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Gravitational quantum collapse in dilute systems

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Abstract. Penrose has suggested that large fluctuations of the gravitational energy of quantum systems, resulting from fluctuations of its density in space, may induce a quantum collapse mechanism [1], but he did not propose a precise dynamics for this process. We use the GBC (Gravitational Bohmian Collapse) model [2], which provides such a dynamics. The effects of collapse in dilute quantum systems are investigated, both in ordinary 3D space and in configuration space. We first discuss how a single result appears during a quantum measurement. The GBC model predicts a continuous but very fast evolution of the state vector that, at the end of the measurement, reproduces the von Neumann projection postulate. This ensures that the model remains compatible with the relativistic nosignaling constraint. In the absence of any measurement, we study the spontaneous effects of the GBC process, which depend on the quantum correlation function of observables with the spatial density operator. If the selected observable is the local current of the density fluid, we show that the collapse term leads to modifications of the Newton force, in a scalar or tensor form.

Keywords. quantum collapse, quantum measurement, modified Schrödinger dynamics, modified gravity, Penrose model.

Mots-clés. mesure quantique, collapse dynamique, dynamique de Schrödinger modifiée, gravitation.

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In quantum mechanics, the so called "measurement problem" arises because the Schrödinger equation alone is not able to predict the emergence of a single result in a quantum measurement experiment. Instead, and as noted in 1933 by von Neumann in his famous book [3], it predicts the appearance of a QSMDS (Quantum Superposition of Macroscopically Different States) containing all possible results at the same time. This superposition propagates further and further into the environment, without ever resolving into a single macroscopic component. The difficulty can be overcome by various interpretations. The Copenhagen interpretation for instance limits the validity of the Schrödinger equation by introducing a "cut", or "shifty split", between the measured system and the macroscopic measurement apparatus, and then stating that the equation should not be used beyond this cut. Von Neumann proposed another solution by introducing his "projection postulate", which is nowadays introduced in most textbooks on quantum mechanics. Many different interpretations have been proposed by various authors. Still another solution is to modify the theory and the Schrödinger equation in order to resolve QSMDS.

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In 1996, Penrose [1] suggested that QSMDS could indeed be spontaneously resolved into a single macroscopic component under the effect of gravitational attraction. Introducing such a mechanism of gravitational quantum collapse into the theory would certainly solve the difficulties mentioned above; for instance the famous Schrödinger cat paradox would immediately vanish. Penrose nevertheless proposed no specific mechanism for this collapse, relying just on a qualitative argument involving the fluctutations of the gravitational energy of the system and the time-energy Heisenberg uncertainty relation. A more precise mechanism was proposed in [2], with a simple model where the source of gravitational attraction is the Bohmian positions of the particles, and where the gravitational constant contains a small imaginary part to produce the collapse. As it is, the model remains naive since, for instance, it uses the dBB (de Broglie–Bohm) theory to obtain the motion of the particles, which means that it is non-relativistic. We will call it the GBC (Gravitational Bohmian Collapse) model.

Modifying the standard Schrödinger dynamics to include a collapse mechanism is, of course, not a new idea. In 1986, Ghirardi, Rimini and Weber introduced the GRW theory [4], where sudden random spatial localization processes are added into the quantum dynamics; a stochastic non-Hamiltonian term is introduced into the Schrödinger equation. Soon after, a similar but continuous dynamics was proposed in 1989 by Pearle with the CSL theory [5, 6], based on the inclusion of Wiener processes in the quantum dynamics. An interesting feature of these theories is that they make predictions that differ from those of standard quantum mechanics, which means that they can be tested experimentally [7]. Another common feature is that they require the introduction of two dimensional constants, usually a localization length and a time constant.

In 1989, Diósi proposed to relate dynamical collapse theories to gravity in order to suppress the necessity for the introduction of these new constants, and derive a collapse dynamics depending only of the Newton constant. Nevertheless, Ghirardi, Grassi and Rimini soon showed [8] that this theory makes predictions that are contradictory with known facts, in particular in nuclear physics. The conclusion at the time was that, even if the Newton gravitational constant is used in the theory, at least another dimensional constant has to be introduced into the dynamics in order to avoid contradictions with known experimental facts.

By contrast, the GBC gravitational collapse mechanism introduces no dimensional parameter, but just one dimensionless constant ε: the Newton gravitational constant G is replaced by $G' = G(1 - i\varepsilon)$ where $\varepsilon \simeq 10^{-3}$. Another difference is that the dynamics of the collapse is a collective effect, while in the GRW theory all particles in the physical system undergo independent localization processes in parallel. In GBC dynamics, all Bohmian positions of the particles contribute to the gravitational potential, which in turn tends to localize each particle in the regions of low values of the potential. Several consequences result from this collective character. First, for macroscopic systems, small values of ε are sufficient to obtain a very fast collapse. Second, the border between microscopic and macroscopic physical systems is very sharp [2], since for a system of typical size ℓ the collapse time constant varies as ℓ^5. Microscopic systems, for which the gravitational self-attraction is extremely small, are almost not perturbed by the collapse process; macroscopic systems undergo very fast collapse. As a result, only situations involving a QSMDS in space (large quantum fluctuation of the density in ordinary space) are subject to rapid collapse.

Another feature of GBC is that the model does not involve any fluctuating term in the dynamics, or Wiener process having an infinite spectrum. The GBC dynamics is continuous and deterministic, the only random component being the initial values of all Bohmian positions in the system. This is similar with the non-Markovian collapse model proposed by Tilloy and Wiseman [9], where the evolution of the conditional state vector of the system S is controlled by the evolution of the Bohmian positions of the particles in a bath entangled with S (but with no role of gravity). But this is also completely different from the other proposals relating gravity and quantum collapse, which involve random functions: for instance Pearle and Squires use a theory where the field causing collapse is the gravitational curvature scalar containing a white noise fluctuating source [10]; Tilloy and Diósi build a semiclassical gravity in the Newtonian limit where the source of gravity contains a white noise component [11]; Adler proposes to add a complex part in the classical space-time metric, which is somewhat similar to GBC, but this part is a fluctuating noise [12]; similar ideas have been developed by Gasbarri et al [13]. Generally speaking, it is well-known that any modification of the Schrödinger dynamics should be combined with some random element in the dynamics to avoid the possibility of faster than light communication [14,15]. With GBC, this random component is the initial Bohmian positions of the particles, which ensures the required nonsignaling property (§ 2.2).

In this article, we study a few physical predictions of the GBC model. The reason why we focus on dilute systems is as follows. Within a piece of solid sitting in a well defined region of space (no QSMDS), the gravitational constant created by all its particles varies slowly inside its volume, with a maximum at the center of the solid. In GRW and CSL theories, every particle inside the solid is subject to localization processes that localize it within a microscopic length (of the order of 10^{-7} m), which increases its energy; this constantly heats up the solid, even if at a very small rate. Moreover the bulk matter properties of the solid are changed [16]. Within the GBC dynamics, the localization length is much larger, since it is of the order of the size ℓ of the macroscopic object, which means that the energy transfer and the heating rate are much smaller. The net effect is rather a collapse of the piece of solid inside itself, an effect that is counterbalanced by the finite compressibility of the solid, so that the density change remains extremely small. In a gas, or in a dilute cloud of solid objects, the situation is different: no restoring force opposes the density condensation induced by the collapse, and much larger effects can be expected.

In § 1, we briefly summarize the basic equations of the GBC dynamics. In § 2, we come back to a subject that was already studied in [2], the effects of gravitational collapse during quantum measurements, and the absence of superluminal signaling. In § 3, we study spontaneous effects occuring in situations other than quantum measurement, and calculate the evolution of the local density and current of the probability fluid in ordinary 3D space. In particular, we show how the localization process may induce a change of the gravitational constant. In § 4, we study the motion of the probability fluid in configuration space, and discuss the validity of quantum equilibrium in various situations.

1. Dynamics of the model

In this section, we briefly recall the main equations of the GBC dynamics. The Hamiltonian H of a physical system is the sum of its usual Hamiltonian H_{int} and of a gravitational Hamiltonian H_G, due to the attraction of masses with mass density $n_G(\mathbf{r})$:

$$H = H_{\text{int}} + H_G \tag{1}$$

where H_G is given by:

$$H_G = -gGm \int \mathrm{d}^3 r\ \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \int \mathrm{d}^3 r'\ \frac{1}{|\mathbf{r}-\mathbf{r}'|} n_G\left(\mathbf{r}'\right) \tag{2}$$

For the moment, we set $g = 1$; G is Newton's constant, m the mass of the particles, and $\Psi(\mathbf{r})$ the quantum field operator of the particles contained in the physical system. For the sake of simplicity, we have written the source of the gravitational field as the mass density $n_G(\mathbf{r}')$ at the same time. An obvious improvement of (2) would be to insert a retarded value of the potential by replacing this mass density by $n_G(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)$, where c is the speed of light. Since this does not change de discussion in the rest of this article, the simple form (2) will suffice.

By setting $n_G(\mathbf{r}')$ equal to the quantum local density average $\langle\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\rangle$, we would obtain the usual Schrödinger–Newton equation [17]; but we will proceed differently. We now introduce two non-standard assumptions concerning the Hamiltonian H_G describing the internal gravitational attraction inside the system. First, we assume that $n_G(\mathbf{r})$ is determined by the Bohmian positions q_n of the N particles of the system:

$$n_G(\mathbf{r}) = m \sum_{n=1}^{N} \delta\left(\mathbf{r} - \mathbf{q}_n\right) \tag{3}$$

Second, as in Ref. [18], we assume that the dimensionless constant g contains a small imaginary part ε:

$$g = 1 - i\varepsilon \tag{4}$$

The operator H_G is now the sum of an Hermitian part $H_G^0 = H_G(\varepsilon = 0)$ and an antiHermitian part iL, where L is the localization operator:

$$L = \varepsilon G m \int \mathrm{d}^3 r\ \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \int \mathrm{d}^3 r'\ \frac{1}{|\mathbf{r} - \mathbf{r}'|} n_G\left(\mathbf{r}'\right) \tag{5}$$

This operator is diagonal in the position representation.

The non-normalized (if $\varepsilon \neq 0$) state vector $|\Phi(t)\rangle$ evolves according to:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left|\Phi(t)\right\rangle = \left[H_{\text{int}} + H_G^0 + iL\right] \left|\Phi(t)\right\rangle \tag{6}$$

After normalization, this state becomes a state $|\overline{\Phi}(t)\rangle$, which evolves according to [2]:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left|\overline{\Phi}(t)\right\rangle = \left[H_{\text{int}} + H_G^0 + i\varepsilon G m \int \mathrm{d}^3 r \int \mathrm{d}^3 r'\ \left[\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) - D_\Phi(\mathbf{r})\right]\ \frac{1}{|\mathbf{r} - \mathbf{r}'|}\ n_G\left(\mathbf{r}'\right)\right] \left|\overline{\Phi}(t)\right\rangle \tag{7}$$

where $D_\Phi(\mathbf{r})$ is the density of particles in state $|\overline{\Phi}(t)\rangle$:

$$D_\Phi(\mathbf{r}) = \left\langle \overline{\Phi}(t) \middle| \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \middle| \overline{\Phi}(t) \right\rangle \tag{8}$$

2. Fast gravitational collapse during measurements

Technically, the GBC model obtains a collapse of the state vector by enriching the standard Schrödinger dynamics with the addition of a point P in configuration space; the components of P are all Bohmian positions of the particles contained in the physical system. The role of P is somewhat similar to that of the wave singularity in the de Broglie theory of the double solution [19]. The position of P is guided by the wave function, but conversely P reacts on it (as opposed to what happens in usual dBB theory). One can then expect that this point should play the role of an attractor and, if the number of particles is sufficient, that a collective effect might occur that forces the N body wave function to remain in the vicinity of P. We will see that, during a measurement process, this creates the equivalent of the von Neumann projection postulate, with a very sudden projection obtained within a continuous dynamics.

For the sake of simplicity, we assume that the particles are spinless; adding spins within the Pauli spin theory would not change much the discussion, while making the notation more complicated.

2.1. *Appearance of a single result during a quantum measurement*

In the initial stage of a quantum measurement experiment, the measured system S becomes entangled with the measurement apparatus M. The usual de Broglie-Bohm (dBB) theory then predicts that the Bohmian positions of the particles of M begin to play an important role in the dynamics [20]. These positions are for instance the positions of the particles inside the pointer of the measurement apparatus. Together with the positions attached to the measured quantum system S, they determine the position of P.

Within standard dBB theory, during the interaction between S and M, the point P can a priori follow several branches of the wave function (regions of the configuration space where it does not vanish). Each branch is associated with a single measurement result. Because of the cohesive forces between the particles inside the pointer, all individual Bohmian positions have to remain grouped together in one branch of the state vector. This is because the position of the point P in configuration space cannot reach points where the many-body wave function vanishes, in particular points where the positions variables of the particles inside the pointer are spread among different regions of space. Indeed, within standard dynamics, the pointer may end up in a superposition of states indicating different results, but cannot spontaneously decompose into a broken pointer state, a superposition of pointer fragments located at different places. Moreover, as soon as the overlap between the wave functions of these branches tends to zero (this happens because the particles inside the pointer move in different directions in different branches), all the Bohmian positions have to remain inside the same branch until the end of the experiment. Finally, if several realizations of the experiment are performed, in successive realizations the positions may follow different branches; the standard dBB dynamics predicts that the proportion of cases where all positions follow a given branch is given by the Born rule.

Within the GBC dynamics, what happens during a single realiztion of an experiment? Before the measurement process starts, the dynamics of an isolated quantum systems S remains in practice indistinguishable from the standard quantum dynamics in the absence of gravity. It is well known that the gravitational interaction between two protons is about 10^{38} times smaller than the electromagnetic interaction; in addition, the localization term is multiplied by a small parameter $\varepsilon \simeq 10^{-3}$. During the early stages of a measurement, and as long as the entanglement between the measured system and its environment (including the measurement apparatus M) remains microscopic, the effect of the localization term still remains completely negligible. But, within M, successive interactions between neighbour particles make the entanglement progress rapidly, over a distance $\ell(t)$. This distance can be assumed to increase roughly linearly in time: $\ell(t) \simeq vt$, where v is for instance the velocity of sound inside a solid. Now, the inverse time constant $1/\tau$ for collapse increases as the fifth power of $\ell(t)$ [2], so that $1/\tau \sim (vt)^5$. This means that the rate of cancellation of an empty branch of the QSMDS (a branch that does not contain Bohmian positions) increases very rapidly, as the fifth power of time. At a certain point, the progression of entanglement in the measurement apparatus crosses the point where, by a collective gravitational effect inside M, the collapse dynamics becomes very fast and selects the only non-empty component; the transition from one regime to the other is almost instantaneous. In the limit where the process is really instantaneous, we obtain a result that is perfectly equivalent to the standard von Neumann projection postulate.

2.2. *No superluminal communication*

As recalled in the introduction, changing the Schrödinger dynamics, for instance by adding a nonlinear term in the equation, may introduce the possibility of superluminal signaling [14, 15], in contradiction with relativity. A first method to avoid this problem is to introduce stochasticity into the dynamics: in this way, the perturbation introduced by the nonlinear term becomes random, and one can show that it cannot carry messages. This is for example the case in GRW and CSL theories. The GBC model makes use of a second possibility: the dynamics remains deterministic, but the random component is introduced by the initial Bohmian positions of all particles, those of S as well as those of M.

Assume that Alice and Bob do experiments in two remote galaxies, and share an entangled pair of particles. As long as neither of them makes her/his particle interact with a macroscopic system, for instance by making a measurement, no difference with standard quantum mechanics can be observed. This is because, as we have seen, for microscopic systems the GBC localization term remains completely negligible when compared to the standard Hamiltonian. In any case, at this stage of the experiment, the nosignaling relativistic condition does not yet really apply: it forbids instantaneous communication at a macroscopic level between macroscopic observers, but not possible influences at a microscopic level.

The situation changes completely when Alice starts a measurement. Her particle then becomes entangled with another quantum system M that is macroscopic, and a QSMDS is initiated. We have seen above that the rapid propagation of entanglement within M, by successive interactions between neighbour particles, introduces a resolution rate of the QSMDS that varies as the fifth power of time. This results in a projection dynamics that is very sudden. Moreover, it is governed by the initial positions of the Bohmian particles in Alice's laboratory, which she cannot control or determine (and which Bob cannot know either), and which reproduce the Born rule statistically. At the end, the evolution of the state vector is in practice equivalent to that predicted by the instantaneous von Neumann projection postulate. Since this postulate is known to be free of the superluminal communication problem, the GBC model does not have this problem either.

Another way to reach the same conclusion is to analyse how Alice, by choosing the kind of measurement she will perform, can influence the collapse process of the common state vector she shares with Bob, and send him a message in this way. But neither she, nor Bob, can know or control the initial value of the Bohmian positions in her lab. So, the information Alice can send to Bob by projecting the state vector has to be averaged over these positions. But this average is mathematically equivalent to the usual partial trace operation that determines, in standard theory, the density operator received by Bob. It is known that this leads to an operator that is independent of what Alice does, which means that no information is transmitted instantaneously.

2.3. *Quantum equilibrium*

Standard dBB theory predicts that the position distribution of the particles, averaged over many realizations of an experiment, obeys the so called "quantum equilibrium" property: it remains constantly equal to the quantum distribution of probability given by the square of the wave function (in configuration space). Valentini has shown [21, 22] that any deviation from quantum equilibrium would introduce a possibility of superluminal communication: this equilibrium is a necessary condition for avoiding a contradiction with relativity. Moreover, Valentini et al. have shown [23, 24] that, within the dBB dynamics, the quantum equilibrium is an attractor of the dBB dynamics: if, for some reason, the distribution of the positions deviates from the quantum distribution, a fast quantum relaxation process constantly restores quantum equilibrium.

Since GBC changes the evolution of the wave function, and therefore the quantum distribution, quantum equilibrium is no longer necessarily obeyed at any time. In a "normal" situation (no QSMDS), the deviation from quantum equilibrium remains completely negligible: it is created by a gravitational attraction term, which is usually completely ignored in quantum mechanical calculations, and moreover multiplied by a small coefficient ε. In addition, it is constantly counterbalanced by the quantum relaxation process, which tends to restore the equilibrium.

But now consider a situation where standard quantum theory predicts the appearance of a QSMDS. As we have seen above, this is the case, for instance, in the initial stage of a quantum measurement experiment, when the measured physical system S becomes entangled with the various components of the measurement apparatus M. The wave function Φ is strongly attracted towards the region of space where all Bohmian positions move together. This of course completely changes the quantum distribution $|\Phi|^2$ in configuration space, which suddenly destroys the quantum equilibrium between the two distributions. But this disappearance of the quantum equilibrium remains a short transient effect: immediately after the collapse of Φ has taken place, the QSMDS has disappeared, and the quantum relaxation process discussed by Valentini et al. [23, 24] tends to restore the equilibrium. A situation similar to the one before measurement is then quickly recovered.

To summarize, strong deviations from quantum equilibrium may happen during a short time, when the two components of Φ located in two different regions of space begin to appear. But, as soon as the spatial separation becomes large, the equilibrium is promptly restored. In § 4, we come back in more detail to this phenomenon in configuration space.

3. Slow localization effects in dilute systems

The effects of the Hermitian part of the Bohmian gravitational Hamiltonian have already been discussed in several articles, see for instance [25–29] and references contained. Here we focus on the physical effects of the antiHermitian part in situations other than quantum measurements, already discussed in § 2.1. Instead of liquid or solid systems, we consider dilute physical systems having lower densities, so that the effects of the gravitational collapse are much weaker. Moreover, in dense systems, forces between the particles constantly and efficiently tend to oppose changes of the density, which does not happen in dilute systems such as gases. As a result we can expect that, instead of producing a very fast collapse as during a quantum measurement, the gravitational localization term in a dilute system will produce softer and more continuous effects, occurring over much longer time scales.

3.1. *Localization terms*

The Bohmian positions q_n of the N particles of the system are the sources of the gravitational potential $\mathcal{V}_G(\mathbf{r})$:

$$\mathcal{V}_G(\mathbf{r}) = -Gm^2 \sum_{n=1}^{N} \frac{1}{|\mathbf{r}-\mathbf{q}_n|} \tag{9}$$

Equation (7) shows that, in addition to the standard Hamiltonian term (including the gravitational potential $\mathcal{V}_G(\mathbf{r})$), the evolution of the normalized state vector $|\overline{\Phi}(t)\rangle$ contains a localization term that reads:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \left|\overline{\Phi}(t)\right\rangle = \frac{\varepsilon}{\hbar}\int \mathrm{d}^3 r' \;\left[D\left(\mathbf{r}'\right) - \left\langle D\left(\mathbf{r}'\right)\right\rangle\right]\; \left|\mathcal{V}_G\left(\mathbf{r}'\right)\right| \left|\overline{\Phi}(t)\right\rangle \tag{10}$$

where $D(\mathbf{r})$ is the operator associated with the local density of particles:

$$D(\mathbf{r}) = \Psi^{\dagger}(\mathbf{r})\Psi(\mathbf{r}) \tag{11}$$

and $\langle D(\mathbf{r})\rangle$ its average value in state $|\overline{\Phi}(t)\rangle$. Relation (10) indicates that the spontaneous localization process is more effective in regions of space where $|\mathcal{V}_G(\mathbf{r})|$ takes large values. It tends to increase the value of the wave function if the action of the quantum density operator $D(\mathbf{r})$ exceeds that of the multiplication by the local average of this density, to decrease it otherwise.

The evolution of $\langle D(\mathbf{r})\rangle$ due to the localization process is then given by the relation:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \left\langle D(\mathbf{r})\right\rangle = \frac{2\varepsilon}{\hbar}\int \mathrm{d}^3 r' \;\left[\left\langle D(\mathbf{r})D\left(\mathbf{r}'\right)\right\rangle - \langle D(\mathbf{r})\rangle\left\langle D\left(\mathbf{r}'\right)\right\rangle\right]\; \left|\mathcal{V}_G\left(\mathbf{r}'\right)\right| \tag{12}$$

Since:

$$\int \mathrm{d}^3 r \; D(\mathbf{r}) = N \tag{13}$$

it is easy to check that the localization term does not change the total number of particles, as expected. Equation (12) shows that all points of space $\mathbf{r}'$ contribute to the variation ot the number density at any point $\mathbf{r}$, with a weight proportional to the gravitational potential at point $\mathbf{r}'$, but also with a coefficient that is proportional to the density correlation function at the two points of space; the average over all space of this correlation coefficient vanishes.

3.2. *Evolution of the density*

We introduce the density correlation function $\langle F(\mathbf{r},\mathbf{r}')\rangle$ by:

$$\left\langle D(\mathbf{r})D\left(\mathbf{r}'\right)\right\rangle = \langle D(\mathbf{r})\rangle \left\langle D\left(\mathbf{r}'\right)\right\rangle + \left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle \tag{14}$$

If we sum this relation over $\mathrm{d}^3 r$ and use relation (13), we obtain:

$$\int \mathrm{d}^3 r\ \left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle = \int \mathrm{d}^3 r'\ \left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle = 0 \tag{15}$$

This shows that the function $\langle F(\mathbf{r},\mathbf{r}')\rangle$ necessarily takes positive and negative values. If $\mathbf{r}' = \mathbf{r}$, it is positive since:

$$\left\langle \left[D(\mathbf{r})\right]^2\right\rangle > \left\langle D(\mathbf{r})\right\rangle^2 \tag{16}$$

so that:

$$\left\langle F(\mathbf{r},\mathbf{r})\right\rangle > 0 \tag{17}$$

When $|\mathbf{r}'-\mathbf{r}| \to \infty$, the quantum correlations disappear, and:

$$\left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle \underset{|\mathbf{r}'-\mathbf{r}|\to\infty}{\to} 0 \tag{18}$$

Equation (12) reads:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \left\langle D(\mathbf{r})\right\rangle = \frac{2\varepsilon}{\hbar}\int \mathrm{d}^3 r' \left|\mathcal{V}_G\left(\mathbf{r}'\right)\right| \left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle \tag{19}$$

$\langle F(\mathbf{r},\mathbf{r}')\rangle$ has a positive peak at $\mathbf{r}' = \mathbf{r}$, then negative values when $|\mathbf{r}'-\mathbf{r}| \simeq \lambda_c$ (where λ_c is a correlation length), and then tends to zero when $|\mathbf{r}'-\mathbf{r}| \gg \lambda_c$. If we integrate this relation over $\mathrm{d}^3 r$ and use (15), we recover the conservation law of the total number of particles. The same relation shows that, if the gravitational potential $\mathcal{V}_G(\mathbf{r}')$ is constant over space, no evolution of the local density takes place. This evolution is therefore driven by the gravitational gradient, i.e. the gravitational force, rather than the potential itself.

3.3. *Short correlation range*

To have an idea of the origin of the correlation length λ_c, let us for instance assume that the system is described by a number state containing many individual states with populations $n_1, n_2, .., n_p$:

$$\left|\overline{\Phi}_0\right\rangle = \frac{1}{\sqrt{n_1! n_2! \dots n_p!}} \prod_k \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} \dots \left(a_p^\dagger\right)^{n_p} |\mathrm{vac.}\rangle \tag{20}$$

where $|\mathrm{vac.}\rangle$ is the vacuum state. The density-density correlation operator is:

$$\widehat{D}(\mathbf{r})\widehat{D}\left(\mathbf{r}'\right) = \sum_{k,l,k',l'} \varphi_k^*(\mathbf{r})\varphi_l(\mathbf{r})\varphi_{k'}^*\left(\mathbf{r}'\right)\varphi_{l'}(\mathbf{r}')\ a_k^\dagger a_l\ a_{k'}^\dagger a_{l'} \tag{21}$$

where $\varphi_k(\mathbf{r})$ is the wave function of the individual state created by $(a_k)^\dagger$. This leads to:

$$\left\langle F\left(\mathbf{r},\mathbf{r}'\right)\right\rangle = \sum_{k\neq k'} n_k\left(n_{k'}+1\right)\varphi_k^*(\mathbf{r})\varphi_{k'}(\mathbf{r})\varphi_{k'}^*\left(\mathbf{r}'\right)\varphi_k\left(\mathbf{r}'\right) \tag{22}$$

If $\mathbf{r}-\mathbf{r}'$ becomes large, the sum of the products $\varphi_k^*(\mathbf{r}')\varphi_k(\mathbf{r})$ tends to zero by destructive interference between all the oscillating wave functions $\varphi_k(\mathbf{r})$. The characteristic length of this cancellation at large distances is determined by the dispersion of the momenta of the various wave functions $\varphi_k(\mathbf{r})$. Its value depends on the physical system studied, but in many cases we can assume that λ_c is a microscopic length.

We then assume that λ_c is smaller than the charactetistic distances over which $\mathcal{V}_G(\mathbf{r}')$ varies; in particular, we assume the absence of QSMDS such as those appearing for a short time during a quantum measurement process, as discussed in § 2. We can then expand the spatial variations of the potential and obtain:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \langle D(\mathbf{r})\rangle = \frac{2\varepsilon}{\hbar}\boldsymbol{\nabla}\left|\mathcal{V}_G(\mathbf{r})\right|\cdot\int \mathrm{d}^3r'\ \left(\mathbf{r}'-\mathbf{r}\right)\ \langle F\left(\mathbf{r},\mathbf{r}'\right)\rangle \tag{23}$$

where $\boldsymbol{\nabla}\left|\mathcal{V}_G(\mathbf{r})\right|$ is nothing but the local gravitational force $\mathbf{F}_G(\mathbf{r})$.

What appears in the integral in the right side of (23) is only the component z'' of $\mathbf{r}'-\mathbf{r}$ along the direction of the gravitational force. If the function $F(\mathbf{r},\mathbf{r}+\mathbf{r}'')$ is symmetric (even) with respect to the plane $z''=0$, no evolution of the local density takes place under the effect of spontaneous localization; it is therefore the asymmetry of this function with respect to the plane $z''=0$ that drives the evolution of $\langle D(\mathbf{r})\rangle$.

The order of magnitude of the maximum value of the right hand side of relation (23) is then:

$$\frac{\varepsilon}{\hbar}\mathbf{F}_G(\mathbf{r})\langle D(\mathbf{r})\rangle\int_{|\mathbf{r}|\leq\lambda_c}\mathrm{d}^3\mathbf{r}'\,\lambda_c\,\langle D\left(\mathbf{r}'\right)\rangle \simeq \frac{\varepsilon}{\hbar}\mathbf{F}_G(\mathbf{r})\,\lambda_c^4\langle D(\mathbf{r})\rangle^2 \tag{24}$$

The time constant of the relative variation of the local density is obtained by dividing this result by $\langle D(\mathbf{r})\rangle$. To obtain an order of magnitude, we may assume that the system is a gas in standard conditions with a number density of $10^{26}\mathrm{m}^{-3}$, moving in the Earth gravitational field with $m=10^{-26}$kg; we take $\varepsilon=10^{-3}$ and choose the microscopic length $\lambda_c=10^{-9}$ m. We then get a large time constant of the order of 10^6s, probably much too long to be experimentally detected. But this time constant varies very rapidly as a function of λ_c: if we assume that $\lambda_c\simeq 10^{-7}$m, we obtain a time constant of 10^{-2}s, and of course even much shorter time constants if λ_c is larger. The conclusion is that the results are extremely sensitive to the length over which the quantum fluctuations of the density extend. It is therefore difficult to make predictions without a more precise quantum model of the physical system.

3.4. *Evolution of the local current of particles*

The operator associated with the local current of particles is:

$$\mathbf{J}(\mathbf{r}) = \frac{\hbar}{2mi}\left[\Psi^\dagger(\mathbf{r})\boldsymbol{\nabla}\Psi(\mathbf{r})-\boldsymbol{\nabla}\Psi^\dagger(\mathbf{r})\ \Psi(\mathbf{r})\right] \tag{25}$$

The evolution of its average value due to the localization term is:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \langle\mathbf{J}(\mathbf{r})\rangle = \frac{\varepsilon}{\hbar}\int \mathrm{d}^3r'\left[\langle\mathbf{J}(\mathbf{r})D\left(\mathbf{r}'\right)+D\left(\mathbf{r}'\right)\mathbf{J}(\mathbf{r})\rangle-2\langle\mathbf{J}(\mathbf{r})\rangle\langle D\left(\mathbf{r}'\right)\rangle\right]\left|\mathcal{V}_G\left(\mathbf{r}'\right)\right| \tag{26}$$

Now, what determines the evolution is the quantum correlation function between the local density operator $D(\mathbf{r}')$ and the local current $\mathbf{J}(\mathbf{r})$. We then set:

$$\langle \boldsymbol{J}(\boldsymbol{r})D\left(\boldsymbol{r}'\right)+D\left(\boldsymbol{r}'\right)\boldsymbol{J}(\boldsymbol{r})\rangle = 2\left[\langle\mathbf{J}(\mathbf{r})\rangle\langle D\left(\mathbf{r}'\right)\rangle+\langle\mathbf{K}\left(\boldsymbol{r},\boldsymbol{r}'\right)\rangle\right] \tag{27}$$

where $\langle\mathbf{K}(\boldsymbol{r},\boldsymbol{r}')\rangle$ depends on the quantum correlations between $\langle\mathbf{J}(\mathbf{r})\rangle$ at the current point of space $\mathbf{r}$ and the value of the local density $D(\mathbf{r}')$ at all other points of space. We then obtain:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{loc}} \langle\mathbf{J}(\mathbf{r})\rangle = \frac{\varepsilon}{\hbar}\int \mathrm{d}^3r'\left|\mathcal{V}_G\left(\mathbf{r}'\right)\right|\ \langle\mathbf{K}\left(\boldsymbol{r},\boldsymbol{r}'\right)\rangle \tag{28}$$

An integration of (27) over $\mathrm{d}^3 r'$ provides, since the integral of $D(\mathbf{r}')$ over $\mathrm{d}^3 r'$ is equal to a constant N:

$$\int \mathrm{d}^3 r' \, \langle \mathbf{K}(\boldsymbol{r}, \boldsymbol{r}') \rangle = 0 \tag{29}$$

which is similar to (15). This shows that the components of $\mathbf{K}(\boldsymbol{r}, \boldsymbol{r}')$ take positive and negative values when $\mathbf{r}'$ varies. Nevertheless, we do not have the equivalent of the positivity relation (17) with the local current. In any case, relation (29) ensures that the right hand side of (28) vanishes if the gravitational potential $\mathscr{V}_G(\mathbf{r}')$ is independent of $\mathbf{r}'$. Therefore, if $\langle \mathbf{K}(\boldsymbol{r}, \boldsymbol{r}') \rangle$ has a limited range, we can locally expand the variation of the gravitational potential and write:

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{\mathrm{loc}} \langle \mathbf{J}(\mathbf{r}) \rangle = \frac{\varepsilon}{\hbar} \int \mathrm{d}^3 r' \, \left[\mathbf{F}_G(\mathbf{r}) \cdot (\mathbf{r}' - \mathbf{r}) \right] \langle \mathbf{K}(\boldsymbol{r}, \boldsymbol{r}') \rangle \tag{30}$$

In this expression, the gravitational force $\mathbf{F}_G(\mathbf{r}) = \boldsymbol{\nabla} |\mathscr{V}_G(\mathbf{r})|$ can be moved outside of the integral. This shows that the localization process creates an additional term in the time evolution of $\langle \mathbf{J}(\mathbf{r}) \rangle$ that is proportional to the usual gravitational force.

The gravitational force is therefore modified by the localization process, but the additional term is not necessarily collinear with $\mathbf{F}_G(\mathbf{r})$: the relation is tensorial in general, and depends on the properties of the correlation function $\langle \mathbf{K}(\boldsymbol{r}, \boldsymbol{r}') \rangle$. Nevertheless, if the correlation function is invariant by rotation around $\mathbf{F}_G(\mathbf{r})$, the additional force is collinear to the standard gravitational force; the localization term then just appears as a correction of the value of the Newton constant.

As above, evaluating the effect of the localization term quantitatively is difficult, because of the very fast dependence of the time constants on the correlation length λ_c. Nevertheless, effects on the local current seem more accessible to observation than direct effects on the density. In astrophysical systems for instance, even a small change of the velocity may, after propagation during a very long time, lead to larger changes of the density.

4. Probability fluid in configuration space

We now consider the evolution of the probability fluid in the configuration space. For simplicity, we consider an ensemble of spinless particles described by the wave function $\Phi(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N)$. In configuration space, the density of the probability fluid $\rho_N(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N)$ is given by:

$$\rho_N(\mathbf{r}_1, .., \mathbf{r}_N) = \left| \Phi(\mathbf{r}_1, .., \mathbf{r}_N) \right|^2 \tag{31}$$

and the standard expression of the probability current $\mathbf{J}_N(\mathbf{r})$ in this space is:

$$\mathbf{J}_N(\mathbf{r}_1, .., \mathbf{r}_N) = \frac{\hbar}{2mi} \left[\Phi^*(\mathbf{r}_1, .., \mathbf{r}_N) \boldsymbol{\nabla}_N \Phi(\mathbf{r}_1, .., \mathbf{r}_N) - \mathrm{c.c.} \right] \tag{32}$$

where c.c. means "complex conjugate", and $\boldsymbol{\nabla}_N$ denotes the N dimensional gradient. We now proceed to calculate the value of the expression:

$$\frac{\partial}{\partial t} \rho_N(\mathbf{r}_1, .., \mathbf{r}_N) + \boldsymbol{\nabla}_N \cdot \mathbf{J}_N(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N) \tag{33}$$

In standard theory, the value of this expression is zero, which expresses the local conservation of probability. In the gravitational collapse model, since the Schrödinger equation is modified, the expression no longer vanishes in general.

Within this model, the wave function Φ evolves according to:

$$\begin{aligned} \frac{\partial}{\partial t} \Phi(\mathbf{r}_1, .., \mathbf{r}_N) = i \left[\frac{\hbar}{2m} \Delta_N - V(\mathbf{r}_1, .., \mathbf{r}_N) \right] \Phi(\mathbf{r}_1, .., \mathbf{r}_N) \\ + \varepsilon \left[\sum_{n=1}^{N} |\mathscr{V}_G(\mathbf{r}_n)| - \int \mathrm{d}^3 r' \langle D(\mathbf{r}') \rangle \left| \mathscr{V}_G(\mathbf{r}') \right| \right] \Phi(\mathbf{r}_1, .., \mathbf{r}_N) \end{aligned} \tag{34}$$

where Δ_N denotes the N dimensional Laplacian (the sum of N individual Laplacians with respect to $\mathbf{r}_n$). This equation is explicitly nonlinear since $\langle D(\mathbf{r}')\rangle$ depends on Φ:

$$\left\langle D\left(\mathbf{r}'\right)\right\rangle = \sum_{n=1}^{N} \int \mathrm{d}^3 r_1 .. \int \mathrm{d}^3 r_{n-1} .. \int \mathrm{d}^3 r_{n+1} .. \int \mathrm{d}^3 r_N \left|\Phi\left(\mathbf{r}_1, .., \mathbf{r}_n = \mathbf{r}', .., \mathbf{r}_N\right)\right|^2 \tag{35}$$

Relation (32) provides:

$$\boldsymbol{\nabla}_N \cdot \mathbf{J}_N = \frac{\hbar}{2mi}\left[\Phi^* \boldsymbol{\Delta}_N \Phi - \text{c.c.}\right] \tag{36}$$

If we replace $\boldsymbol{\Delta}_N\Phi$ in this expression by the one resulting from (34), we obtain:

$$\boldsymbol{\nabla}_N \cdot \mathbf{J}_N = \left\{-\frac{\partial}{\partial t} + 2\varepsilon\left[\sum_{n=1}^{N}\left|\mathcal{V}_G\left(\mathbf{r}_n\right)\right| - \int \mathrm{d}^3 r' \left\langle D\left(\mathbf{r}'\right)\right\rangle \left|\mathcal{V}_G\left(\mathbf{r}'\right)\right|\right]\right\} \rho_N\left(\mathbf{r}_1, .., \mathbf{r}_N\right) \tag{37}$$

Consider now a distribution of Bohmian positions in configuration space (distribution over many realizations of the experiment) that coincides with the quantum distribution ρ_N. If ε is equal to zero, the two distributions coincide again at time $t+\mathrm{d}t$; this is no longer true if ε does not vanish. The term in ε contains the difference between two different samplings of the gravitational potential: the sum of all $|\mathcal{V}_G(\mathbf{r}_n)|$ at the various 3D components $\mathbf{r}_n$ of the current point P in configureation space, and the 3D average of $|\mathcal{V}_G(\mathbf{r}')|$ over the density $\langle D(\mathbf{r}')\rangle$ associated with the state vector at time t. Two different regimes are therefore possible.

4.1. *Quantum equilibrium in ordinary situations*

In ordinary situations (no QSMDS) one can expect that, for a macroscopic physical system, the very large value of N ensures that, at least for typical configurations, the two samplings provide the same value (within a fluctuation proportional to $\sqrt{N}$). So, in this case, two reasons contribute to make the effect of the term in ε completely negligible:

- the sampling of the gravitational potential in the two terms is almost the same.
- the remaining component is proportional to the gravitational interaction. This interaction is much smaller than the terms in the standard Hamiltonian, which constantly tend to restore the equality between the distribution of Bohmian positions and ρ_N [23, 24].

Therefore, in ordinary situations, departure from quantum equilibrium remains completely negligible. Quantum equilibrium is still an excellent approximation.

4.2. *Transient departures from quantum equilibrium in QSMDS*

If a QSMDS occurs, and if the various components of this superposition occupy different regions of space, the situation is completely different. The current variables $\mathbf{r}_1, .., \mathbf{r}_N$ cannot sample simultaneously all regions where $\langle D(\mathbf{r})\rangle$ is non-zero (if they did, they would sit at a point where the wave function Φ vanishes). The reason has already been discussed in § 2.1: because of the cohesive forces inside the object that enters a QSMDS (the pointer of a measurement apparatus for instance), Φ vanishes unless all $\mathbf{r}_1, .., \mathbf{r}_N$ simultaneously belong to a single region of space corresponding to one single component. By contrast, the sampling by $\langle D(\mathbf{r})\rangle$ includes all regions. In this case, the term in ε of (37) becomes large, so that the local variation of the quantum density fluid $\partial\rho_N/\partial t$ significantly differs from the local variation of the density of Bohmian variables $\boldsymbol{\nabla}_N \cdot \mathbf{J}_N$. The result is that, during a short transient time, a large fraction of the quantum probability fluid is transferred towards the region of lowest potential energy $\mathcal{V}_G(\mathbf{r})$ (largest value of its absolute value). In each realization of the experiment, the Bohmian positions are insensitive to this transfer of probability density: they smoothly continue along their continuous trajectories. In other words, it is now the Bohmian position that pilot the wave function.

As soon as the QSMDS is resolved by this process, one returns to the previous regime where the Schrödinger dynamics is an excellent approximation. The usual relaxation process [23, 24] takes place, and very soon quantum equilibrium is restored.

5. Conclusion

In standard quantum mechanics, solutions of the Schrödinger equation become physically meaningless, when they have been left to propagate "too far" and involve a coherent superposition of states that are macroscopically different and never observed. In standard theory, this problem is considered irrelevant: one postulates that the proper use of the equation is to compute the evolution of quantum states only as long as they do not involve QSMDS and macroscopic entanglement with the environment (for instance during a quantum measurement); one merely ignores any further evolution of these states when this condition is no longer met. The interest of modified Schrödinger dynamics such as GRW, CSL, etc. is to propose a universal dynamics that applies all the time, without any limit or selection of particular solutions of the equation.

The form of the modified dynamics contained in GBC dynamics is particularly simple. It relates the dynamical projection to a well-known field, the gravity, and introduces no new dimensional constant. It quickly resolves QSMDS if the quantum states of the superposition are distinguishable in ordinary 3D space, corresponding to different spatial densities of matter; by contrast, it would not resolve QSMDS occurring only in momentum space, such as two macroscopic currents flowing in opposite direction in a superconducting ring. This is because they correspond to the same distribution of densities is space, which makes them insensitive to gravitational collapse.

Mathematically, what determines the importance of the effects of a gravitational quantum collapse is the quantum correlations of the local density with itself, or with other observables. In ordinary space, and then in configuration space, we have studied two situations: spontaneous appearance of quantum fluctuations of density, or fluctuations induced by quantum measurement and described by a transient QSMDS. The latter situation is simpler, since the crossover region between the standard quantum regime and the fast localization regime is crossed very suddenly, while the entanglement quickly progresses towards larger and larger scales. The standard von Neumann projection postulate then appears as an extremely good approximation, a sort of "sudden approximation". In the absence of quantum measurement, there is not reason why, in general, such a rapid growth of the scale of entanglement should occur. The localization process is then more complex, since it depends on the details of various quantum correlation functions. In the frame of simple approximations, we have seen that the gravitational localization process leads to modified expressions of the gravity, either in a scalar form with a change of the Newton constant G, or in a tensor form where the direction of the Newton force is modified.

Obviously, the GBC model, as discussed in the present article, remains rather naive: it is not expressed in terms of a field theory, and not even relativistic (it relies on the standard dBB theory, which assumes a preferred reference frame). As elementary as it is, the model nevertheless illustrates how the enrichment of the quantum dynamics, by adding a single point in the configuration space to the N-body wave function, may lead to a completely different dynamics of QSMDS. One may actually be surprised by the fact that such an elementary model should work so well, leading to an evolution of the state vector that is so similar to that of standard theory for microscopic systems, while also naturally including the von Neumann projection for QSMDS. Moreover, since the GBC localization process is a continuous collective effect, the spontaneous heating of macroscopic systems is expected to be many orders of magnitude lower than that predicted by stochastic localization theories such as GRW of CSL. The GBC equations are particularly simple, and do not require the introduction of stochastic processes or of the definition of a probability rule to specify them. Finally, if for instance one arbitrarily chooses the fine structure constant for the value of ε, the dynamics requires the addition of no new physical constant into the equations.

From a historical point of view, one may notice some similarity with the ideas developed by de Broglie when he worked on a double solution theory [30, 31], which also involved a singular solution localized at a precise point of configuration space. In GBC the singular solution is replaced by a point P, and the two variables Ψ and P do not obey the same equation. Otherwise they constantly guide each other, the reaction of P on Ψ being significant only during the resolutions of QSMDS. Another analogy can be found with the "shifty split" [32] contained in the Copenhagen interpretation, i.e. the moveable split (or cut) between the quantum world of S and the macroscopic world of measurement apparatuses M. We have seen that, because of the very rapid variation of the collapse time constant during the interaction between S and M, the exact value of the constant ε introduced in the localization term (5) is not so important: changing ε shifts the time at which the sudden collapse occurs, but the change is only very small, so that a large range of values provides physically acceptable results. In other words, we can arbitrarily move the split, or cut, between the two worlds without changing much the predictions of the model.

Conceptually, the GBC model remains relatively neutral about interpretations and ontology. Within the dBB theory, the traditional view is that the Bohmian positions provide the "beables" [33, 34] of the theory. The state vector is then seen as a mathematical object, similar to a Lagrangian, which pilots the motion of the real positions. But an opposite point of view is possible witht the GBC model. One can group all Bohmian positions into a single position in configuration space and consider this position just as a mathematical tool to enrich the quantum dynamics of spinless particles. Since the N-particle wave function constantly follows the observations, and since every particle keeps a conditional wave function at any time in ordinary 3D space, one can consider that it directly represents the physical reality of every particle, as initially envisaged by Schrödinger when he introduced the dynamical equation of wave mechanics.

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