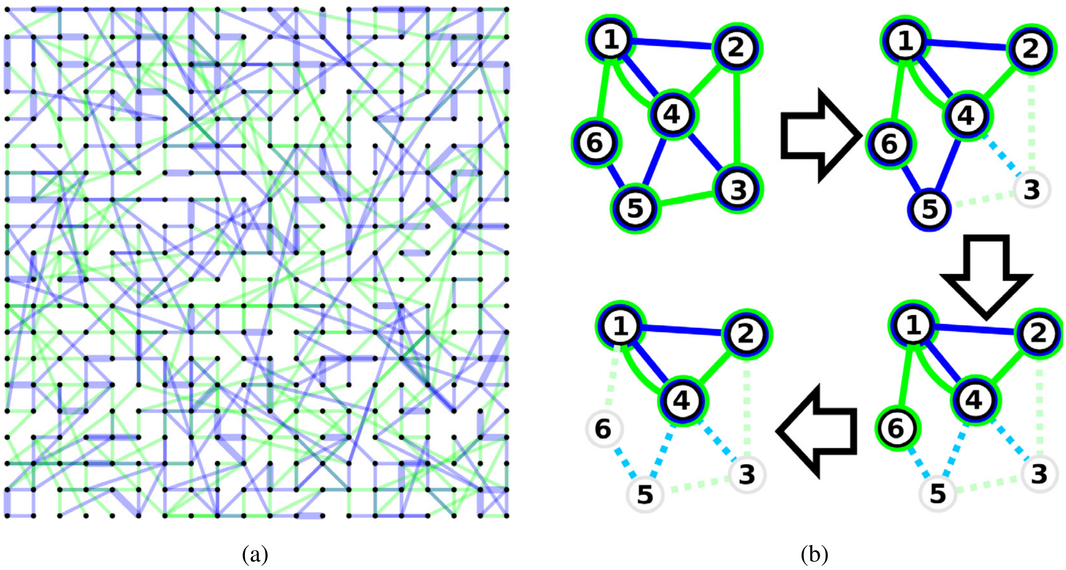

Fig. 4. A new model for spatial multiplexes. (**a**) Each node appears only once (as in a multiplex), but the network contains two sets of links. Each set of links is created such that the distribution of link lengths forms an exponential distribution. In order for a node to be functional it must be connected to the giant component in *both* sets of links. (**b**) Cascades can occur in this network when nodes fail due to a lack of connection to one set of links. Starting from the top left we see that node 3 fails due to lack of connection in the blue links. Its failure then disconnects node 5 from the green links, whose failure in turn disconnects node 6 from the blue links. Finally we reach a stable state at the bottom left. After [62].

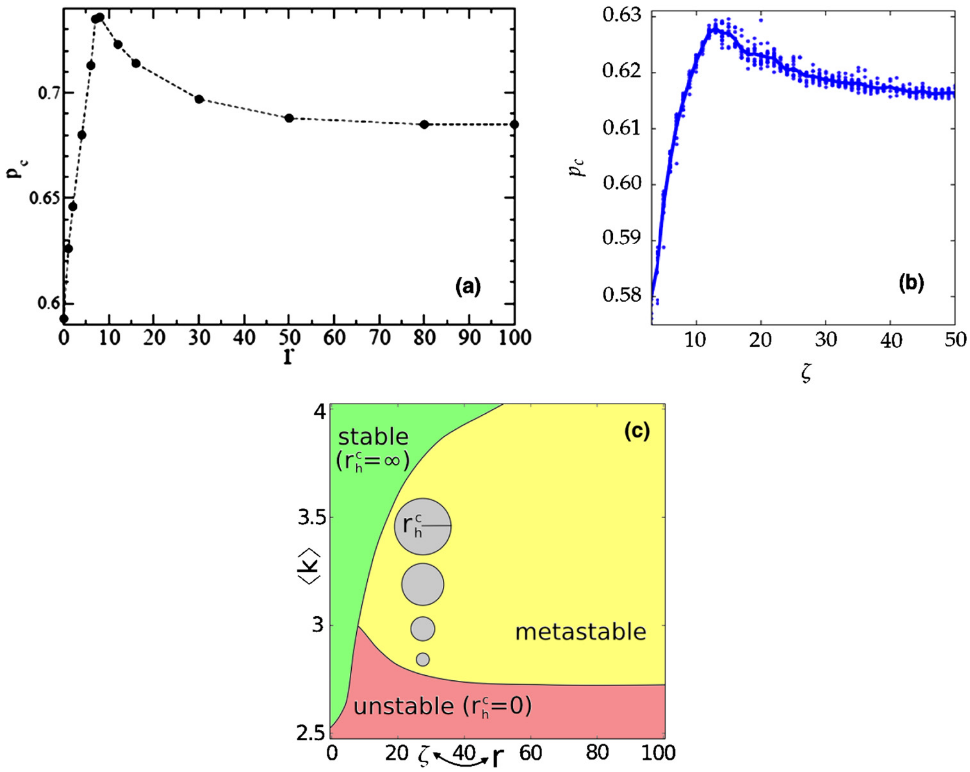

Fig. 5. A comparison of dependency links of length r to spatial multiplexes with connectivity links of length ζ. The critical threshold p_c as a function of r (**a**) and ζ (**b**). In both, below the maximal p_c the transition is continuous and becomes abrupt when p_c reaches a peak and remains first order as p_c declines. Note the similarity in how both plots involve an increase in p_c as r (ζ) increase until reaching a peak after which the value of p_c decreases with further increasing r (ζ). After [59] and [62]. **(c)** For localized attacks there exists a region in the phase space of $\langle k \rangle$, the average degree, and r or ζ for which the system is metastable. This means that the system collapses under localized attacks but is robust to random attacks. The radius, r_h^c, represents the critical radius above which cascading occurs and the system collapses. This region exists both for the dependency link model with r and for the spatial multiplex with connectivity links of length ζ. After [68].

4. Localized attacks

In many of the real-world complex systems being discussed, all of the nodes in a certain region could fail at once due to a natural disaster or targeted attack. Here we will focus on the consequences of such localized attacks.

While our primary goal is to study such attacks on spatial networks, we will first introduce results for localized attacks on random networks. This problem was first studied by Shao et al. [69] who considered a localized attack as one where a node and its neighbors up to some number of steps from it (chemical distance), being a fraction $1-p$ of the network, are all removed at once. They proved that for Erdős–Rényi networks there is no difference between localized or random attacks in that both p_c and the giant component remaining are identical. However, for random-regular networks the network is more robust against localized attacks than random attacks, while for scale-free networks localized attack is usually more damaging than random attacks.

In spatial networks a localized attack has a more natural interpretation as it suggests that all nodes in some spatial region fail at once. All nodes within some region are removed creating a hole. The size of this 'hole' is defined by its radius r_h meaning all nodes within a distance r_h of some node fail due to an attack.

The first study to consider the consequences of localized attacks on spatial networks, considered interdependent square lattices [68] with dependency links defined up to some distance r as shown above in Fig. 3b. Note that for single isolated spatial networks, such as lattices, localized attacks will not cascade and $p_c = 0$. However, for interdependent lattices the system is extremely vulnerable. Since many infrastructure systems such as power grids have an average degree, $k < 4$, the authors in [68] removed links in a square lattice in order to dilute the degree to lower values. They found that, depending on the lengths of the dependency links r and the degree $\langle k\rangle$, as the radius of the hole removed, r_h, is increased, there exists a critical value r_h^c for which the damage propagates radially and failures cascade throughout the entire system, whereas below r_h^c, failures do not propagate.

Further, they found that this r_h^c does not increase as the system size is increased. This means that a localized attack of zero fraction of the system size ($p_c = 1$) will lead to a full collapse. That is, there is a metastable region where for random attacks the network is resilient, yet for localized attacks, the network collapses. The reason for this is that a nucleation phenomenon takes place where the damage spreads radially outward from the site of attack through the entire system. In addition to the numerical work, Berezin et al. [68] developed a theory for predicting the value of r_h^c for every set of parameters. The theory is based on gradient percolation [70] and the calculation of the characteristic size of clusters near criticality [68]. Thus, localized attacks in interdependent spatial networks are significantly more damaging than random failures and represent a significant threat to the resilience of these systems.

Further work by Vaknin et al. [71], considered localized attack on the spatial multiplex model discussed in Sec. 3. For that case, the nodes are placed spatially on a lattice but the links are formed by connecting nodes such that the distribution of link lengths is exponential with an average characteristic length ζ. Dependency links are again between nodes in the same locations as in Fig. 4a.

Due to the more complex structure of the network, analytic solutions were not possible, but numerical results were obtained. Despite having zero distance for the dependency links, a metastable region was also found in this study, though in this case the parameters are the degree $\langle k\rangle$ and the characteristic length ζ (see Fig. 5). Thus, both connectivity links of length ζ or dependency links of length r yield similar results since both are effectively spreading the damage to a distance ζ or r respectively. Fundamentally, this is the main mechanism for the cascading failures.

To gain some intuition into the model behavior, every region of order ζ can be considered like an Erdős–Rényi network with random connections within that region. The critical threshold for an interdependent Erdős–Rényi network is $p_c = 2.445/\langle k\rangle$ [22] and the spatial multiplex model can be likened to an Erdős–Rényi network in regions that are of size ζ. Thus, if within a region of size ζ, the fraction of surviving nodes is less than the p_c value above, the entire region can be expected to fail due to the cascade. The failure of this region can then spread to a neighboring region of size ζ which may in turn also reach a level of local failure below p_c leading to this next region's collapse. This process can then continue cascading until the entire network collapses.

At first the failures take place within a region of size proportional to ζ, then later the failures cascade spatially outward to the rest of the network (see Fig. 6). In addition, the existence of a novel scaling behavior with respect to the necessary hole size, r_h^c, as a function of average degree has been observed. In contrast to typical scaling behavior, this relation emerges from considerations of the nucleation size itself, and does not require the usual assumptions of second-order phase transitions. This finding also suggests universal mechanisms behind first-order phase transitions where nucleation processes are involved and shows the unique behavior of cascading failures in spatial networks.

5. Overload failures, other realistic features, and real-world networks

In this section we will review research on the robustness of spatial networks that is slightly different from the framework presented above of interdependence between nodes in networks of spatial networks. This work is similar in that it also involves robustness of spatially embedded networks, yet it considers different types of dependence such as overloads, incorporates new possibilities like recovery of components, or opens the potential for considering novel concepts such as inhomogeneity in the distribution of nodes. These results are even more recent and represent novel avenues of research.

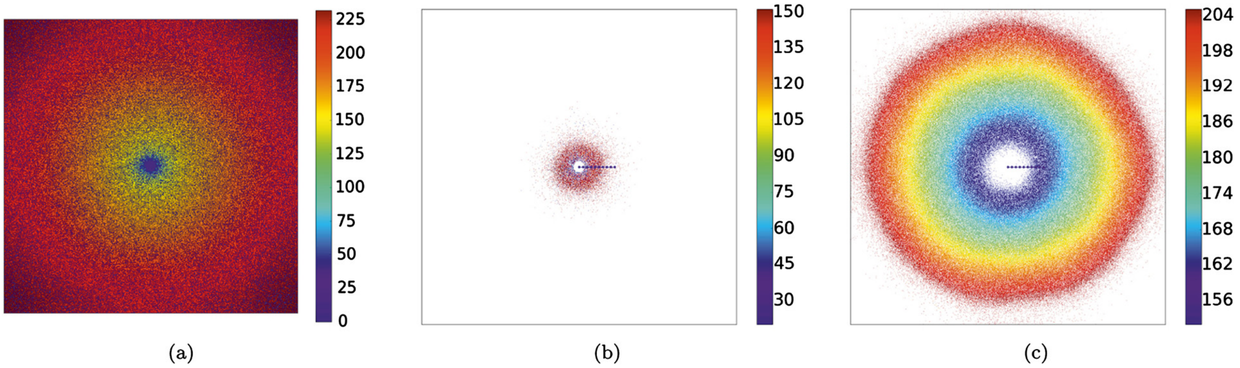

Fig. 6. Cascading failures for a localized attack of size r_h^c. The colors represent the iterations of the cascades (from blue earlier to red later) — until the whole system collapses. (a) The entire propagation of failures, (b) focus on the branching process in a local region of order ζ, (c) focusing on the spatial spreading cascade. After [71].

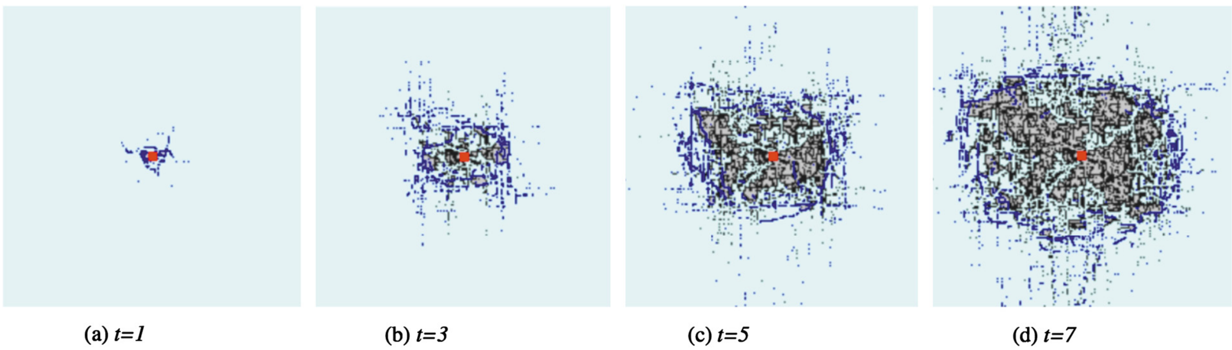

Fig. 7. Here we show the spreading of overload failures in a spatial network. We assume that each node has a given load and some tolerance level α for which it will not fail even if its load is increased. However, once the load passes α the node fails. We begin by removing a small region of nodes in **(a)**, the red region. This leads the flow going through them to be diverted to the neighboring nodes which in turn leads some of these nodes to exceed their tolerance level in **(b)**. The failure of these nodes leads additional load to be diverted leading to further overload failures in **(c)** and **(d)**. After [72].

The first topic we will discuss is the importance of overload failures in spatial networks and their analogy to interdependent networks. Such failures are common in the context of power grids where the flow would represent electrical current passing through a substation [73], the internet where denial of service attacks may cause a server to become overloaded [74], and other infrastructure contexts. When too much flow is forced to pass through a substation, server, or other node, the node becomes 'overloaded' and fails. We assume that each node in a lattice has some load, l, passing through it, defined by the value of the betweenness of the node [75]. This definition implies that flow is equally likely from any one node to any other node in the network. Further we assume that the system has some tolerance level, $\alpha > 1$, for which even if the load on a node increases to αl, the node remains functional. Given such a system, Zhao et al. [72] considered what happens after a local failure in the spatial network (square lattices were used as the model). It was found that after the initial failure, load is redistributed to the nodes surrounding the failed nodes. This leads some of those surrounding nodes to fail, in turn redistributing load further out and causing further cascading failures, etc. The rate of spread of the cascade was found to be constant in time and radially outward in space (see Fig. 7). These results not only provide insight into the spatial and temporal dynamics of cascades of failures, which is crucial for stopping such cascades, but also show evidence of hidden dependencies that exist in the network due to overload dependencies. For low values of tolerance, the overload failures spread to large r distances while for high tolerance they spread only short distances. Thus, this distance r is analogous to the r of Wei et al. discussed above [59].

Another important aspect of infrastructure resilience is the possibility for recovery of components in the network. This concept was only quite recently introduced even for single random networks [76]. The model for recovery involved a fraction of nodes that fail due to internal reasons (random failures essentially) and a fraction of nodes that fail due to a lack of sufficient connections to functional nodes. Furthermore, it was assumed that nodes would recover after some given time, which is a parameter of the model. It was found that there exist 2 distinct states in the model. First, is one where most nodes are functional and the network overall is considered to be in a functional state; and second is a failed state where most nodes are non-functional. In addition, there was a middle, metastable (hysteresis), region where the system could spontaneously transition from failed to recovered states and vice versa. Further work expanded this model to a pair of interdependent networks [77], for which it is found that there are 4 distinct states: one where both networks are in failed states, another two states where one of the networks is functional and the other is failed, and finally a state where both

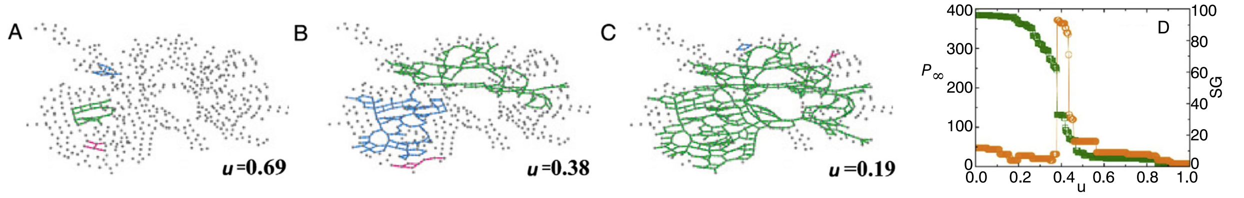

Fig. 8. (a)–(c) We observe that as the congestion threshold u is decreased the network becomes connected and merges into a single giant connected component encompassing most of the network. Similar to percolation on model networks, in **(b)** one can see that near criticality there is a second largest connected component which is of nearly similar size to the largest component. After [8]. **(d)** Here we show both P_∞ and the second largest component, SG, as a function of u. There is a clear spike in SG at the critical point, u_c.

networks are functional. This work has also been extended to spatial networks where it was found that the hysteresis region is much smaller than for random networks [78]. Additional work considered localized recovery in networks [79], recovery specifically after localized attacks [80], and recovery near the functional giant component [81].

All of the models discussed thus far are isotropic, in that all of the nodes were distributed equally throughout space. In real-world networks however this is usually not the case as natural boundaries such as mountains, rivers, etc. may lead certain areas to contain more nodes or links than others. Preliminary work on heterogeneity has considered the robustness of modular networks (or networks with communities) [82,83]. In this model, nodes are connected with one level of connectivity to other nodes in the same community (for example on the same side of a natural boundary) with fewer connections to nodes in other communities (on the other side of a natural boundary). Furthermore, it was assumed that those nodes connecting across the boundary (interconnected nodes) are more likely to fail as they have longer links and also have a higher betweenness (load) passing through them [82,83]. As these interconnected nodes fail, there are two cases depending on the levels of connectivity within and outside the individual communities. (i) In the first case, there is sufficiently small connectivity between the communities such that as the interconnected nodes fail, the communities end up being divided from one another, yet still sufficiently connected within their own community. If additional nodes fail, the individual communities will also disintegrate. (ii) In the second case, there is sufficiently high connectivity between the communities such that the entire system collapses together with no regime for which the communities are functional separately [82]. This same pattern has been seen for a NON where the individual networks have the same community structure [83]. Future work will hopefully consider additional realistic heterogeneity and anisotropy incorporating them into models for spatial networks as well as random ones.

Having shown the advancements made for models of spatial networks, we now present recent work that applied the methods of percolation to traffic systems. The application of percolation to a traffic network can be done as follows [8]: links are formed between junctions and the speeds between the various junctions are measured for an entire day. The weights on the links throughout the day are then given by the speed at the given time divided by the maximum speed throughout the day. This essentially gives a measure of traffic quality at a given time on a particular road. For a given time, a threshold u for the relative speed can be defined and links that have a weight value below u are considered failed (or stalled). As u is increased from $u = 0$ to $u = 1$ a percolation-like process occurs on the city traffic network and the size of the giant connected component decreases [8], see Fig. 8. Furthermore, similar to percolation on network models, the network breaks into multiple clusters near a particular $u = u_c$ and the size of the second largest connected component reaches a peak at u_c [84]. Essentially, this means that the traffic network has broken into distinct regions where traveling from one region to another is highly burdensome due to extreme congestion. Thus, the value of u_c can be used as a parameter indicating the global traffic quality in the city at a given time. Even more recent work has shown that the distribution of cluster sizes at u_c shows different scaling properties depending on the time of day [85]. There it is found that during weekday rush-hours, the distribution of cluster sizes at criticality is similar to that for spatially embedded networks (lattices). In contrast, during weekends or non rush-hours at the critical u_c, the distribution of cluster sizes is similar to that of non-embedded networks. This surprising phenomena can be understood as follows: During rush hours the highways, which can be regarded as long-range links, are highly congested while during non-rush hours they are effective and the traffic system behaves as a high dimensional system. These results show how the robustness of a real-world spatially embedded system can be described and modeled using methods based on percolation theory.

6. Conclusion

In this manuscript we have provided a current overview of research on the robustness of spatial networks. In general, it is clear that these networks have unique vulnerability properties that contrast significantly with those of non-spatial networks. Only by understanding these unique properties will it be possible to design more robust infrastructure since these systems are inherently embedded in the two-dimensional space that is the surface of the earth. Future research will likely move towards the direction of more realistic models that incorporate the novel structure of these systems such as heterogeneity and anisotropy, important processes taking place such as overload failures and recovery, a focus on real-data, and other aspects of the system crucial for resilience.

By advancing models of infrastructure networks to include these features it will be possible to better understand and hopefully prevent large cascades of failure in these systems. As our world becomes more interconnected and interdependent, it is crucial to understand the vulnerabilities of all of these new systems which we are creating. One big step in this direction is understanding how the spatial embedding and spatial properties of these systems affect their vulnerabilities.

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