C. R. Physique 20 (2019) 22–32

Contents lists available at ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

The new International System of Units / Le nouveau Système international d'unités

A consistent unified framework for the new system of units: Matter–wave optics

Un cadre unifié cohérent pour le nouveau système d'unités : l'optique des ondes de matière

Christian J. Bordé [a,b,*,1]

[a] *LNE-SYRTE, UMR 8630 CNRS, Observatoire de Paris, 61, avenue de l'Observatoire, 75014 Paris, France*

[b] *Laboratoire de physique des lasers, UMR 7538 CNRS, Université Paris-Nord, 99, avenue Jean-Baptiste-Clément, 93430 Villetaneuse, France*

ARTICLE INFO

Article history:
Available online 11 January 2019

Keywords:
SI units
Metrology
Geometry
Action
Entropy
Matter–waves

Mots-clés :
Unités SI
Métrologie
Géométrie
Action
Entropie
Ondes de matière

ABSTRACT

Thanks to considerable progress in quantum techniques, it has been possible to redefine all SI base units from fundamental constants. A last challenge is to produce a unified framework for fundamental metrology in which all base quantities and relevant fundamental constants appear naturally and consistently. We suggest a generalized 5D framework in which both gravito-inertial and electromagnetic interactions have a natural geometrical signification and in which all measurements can be reduced to phase determinations by optical or matter–wave interferometry. The corner stones of this unification are action and entropy. The connection is made with Kaluza's 5D theory and Planck's natural units.

RÉSUMÉ

L'année 2018 a été celle du renouvellement complet du système d'unités, avec l'ambition d'établir un système pérenne, universel et cohérent. Ce système se caractérise par l'abandon des artefacts pour se fonder uniquement sur les constantes fondamentales de la Physique [1–3]. La cohérence sous-jacente du système proposé et la notion de constante fondamentale seront discutées. Un système d'unités naturelles a été introduit par Max Planck en 1899. Il est fondé sur cinq constantes fondamentales, $\hbar$, k_B, c, G, ε_0. Les deux premières concernent respectivement les mouvements cohérents et la décohérence thermique dans l'espace des phases au moyen des concepts d'action et d'entropie. Les trois dernières précisent la géométrie de cet espace en présence d'interactions gravitationnelles et électromagnétiques. Cette géométrie est celle d'un espace à cinq dimensions introduit par Theodor Kaluza en 1921 [4]. Les progrès considérables des interféromètres à ondes de matière dans les domaines atomique et électrique imposent aujourd'hui un nouveau système, dans lequel les deux dernières constantes sont plutôt une différence de masse entre les niveaux d'un même atome et la charge de l'électron. La cinquième dimension est alors le temps propre donné par les horloges atomiques. Le lien entre les deux systèmes fait in-

* Correspondence to: LNE-SYRTE, UMR 8630 CNRS, Observatoire de Paris, 61, avenue de l'Observatoire, 75014 Paris, France.
E-mail address: christian.borde@obspm.fr.

[1] Member of the "Institut de France", Académie des Sciences, President of the "Conférence générale des poids et mesures" (CGPM) in 1999, 2003, 2007, and 2011.

https://doi.org/10.1016/j.crhy.2018.12.004

tervenir la constante de structure fine et son équivalent gravitationnel. Le système proposé apparaît alors comme le meilleur compromis, dans l'état actuel de nos connaissances.

1. Introduction: what framework for fundamental metrology?

This framework is naturally the one imposed by the two great physical theories of the 20th century: relativity and quantum mechanics. These two major theories themselves have given birth to quantum field theory, which incorporates all their essential aspects and adds those associated with quantum statistics. The quantum theory of fields allows a unified treatment of fundamental interactions, especially, of electroweak and strong interactions within the standard model. General Relativity is a classical theory; hence, gravitation remains apart and is reintegrated into the quantum world only in the recent theories of strings. We do not wish to go that far and we will keep to quantum electrodynamics and to the classical gravitation field. Such a framework is sufficient to build a modern metrology, taking into account an emerging quantum metrology [1–3]. Of course, quantum physics has been operating for a long time at the atomic level, for example in atomic clocks, but now it also fills the gap between this atomic world and the macroscopic world, thanks to the phenomena of quantum interferences, whether concerning photons, electrons, Cooper pairs or, more recently, atoms in atom interferometers [5–10]. The main consequence of this evolution is that all base units can now be redefined by fixing the values of fundamental constants having a dimension, such as c, h, k_B... whereas constants without dimension such as the fine structure constant α cannot be arbitrarily fixed.[2]

1.1. Fundamental constants and symmetries

Some quantities transform into each other in symmetry transformations, thus reflecting a common nature. If independent units have been chosen for the corresponding quantities, a fundamental constant appears as a result of this arbitrary choice. This is the case for space–time coordinates connected together by rotations or by Lorentz transformations or else to proper time as a result of gauge transformations. Other pairs of quantities can be shown to be equivalent once their common nature is understood and recognized. This is the case of energy and thermal excitation. This equivalence is responsible for the Boltzmann constant. Similarly, mass is equivalent to a proper frequency through Planck's constant. In all these cases, the resulting constant may be fixed (and even in some cases set equal to unity). Finally nature imposes some systems for which a quantity is quantized hence becoming universal and reproducible for identical objects. This is the case of atomic levels or of the mass of elementary particles, thus imposing a unit of time.

1.2. Geometry and numbers

Most base quantities of metrology, length, time, mass, electrical quantities, temperature, are ultimately measured by optical or matter–wave interferometers. Optics and quantum mechanics play a central role in the description of these devices. As a consequence, future fundamental metrology will deal essentially with phase measurements i.e. invariant numbers. One should also emphasize the non-commuting character of quantities like mass and proper time, which is a reason why Planck's constant has such a special place in the system of units. Base quantities should be quantum observables. Some appear as base quantities with their conjugate partner (e.g., mass and proper time), others do not (e.g., position coordinate and momentum). The quantum-mechanical link between conjugate quantities does not allow any more to leave Planck's constant out of the system of units. Quantum measurement theory becomes essential to explore the limits of the new metrology.

A natural 5D theoretical framework for the redefinition of the SI is presented, in which a clear separation is made between proper time (observable!) and time coordinate (not observable!) as distinct quantities sharing the same unit. The role of the electromagnetic field is to couple space–time and proper time coordinates through the corresponding off-diagonal components of the metric tensor. The 5D action gathers all phenomena and constants of interest for a fully relativistic quantum metrology in an invariant phase through Planck's constant and this includes the dephasing arising from gravito-inertial fields (e.g., the Sagnac effect or the effect of gravitational waves) as well as those of electromagnetic origin (such as the Aharonov–Bohm or the Aharonov–Casher effect).

1.3. Action and entropy

Owing to its link with the quantum mechanical phase through Planck's constant, the concept of action therefore plays the central role in the new metrology. Mass is directly connected to Planck's constant, not only in atomic clocks phases but

[2] These constants are ratios of quantities with identical units and they have to be measured and in some cases can be calculated.

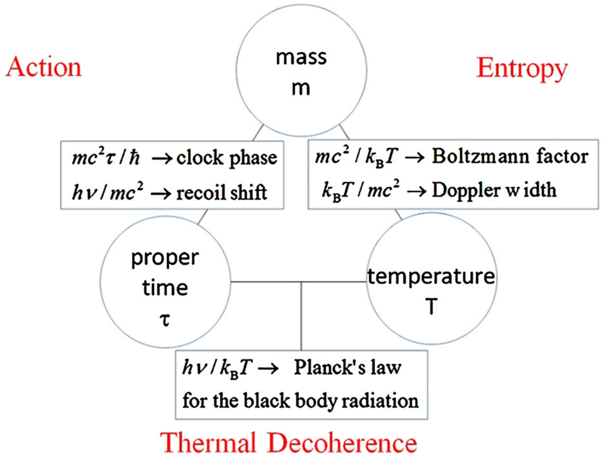

Fig. 1. Coherent and incoherent motions lead to adimensional factors that may be expressed and classified with Planck and Boltzmann constants along the sides of this triangle.

also in the recoil shift [11,12,16], now accessible with high accuracy by atom interferometry [14,15].[3] Our new system of units cannot escape the fact that mass and time units are directly coupled through the natural standard of action which is Planck's constant. This link is further reinforced by the possibility today to extend this connection to the macroscopic world thanks to silicon spheres [19,20] and to the watt balance [21].

Statistical mechanics permits to go from probabilities to entropy thanks to another dimensioned fundamental constant, Boltzmann's constant k_B. By analogy with the case of Planck's constant, it seems natural to fix Boltzmann's constant k_B. Indeed, there is a deep analogy between the two "S's" of physics, which are action and entropy (illustrated in Fig. 1). They provide respectively the phases and the amplitudes for the density operator [22]:

$$\rho = \sum_{\psi} p_{\psi} \, |\psi\rangle \, \langle\psi|$$

The corresponding conjugate variables of energy are respectively time for the Liouville–von Neumann equation and reciprocal temperature in the Bloch equation.

The link between atom interferometry and the Doppler broadening of line shapes by the thermal motion of atoms is established in reference [13] which brings the connection between phase and temperature measurements. The thermal motion of atoms is responsible for a loss of phase coherence and the Doppler broadening may be seen as a limited visibility of interference fringes. The relative Doppler width is indeed a measure of entropy[4] and is one of the methods that have led to a satisfactory determination of the Boltzmann constant [23].

In order to compute action and entropy, we need to know the full 5D geometry and then from the expressions of action and entropy, we can obtain the density operator and hence the average value of any quantity.

2. From 4 to 5 dimensions: proper time interval and mass as a conjugate pair for a 5-dimensional optics

Matter–wave interferometry started with electrons and neutrons, for which a pure space–time description is sufficient in analogy with ordinary interferometry of light waves. However, an extra phase factor is required for these massive particles, which is provided by the action. Since, for these particles, mass is a constant of motion, the corresponding phase is simply proportional to the proper time interval along each path. When it comes to atoms or molecules, mass varies with internal excitation and it plays the same role as any external momentum component as a new dynamical variable to achieve mode coupling and generate new optical paths in an interferometer. The internal phase contributes to the overall dephasing and is responsible for the clock term in atoms. Atomic clocks can then be seen as genuine atom interferometers. To account for this internal motion, we may enlarge our space–time with the additional dimension $s = c\tau$ (see Fig. 2)

$$\mathrm{d}\hat{x}^{\hat{\mu}} = (c\,\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, c\,\mathrm{d}\tau) = (\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3, \mathrm{d}x^4) \tag{1}$$

[3] The ratio of Planck's constant to electron mass was already measured with much lower accuracy using interferometry in rotating superconductors [17,18].

[4] An interesting analogy may be drawn for the two inaccessible limits that are the velocity of light c and the absolute zero temperature $T = 0$. In both cases, internal motion stops and both velocities $\mathrm{d}\tau/\mathrm{d}t$ (cf. Langevin twins) and $u = \sqrt{2k_BT/m} \longrightarrow 0$. (The Doppler width and the black body radiation shift vanish as the thermal decoherence time decreases.)

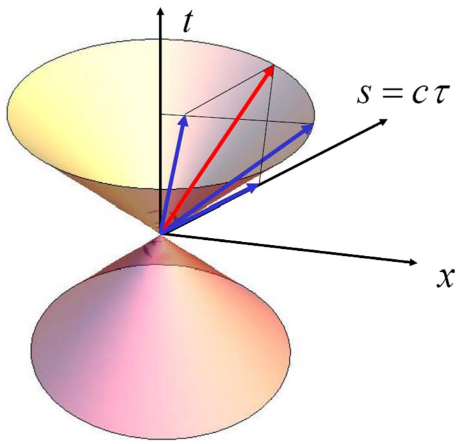

Fig. 2. In 5D coordinates, the 5D velocity is always c, but the projection on space is the particle's velocity $\overrightarrow{v}$, whereas the component along the proper time axis is reduced accordingly.

and introduce a generalized light cone for massive particles

$$\mathrm{d}\sigma^2 = G_{\hat{\mu}\hat{\nu}}\mathrm{d}\widehat{x}^{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\nu}} = 0 \text{ with } \hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4 \tag{2}$$

where $G^{\mu\nu} = g^{\mu\nu}$; $G^{\hat{\mu}4} = G^{4\hat{\nu}} = 0$; $G^{44} = G_{44} = -1$.

In flat space–time, since $g^{00} = 1$ and $g^{ii} = -1$, this relation reduces to Pythagoras' theorem:

$$c^2\mathrm{d}t^2 = c^2\mathrm{d}\tau^2 + \mathrm{d}\overrightarrow{x}^2 \tag{3}$$

Proper time is not defined by this equation from other coordinates, but is a true evolution parameter representative of the internal evolution of the object. It coincides numerically with the time coordinate in the frame of the object. Since the time unit is a proper time unit provided now by an atomic clock, this choice determines the time coordinate scale and the length unit by fixing the velocity of light.

Mass is conserved when the system under consideration is invariant in a proper time translation and will become the generator of such translations in the quantum theory. In the case of atoms, the internal degrees of freedom give rise to a mass that varies with the internal excitation. For example, in the presence of an electromagnetic field inducing transitions between internal energy levels, the mass of atoms becomes time-dependent (Rabi oscillations). It is thus necessary to enlarge the usual framework of dynamics to introduce this new dynamical variable as a fifth component of the energy–momentum vector.

The energy–momentum relation

$$E^2 = \overrightarrow{p}^2c^2 + m^2c^4 \tag{4}$$

can be written with a five dimensional notation:

$$G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\nu}} = 0 \tag{5}$$

where $\widehat{p}_{\hat{\mu}} = (p_\mu, p_4 = -mc)$ (see Fig. 3).

Finally, if we combine momenta and coordinates to form a mixed scalar product, we obtain a new relativistic invariant, which is the differential of the action. In 5D, we shall therefore introduce the superaction:

$$\widehat{S} = -\int \widehat{p}_{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\mu}} \tag{6}$$

equivalent to

$$\widehat{p}_{\hat{\mu}} = -\frac{\partial\widehat{S}}{\partial\widehat{x}^{\hat{\mu}}} \text{ with } \hat{\mu} = 0, 1, 2, 3, 4 \tag{7}$$

If this is substituted in

$$G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\mu}} = 0 \tag{8}$$

we obtain the Hamilton–Jacobi equation in 5D

$$G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\widehat{S}\,\partial_{\hat{\nu}}\widehat{S} = 0 \tag{9}$$

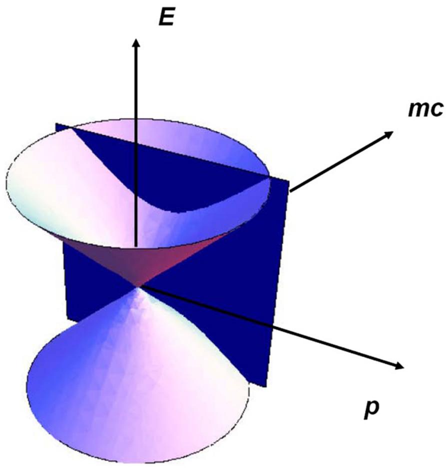

Fig. 3. 5D energy–momentum picture.

which has the same form as the eikonal equation for light in 4D. It is already this striking analogy that pushed Louis de Broglie to identify action and the phase of a matter wave in the 4D case [24]. We shall follow the same track for a quantum approach in our 5D case.

What is the link between the three previous invariants given above? As in optics, the direction of propagation of a particle is determined by the momentum vector tangent to the trajectory. The 5D momentum can therefore be written in the form:

$$\widehat{p}^{\hat{\mu}} = \mathrm{d}\widehat{x}^{\hat{\mu}}/\mathrm{d}\lambda \tag{10}$$

where λ is an affine parameter varying along the ray. This is consistent with the invariance of these quantities for uniform motion.

In 4D the canonical 4-momentum is:

$$p_\mu = mc\frac{g_{\mu\nu}\mathrm{d}x^\nu}{\sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}} = m\,c\,g_{\mu\nu}u^\nu \tag{11}$$

where $u^\nu = \mathrm{d}x^\nu/c\,\mathrm{d}\tau$ is the normalized 4-velocity with $c\,\mathrm{d}\tau = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$.

We observe that $\mathrm{d}\lambda$ can always be written as the ratio of a time to a mass:

$$\mathrm{d}\lambda = \frac{\mathrm{d}\tau}{m} = \frac{\mathrm{d}t}{m^*} = \frac{\mathrm{d}\theta}{M} = \ldots \tag{12}$$

where τ is the proper time of individual particles (e.g., atoms in a clock or in a molecule), t is the time coordinate of the composed object (clock, interferometer, or molecule) and θ its proper time; m, m^*, M are, respectively, the mass, the relativistic mass of individual particles and their contribution to the scalar mass of the device or composed object.

In the usual paradigm of relativity, the time t is a coordinate variable and the proper time τ is taken as the evolution parameter to describe the motion of particles in space–time. In this presentation however, proper time is an independent coordinate describing the internal motion of massive particles, so that we shall rather choose the coordinate time as the evolution parameter. We shall therefore write in 5D:

$$\widehat{p}_{\hat{\mu}} = m^* G_{\hat{\mu}\hat{\nu}}\dot{\widehat{x}}^{\hat{\nu}} = m^*\dot{\widehat{x}}_{\hat{\mu}} \tag{13}$$

expressed with the “relativistic mass“:

$$m^* = m\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{m\,c}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \tag{14}$$

and where the dot refers to derivation with respect to a “laboratory time” (identical to the proper time θ of the apparatus only in the absence of gravitation or inertial effects). With this choice $\dot{\widehat{x}}^0 = c$ and $\widehat{p}^0 = m^*c$. An alternate choice could be to take the proper time θ of the full device as the evolution parameter. In which case:

$$c\,\mathrm{d}\theta = \sqrt{G_{00}}\,\mathrm{d}x^0 \text{ and } M = m^*\sqrt{G_{00}} \tag{15}$$

From

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\widehat{x}^{\hat{\mu}} d\widehat{x}^{\hat{\nu}} = 0 \tag{16}$$

we infer in 5D

$$d\widehat{S} = 0 \tag{17}$$

We shall generalize these relations to an object, such as a clock, a molecule..., composed of a number of subparticles and illustrate the origin of proper time as coming from the inner structure of the object. For such composite objects, mass is given by the eigenvalues of their internal Hamiltonian and thus becomes an operator in the quantum theory.

This picture can be generalized with the introduction of the electromagnetic interaction potentials in the 5D metric tensor.

3. Generalization in the presence of gravitational and electromagnetic interactions

3.1. Generalization of the metric tensor from the generalized interval

The previous 5D scheme can be extended to general relativity with a 4D metric tensor $g^{\mu\nu}$ and an electromagnetic 4-potential A_μ

$$g^{\mu\nu}\left(p_\mu - qA_\mu\right)(p_\nu - qA_\nu) = m^2c^2 \tag{18}$$

($q = -e$ for the electron).

We shall search for a metric tensor $G_{\hat{\mu}\hat{\nu}}$ for 5D such that the generalized interval given by:

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\widehat{x}^{\hat{\mu}} d\widehat{x}^{\hat{\nu}}$$

is an invariant.

Let us recall that, from the equivalence principle, the metric tensor $g^{\mu\nu}$ can be obtained from the Minkovski flat space–time tensor $\eta^{\mu\nu}$ using infinitesimal frame transformations from a locally inertial frame. Quite generally, any infinitesimal coordinate transformation considered as a gauge transformation can be used to introduce a component of the gravito-inertial field. As an example, in 4D, the transformation (case of a rotation):

$$\begin{aligned} dx'^i &= dx^i + \alpha_0^i dx^0 \\ dx'^0 &= dx^0 \end{aligned} \tag{19}$$

transforms the interval

$$ds^2 = g'_{00}(dx'^0)^2 + g'_{ij} dx'^i dx'^j \tag{20}$$

into

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i} dx^0 dx^i + g_{ij} dx^i dx^j \tag{21}$$

with

$$g_{00} = g'_{00} + \alpha_0^i \alpha_0^j g'_{ij} \tag{22}$$

$$g_{0j} = \alpha_0^i g'_{ij} \tag{23}$$

$$g_{ij} = g'_{ij} \tag{24}$$

$$g^{00} = g'^{00} = 1/g'_{00} \tag{25}$$

$$g'^{ij} = 1/g'_{ij} \tag{26}$$

Using

$$g_{ij} g^{i0} = -g^{00} g_{j0} \tag{27}$$

we find

$$\alpha_0^i = -\frac{g^{i0}}{g^{00}} \tag{28}$$

$$\alpha_0^i \alpha_0^j g'_{ij} = -\frac{g_{i0} g^{i0}}{g^{00}} \tag{29}$$

In the case of rotation we recover the usual metric tensor in the rotating frame.

The action S becomes

$$S = -\int p'_\mu \mathrm{d}x'^\mu = -\int p'_0 \mathrm{d}x'^0 - \int p'_i \mathrm{d}x'^i \tag{30}$$

$$= -\int p'_0 \mathrm{d}x^0 - \int p'_i (\mathrm{d}x^i + \alpha^i_0 \mathrm{d}x^0) \tag{31}$$

$$S = -\int \left(p'_0 + p'_i \alpha^i_0\right) \mathrm{d}x^0 - \int p'_i \mathrm{d}x^i = -\int p_\mu \mathrm{d}x^\mu$$

which gives the Sagnac phase as $\int \left(p_i g^{i0}/g^{00}\right) \mathrm{d}x^0$.

The same approach can be used with the fifth dimension by introducing the gauge transformation

$$\begin{aligned} \mathrm{d}x'^4 &= \mathrm{d}x^4 + \beta^4_\mu \mathrm{d}\widehat{x}^\mu \\ \mathrm{d}\widehat{x}'^\mu &= \mathrm{d}\widehat{x}^\mu \end{aligned} \tag{32}$$

to generate the off-diagonal elements $G_{\mu 4}$

$$\mathrm{d}\sigma^2 = G_{44}\left(\mathrm{d}x^4\right)^2 + 2\, G_{44}\beta^4_\mu \mathrm{d}x^4 \mathrm{d}\widehat{x}^\mu + \left(g_{\mu\nu} + \beta^4_\mu \beta^4_\nu G_{44}\right) \mathrm{d}\widehat{x}^\mu \mathrm{d}\widehat{x}^\nu$$

$$G_{44} = G'_{44} \tag{33}$$

$$G_{\mu 4} = \beta^4_\mu G_{44} \tag{34}$$

$$G_{\mu\nu} = g_{\mu\nu} + \beta^4_\mu \beta^4_\nu G_{44} \tag{35}$$

The superaction $\widehat{S}$ given by (6) becomes

$$\widehat{S} = -\int \widehat{p}_{\hat{\mu}} \mathrm{d}\widehat{x}'^{\hat{\mu}} = -\int p_\mu \mathrm{d}\widehat{x}'^\mu - \int \widehat{p}_4 \mathrm{d}x'^4 \tag{36}$$

$$= -\int p_\mu \mathrm{d}\widehat{x}^\mu + \int m\, c\, (\mathrm{d}x^4 + \beta^4_\mu \mathrm{d}\widehat{x}^\mu) \tag{37}$$

$$\widehat{S} = -\int \left(p_\mu - m\, c\, \beta^4_\mu\right) \mathrm{d}\widehat{x}^\mu + \int m\, c^2 \mathrm{d}\tau \tag{38}$$

which yields the Aharonov–Bohm phase if $m\, c\, \beta^4_\mu = q\, A_\mu$.

The metric tensor in five dimensions $G_{\mu\nu}$ is thus written as in Kaluza's theory to include the electromagnetic gauge field potential A_μ

$$\begin{aligned} G_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{4\nu} & G_{44} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 G_{44} A_\mu A_\nu & \kappa\, G_{44} A_\mu \\ \kappa\, G_{44} A_\nu & G_{44} \end{pmatrix} \\ G^{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} G^{\mu\nu} & G^{\mu 4} \\ G^{4\nu} & G^{44} \end{pmatrix} = \begin{pmatrix} g^{\mu\nu} & -\kappa\, A^\mu \\ -\kappa\, A^\nu & G^{44} \end{pmatrix} \end{aligned} \tag{39}$$

where κ is given by the charge-to-mass ratio of the object. This metric tensor is such that

$$\begin{aligned} G^{\hat{\mu}\hat{\lambda}} G_{\hat{\lambda}\hat{\nu}} &= \begin{pmatrix} G^{\mu\lambda} & G^{\mu 4} \\ G^{4\lambda} & G^{44} \end{pmatrix} \begin{pmatrix} G_{\lambda\nu} & G_{\lambda 4} \\ G_{4\nu} & G_{44} \end{pmatrix} = \delta^{\hat{\mu}}_{\hat{\nu}} \\ &= \begin{pmatrix} G^{\mu\lambda} & -\kappa\, A^\mu \\ -\kappa\, A^\lambda & G^{44} \end{pmatrix} \begin{pmatrix} g_{\lambda\nu} + \kappa^2 G_{44} A_\lambda A_\nu & +\kappa\, G_{44} A_\lambda \\ +\kappa\, G_{44} A_\nu & G_{44} \end{pmatrix} \\ &= \begin{pmatrix} G^{\mu\lambda} g_{\lambda\nu} & \kappa\, G_{44} G^{\mu\lambda} A_\lambda - \kappa\, G_{44} A^\mu = 0 \\ -\kappa\, A^\lambda (g_{\lambda\nu} + \kappa^2 G_{44} A_\lambda A_\nu) + \kappa\, G_{44} G^{44} A_\nu & -\kappa^2 G_{44} A^\lambda A_\lambda + G^{44} G_{44} \end{pmatrix} = \delta^{\hat{\mu}}_{\hat{\nu}} \end{aligned} \tag{40}$$

which implies

$$\begin{aligned} G^{\mu\lambda} g_{\lambda\nu} &= \delta^\mu_\nu \\ G^{44} &= 1/G_{44} + \kappa^2 A^\lambda A_\lambda \end{aligned} \tag{41}$$

The equation

$$G^{\hat{\mu}\hat{\nu}} \widehat{p}_{\hat{\mu}} \widehat{p}_{\hat{\nu}} = 0 \tag{42}$$

with

$$\widehat{p}_{\hat{\mu}} = (p_\mu, -mc) \tag{43}$$

and $G_{44} = -1$ is therefore equivalent to equation (18)

$$g^{\mu\nu}\left(p_\mu - qA_\mu\right)(p_\nu - qA_\nu) = m^2c^2 \tag{44}$$

Higher-order electromagnetic interactions are introduced via the multipolar expansion $p_\mu - qA_\mu + Q^\lambda F_{\mu\lambda}$, where dipole moments will become operators in the quantum description.[5]

The fundamental constants associated with this geometry and its metric tensor are:

$$c, m, \text{and } e \tag{45}$$

which together with $\hbar$ and k_B constitute the natural set of fundamental constants to be fixed for the new system of units. The choice of the reference mass m is made according to the best possible clock required to define the unit of proper time. Clearly, today this mass has to be the mass difference of the atomic levels corresponding to the transition defining this unit.

We can derive the direct link between the coordinate $\widehat{x}^4$ and the proper time τ starting from:

$$G_{\hat{\mu}\hat{\nu}}\mathrm{d}\widehat{x}^{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\nu}} = 0$$

we obtain

$$\begin{aligned}\mathrm{d}x^4 &= -\frac{G_{4\mu}}{G_{44}}\mathrm{d}\widehat{x}^\mu + \frac{1}{G_{44}}\left(G_{4\mu}G_{4\nu} - G_{44}G_{\mu\nu}\right)^{1/2}\left(\mathrm{d}\widehat{x}^\mu\mathrm{d}\widehat{x}^\nu\right)^{1/2} \\ &= -\frac{G_{4\mu}}{G_{44}}\mathrm{d}\widehat{x}^\mu + \frac{1}{\sqrt{-G_{44}}}\left(g_{\mu\nu}\mathrm{d}\widehat{x}^\mu\mathrm{d}\widehat{x}^\nu\right)^{1/2}\end{aligned}$$

which, compared to

$$\mathrm{d}x_4 = G_{44}\mathrm{d}x^4 + G_{4\mu}\mathrm{d}\widehat{x}^\mu$$

gives

$$\mathrm{d}x_4 = -\sqrt{-G_{44}}\left(g_{\mu\nu}\mathrm{d}\widehat{x}^\mu\mathrm{d}\widehat{x}^\nu\right)^{1/2} = -\left(g_{\mu\nu}\mathrm{d}\widehat{x}^\mu\mathrm{d}\widehat{x}^\nu\right)^{1/2}$$

3.2. Complex bodies (composite objects)

We can connect this proper time with an intra-atomic or intra-molecular motion and thus link mass and internal kinetic energy. If we consider an object (such as a clock, a molecule...) composed of a number of subparticles, the 5D superaction differential is given by the sum:

$$\mathrm{d}\widehat{S} = \sum_\mathrm{A}\left(-p_{\mathrm{A}\mu}\mathrm{d}x_\mathrm{A}^\mu + m_\mathrm{A}c^2\mathrm{d}\tau_\mathrm{A}\right) \tag{46}$$

where m_A is the mass of particle A. With the following change of coordinates:

$$\mathrm{d}x_\mathrm{A}^\mu = \mathrm{d}X^\mu + \mathrm{d}\xi_\mathrm{A}^\mu \tag{47}$$

$$\mathrm{d}\widehat{S} = -P_\mu\mathrm{d}X^\mu + \sum_\mathrm{A}\left(-p_{\mathrm{A}\mu}\mathrm{d}\xi_\mathrm{A}^\mu + m_\mathrm{A}c^2\mathrm{d}\tau_\mathrm{A}\right) \tag{48}$$

with

$$P_\mu = \sum_\mathrm{A} p_{\mathrm{A}\mu} \text{ and } p_{\mathrm{A}\mu} = \left(m_\mathrm{A}^*/m^*\right)P_\mu + \pi_{\mathrm{A}\mu} \tag{49}$$

The coordinates X^μ and ξ_A^μ are such that

$$\sum_\mathrm{A} m_\mathrm{A}^*\mathrm{d}\xi_\mathrm{A}^\mu = 0 \text{ and } \xi_\mathrm{A}^0 = 0 \tag{50}$$

(common time coordinate for all the particles of the composed object). One obtains for the full object:

$$\mathrm{d}\widehat{S} = -P_\mu\mathrm{d}X^\mu + Mc^2\mathrm{d}\theta = -P_{\hat{\mu}}\mathrm{d}X^{\hat{\mu}} = 0 \tag{51}$$

[5] For atoms interacting with light, these dipole moment operators are off-diagonal with respect to internal states, i.e. mass states and equation (42) leads to coupled equations for the different states. As examples, references [28,10] present coupled Dirac equations for two-level systems.

provided that:

$$Mc^2\mathrm{d}\theta = \sum_{\mathrm{A}} \left(-p_{\mathrm{A}\mu}\mathrm{d}\xi_{\mathrm{A}}^{\mu} + m_{\mathrm{A}}c^2\mathrm{d}\tau_{\mathrm{A}}\right) \tag{52}$$

$$= \sum_{\mathrm{A}} \left(-\pi_{Aj}\mathrm{d}\xi_{\mathrm{A}}^{j} + m_{\mathrm{A}}c^2\mathrm{d}\tau_{\mathrm{A}}\right) \tag{53}$$

This calculation can be easily generalized to take into account the electromagnetic interaction potentials between the constitutive particles:

$$\sum_{\mathrm{A}} \left(-p_{\mathrm{A}\mu}\frac{q_{\mathrm{A}}}{m_{\mathrm{A}}}A^{\mu}\mathrm{d}\widehat{x}_{A4}\right) \tag{54}$$

In all cases, the source of the proper time θ for the object lies in the internal degrees of freedom and its mass Mc^2 is given by its internal Hamiltonian. A well-defined quantum phase for the composed object requires that it should be in an eigenstate of this internal Hamiltonian. For objects like atoms, molecules..., mass is thus an operator with quantized eigenvalues. The description of the interaction with an external electromagnetic field requires to deal with coupled equations corresponding to each of these mass eigenstates. A wave equation will be written for each internal state and each of these is coupled with the other ones through an off-diagonal electric dipole operator.

3.3. Connection with Kaluza's theory

In his search for a unified theory including both gravitational and electromagnetic interactions, Theodor Kaluza has already introduced a 5D space–time in 1921 [4]. His fifth dimension was later interpreted by O. Klein as an internal closed dimension of electrons. We shall now establish the link between this additional dimension and the proper time used in our approach.

The interval squared in Kaluza's theory is written as:

$$\mathrm{d}\sigma^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + \phi^2\left(kA_{\nu}\mathrm{d}x^{\nu} + \mathrm{d}x^5\right)^2$$

corresponding to the metric tensor:

$$\begin{aligned}
\widetilde{G}_{\widetilde{\mu}\widetilde{\nu}} &= \begin{pmatrix} G_{\mu\nu} & G_{\mu 5} \\ G_{5\nu} & G_{55} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} + k^2\phi^2 A_{\mu}A_{\nu} & k\phi^2 A_{\mu} \\ k\phi^2 A_{\nu} & \phi^2 \end{pmatrix} \\
\widetilde{G}^{\widetilde{\mu}\widetilde{\nu}} &= \begin{pmatrix} G^{\mu\nu} & G^{\mu 5} \\ G^{5\nu} & G^{55} \end{pmatrix} = \begin{pmatrix} g^{\mu\nu} & -kA^{\mu} \\ -kA^{\nu} & k^2 g_{\mu\nu}A^{\mu}A^{\nu} + \frac{1}{\phi^2} \end{pmatrix}
\end{aligned} \tag{55}$$

where ϕ is a scalar dilaton field and where

$$k = \frac{4\sqrt{\pi G\varepsilon_0}}{c}$$

We shift to proper time as the new variable:

$$\sqrt{G_{44}}\mathrm{d}x^4 = \mathrm{i}\,\mathrm{d}x^4 = \phi\,\mathrm{d}x^5$$

Kaluza's interval becomes

$$\mathrm{d}\sigma^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + \phi^2\left(kA_{\nu}\mathrm{d}x^{\nu} + \frac{\sqrt{G_{44}}}{\phi}\mathrm{d}x^4\right)^2$$

to be identified with our 5D interval:

$$G_{\hat{\mu}\hat{\nu}}\mathrm{d}\widehat{x}^{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\nu}} = \left(g_{\mu\nu} + \kappa^2 G_{44}A_{\mu}A_{\nu}\right)\mathrm{d}\widehat{x}^{\mu}\mathrm{d}\widehat{x}^{\nu} + 2\kappa G_{44}A_{\nu}\mathrm{d}\widehat{x}^{\nu}\mathrm{d}x^4 + G_{44}\left(\mathrm{d}x^4\right)^2$$

Then:

$$\phi = G_{44}\frac{\kappa}{\mathrm{i}k} = \mathrm{i}\frac{\kappa}{k}$$

is given by the ratio of charge-to-mass ratios of the electron and of Planck's particle, and

$$\frac{\mathrm{d}x^4}{\mathrm{d}x^5} = -\mathrm{i}\phi = \frac{\kappa}{k} = \frac{e}{m\,c}\frac{c}{4\sqrt{\pi\,G\,\varepsilon_0}} = \frac{1}{2}\sqrt{\frac{\alpha}{\alpha_G}}$$

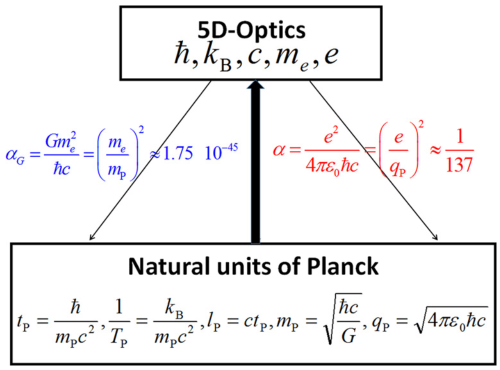

Fig. 4. Link between Planck's units and proposed SI units thanks to the fine structure and to the gravitational constants. The present development of atomic clocks and of quantum electric metrology obliges us to back up from Planck's units to a more realistic choice for time and electrical units. Similarly, the electron mass is not an option for today and should be replaced by the mass difference between the internal energy levels of an atom. This connection involves again a dimensionless constant such as the fine structure constant for the Rydberg constant in the case of hydrogen transitions.

The dispersion relations are respectively:

$$g^{\mu\nu} p_\mu p_\nu - 2kA^\mu p_\mu p_5 + \left(k^2 g_{\mu\nu} A^\mu A^\nu + \frac{1}{\phi^2}\right)(p_5)^2 = 0$$

and

$$g^{\mu\nu}\left(p_\mu - qA_\mu\right)\left(p_\nu - qA_\nu\right) = m^2c^2 \tag{56}$$

which gives the momentum:

$$p_5 = -\mathrm{i}\,\phi\, m\, c = e/k$$

connected to charge as expected.

The metric tensor introduced by Kaluza imposes a set of natural units which coincides with those introduced by Planck in 1899. In this system the properties of electrons are given by two dimensionless constants, which are the fine structure constant and its analog for gravitation. At the present time, Planck's system of units is too far from a realistic system of units, essentially because G cannot be measured with sufficient accuracy and cannot compete with present atomic clocks (see Fig. 4). Our 5D geometry using proper time is more general, since it applies to any composite object, including of course atoms and molecules that are used as clocks. The mass of electrons appears then as the most elegant choice to define the unit of proper time. Unfortunately, we do not know yet how to build a good clock based upon electron–positron annihilation, and an atomic transition is the best compromise combining high accuracy and a close connection with the electron's mass. The future choice of the time unit is a fascinating question that is widely debated today and evolves quickly with the permanent progress of clocks in the optical domain and beyond [25–27]. Finally, given the role of gyromagnetic ratios in the theory, consistency imposes the electron charge to define the unit of electric charge rather than the electric properties of vacuum, which is the case today.

3.4. Conclusions

The most basic mathematical tool of statistical quantum mechanics, which is the density operator ρ, involves both action and entropy and hence requires both Planck's constant $\hbar$ and Boltzmann's constant k_B. This operator acts in a phase space that, in turn, contains the geometry and the three corresponding fundamental constants c, $\Delta m_{at}, e$ thanks to the metric tensor.

Action and entropy are the two corner stones of the future SI. They should be explicitly introduced in its formulation. Mass and time units are coupled through $\hbar$ and both fixed by the choice of the pair of atomic levels having the mass difference Δm_{at} corresponding to the transition Bohr frequency $\Delta m_{at} c^2/h$. The ratio $\Delta m_{at}/m_e$ is obtained with high accuracy from the Rydberg constant.

A natural theoretical framework for the redefinition of the SI is completely provided by the connection between 5D geometry, metric tensor, and metrology that we have outlined. To reduce the theory of measurements to the determination of quantum phases was our primary objective. The direct link between action and matter–wave interferometry can be found in a number of references dealing with all recent applications of this field to metrology. This approach is an attempt to unify all aspects of modern quantum metrology. The final aim is, of course, to adopt a system of units free of arbitrary and artificial features, in harmony with contemporary physics. The perspective that we have adopted incorporates naturally all relevant fundamental constants in a logical scheme, with obvious constraints of economy, aesthetics, and rigour.

Acknowledgement

The author is especially grateful to Dr. Christophe Salomon for very constructive comments about this paper.

References

[1] C.J. Bordé, J. Kovalevsky, Fundamental metrology, C. R. Physique 5 (2004) 789–940, thematic issue.
[2] C.J. Bordé, Métrologie fondamentale: unités de base et constantes fondamentales, C. R. Physique 5 (2004) 813–820.
[3] C.J. Bordé, Base units of the SI, fundamental constants and modern quantum physics, Philos. Trans. R. Soc., Math. Phys. Eng. Sci. 363 (2005) 2177–2202 [p. 2182].
[4] T. Kaluza, Zum Unitätsproblem der Physik, Sitz.ber. Preuss. Akad. Wiss. Phys. Math. K 1 (1921) 966.
[5] F. Piquemal, A. Bounouh, L. Devoille, N. Feltin, O. Thevenot, G. Trapon, Fundamental electrical standards and the quantum metrological triangle, C. R. Physique 5 (2004) 857–879.
[6] C.J. Bordé, Atomic interferometry with internal state labelling, Phys. Lett. A 140 (1989) 10–12.
[7] C.J. Bordé, Atomic clocks and inertial sensors, Metrologia 39 (5) (2002) 435–463.
[8] P. Berman (Ed.), Atom Interferometry, Academic Press, 1997.
[9] C.J. Bordé, Matter–wave interferometers: a synthetic approach, in: P. Berman (Ed.), Atom Interferometry, Academic Press, 1997.
[10] C.J. Bordé, Atom interferometry using internal excitation: foundations and recent theory, in: International School of Physics "Enrico Fermi" – COURSE CLXXXVIII Atom Interferometry, 2014, pp. 143–170.
[11] C.J. Bordé, J.L. Hall, Direct resolution of the recoil doublets using saturated absorption techniques, Bull. Am. Phys. Soc. 19 (1974) 1196.
[12] J.L. Hall, C.J. Bordé, K. Uehara, Direct optical resolution of the recoil effect using saturated absorption spectroscopy, Phys. Rev. Lett. 37 (1976) 1339–1342.
[13] C.J. Bordé, On the theory of linear absorption line shapes in gases, C. R. Physique 10 (2009) 866–882.
[14] M. Cadoret, et al., Combination of Bloch oscillations with a Ramsey–Bordé interferometer: new determination of the fine structure constant, Phys. Rev. Lett. 101 (2008) 230801.
[15] R. Bouchendira, et al., New determination of the fine structure constant and test of the quantum electrodynamics, Phys. Rev. Lett. 106 (2011) 080801.
[16] B. Young, M. Kasevich, S. Chu, Precision atom interferometry with light pulses, in: P. Berman (Ed.), Atom Interferometry, Academic Press, 1997.
[17] J.E. Zimmerman, J.E. Mercereau, Compton wavelength of superconducting electrons, Phys. Rev. Lett. 14 (1965) 887–888.
[18] W.H. Parker, M.B. Simmonds, Measurement of h/m_e using rotating superconductors, in: Precision Measurement and Fundamental Constants, NBS Special Publication, 1971, pp. 243–247.
[19] R.D. Vocke Jr., S.A. Rabb, G.C. Turk, Absolute silicon molar mass measurements, the Avogadro constant and the redefinition of the kilogram, Metrologia 51 (2014) 361–375.
[20] B. Andreas, et al., Counting the atoms in a ^{28}Si crystal for a new kilogram definition, Metrologia 48 (2011) S1–S13.
[21] B.P. Kibble, A measurement of the gyromagnetic ratio of the proton by the strong field method, in: J.H. Sanders, A.H. Wapstra (Eds.), Atomic and Fundamental Constants, vol. 5, Plenum Press, New York, 1975, pp. 545–551.
[22] C.J. Bordé, Density matrix equations and diagrams for high resolution non-linear laser spectroscopy: application to Ramsey fringes in the optical domain, in: Advances in Laser Spectroscopy, Plenum Press, New York, 1983, pp. 1–70.
[23] C.J. Bordé, M.E. Himbert, Experimental determination of Boltzmann's constant, C. R. Physique 10 (2009) 813–915, thematic issue.
[24] L. de Broglie, Ondes et quanta, C. R. Hebd. Séances Acad. Sci. 2 (1923) 507.
[25] C. Salomon, The measurement of time, C. R. Physique 16 (2015) 459–585, thematic issue.
[26] F. Riehle, Towards a redefinition of the second based on optical atomic clocks, C. R. Physique 16 (2015) 506–515.
[27] M. Abgrall, et al., Atomic fountains and optical clocks at SYRTE: status and perspectives, C. R. Physique 16 (2015) 461–470.
[28] C.J. Bordé, A. Karasiewicz, P. Tourrenc, General relativistic framework for atomic interferometry, Int. J. Mod. Phys. D 3 (1994) 157–161.