INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Physique

Andrew T. Young

Did Monge really explain inferior mirages?

Volume 23, Special Issue S1 (2022), p. 467-481

Published online: 20 December 2022
Issue date: 27 October 2023

https://doi.org/10.5802/crphys.106

Part of Special Issue: Astronomie, atmosphères et réfraction
Guest editors: Pierre Léna (Professeur émérite, Observatoire de Paris et Université Paris Cité, membre de l'Académie des sciences) and Luc Dettwiller (Université Jean Monnet Saint-Etienne, CNRS, Institut d Optique Graduate School, Laboratoire Hubert Curien UMR 5516, F-42023, SAINT-ETIENNE, France)

CENTRE MERSENNE

Les Comptes Rendus. Physique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1878-1535

Comptes Rendus
Physique
2022, Vol. 23, Special Issue S1, p. 467-481
https://doi.org/10.5802/crphys.106

Astronomie, atmosphères et réfraction / *Astronomy, Atmospheres and Refraction*

Did Monge really explain inferior mirages?

Monge a-t-il vraiment expliqué les mirages inférieurs ?

Andrew T. Young *, [a]

[a] Astronomy Department, San Diego State University, San Diego, California, 92182-1221, USA

E-mail: ATYoung@sdsu.edu (A. T. Young)

Abstract. Gaspard Monge has received undue credit for his observations of mirages in 1798 in Egypt. He did not discover inferior mirages; they were known to sailors before 1687, and the term *mirage* had entered the French literature in 1753. Monge's "total reflection" explanation for them was seriously flawed; it had already been considered and rejected by Gruber and by Büsch, and was promptly criticized by others. Yet he continues to be given credit for the first scientific explanation of mirages; and his false total-reflection mechanism survives today in some textbooks. Apparently this is due to its superficial simplicity, which appeals to novices.

Résumé. Gaspard Monge a reçu une reconnaissance imméritée pour ses observations de mirages en 1798 en Egypte. Il n'a pas découvert les mirages inférieurs : avant 1687 ils étaient déjà connus des marins, et le terme *mirage* fut introduit dans les publications françaises à partir de 1753. L'explication par Monge en terme de réflexion totale est gravement erronée ; elle avait été déjà examinée puis rejetée par Gruber et par Büsch, et promptement critiquée par d'autres. Cependant, on continue d'attribuer à Monge le mérite de la première explication scientifique des mirages ; et son idée fausse de réflexion totale survit aujourd'hui encore dans certains manuels. Apparemment, cela vient de son indue simplicité, qui n'est qu'un vernis attirant pour les novices.

Keywords. Monge, mirages, refraction, meteorological optics, apparent horizon, horizon dip, history of optics.

Mots-clés. Monge, mirages, réfraction, optique météorologique, horizon sensible, dépression de l'horizon, histoire de l'optique.

Published online: 20 December 2022, Issue date: 27 October 2023

1. Introduction

In 1798, Gaspard Monge, a mathematician acquainted with physics and engineering, accompanied Napoleon Bonaparte's expedition to Egypt. During his time there, he (along with many of the French soldiers, as well as the scientists in the expedition) noticed the common appearance of water on the desert sand. On 28 August, a month after arriving in Cairo, Monge gave a talk [1,2] on this phenomenon, and later published a paper [3, 4] on this subject, attributing the word *mirage* to sailors in the titles of these publications.

* Corresponding author.

ISSN (electronic) : 1878-1535

Neither the word nor the phenomenon was previously unknown. The optical mechanism (total internal reflection) that Monge used to explain the phenomenon had already been considered, and rejected, by people who (unlike Monge) had performed experiments to help understand it. His mirage papers contain neither equations nor numbers, unlike his usual mathematical theses; nor even a ray diagram, though Monge was famous for his skills in geometry and mechanical drawing. His explanation was promptly dismissed by scientists who studied optics, and has continued to be criticized in expert reviews of the subject.

Nevertheless, Monge is still associated with “the mirage of the desert”. One readily finds books — and, of course, websites — repeating his claim that it is caused by total internal reflection, or saying that Monge’s was “the earliest attempt to explain the mirage,” or even that “the term ‘mirage’ was coined by Gaspard Monge” [5], or that he was the first person to explain it correctly. But none of these is true. Why do such errors persist?

Let’s consider how this mirage came to be understood.

2. The word

Long before mirages were studied in deserts, they were a common sight at sea. The French seamen had a phrase that was common enough to appear in a dictionary of nautical terms [6] as early as 1687: *“La terre se mire.”* The definition translates as “This is said when the vapors make the lands appear as if they were raised above the low clouds.” (This was at a time when refraction and other atmospheric effects were routinely attributed to unspecified “vapors” near the horizon.)

A similar definition was offered in a French/Dutch dictionary [7] published in 1702. From its second edition (1736), Le Gentil [8] took the description of this phrase for use in his last published work, in 1789, which has been cited by many later writers.

Meanwhile, the sailors had turned the verb phrase into a noun. By 1753 it had come to the attention of the astronomer Joseph-Bernard de Chabert, who was sent to survey the French settlements in what is now Nova Scotia. His official report [9] says that “I was busy three days in mapping operations, … in which I was often stopped by the effect of *mirage*. This is what many sailors call a change that sometimes occurs in the appearance of coasts a little way off, because they attribute it to the reflection of the sky, which is painted in the sea below the coast, making this coast seem raised in the sky. Instead, this phenomenon seems to come from the large refraction suffered by objects seen through very dense vapors: this density not being at all equal in the whole extent of a coast, the diversely bent rays rendering it unrecognizable. This is the reason given by the late M. Maraldi, in the Memoirs of the Academy for the year 1722.” (In the abstract of de Chabert’s work given in the *Histoire* [10], the term “mirage” is attributed to “the inhabitants” rather than sailors.)

Evidently, “mirage” had been used for half a century or more, mostly by mariners, before Monge used it.

3. The observations

Mirages had been seen in antiquity, but the ancient descriptions are vague and lack specific details. In 1670, Jean Picard [11] went to Uraniborg to determine the position of Tycho’s instruments. One day he found that an island near Copenhagen that he had been using as a reference point unexpectedly appeared to be floating in the sky; he attributed this to the sea being so flat that its surface acted like a mirror. This is clearly an observation of the inferior mirage, but Picard did not realize that atmospheric refraction was the cause.

However, half a century later, J. P. Maraldi [12] studied a series of observations made from a private observatory in Genoa, which showed mirages of features on Corsica, some 260 km away. “From the coasts of Genoa and Provence, one sees the mountains of Corsica, which sometimes appear raised above the apparent horizon, as if they had emerged from the water, and which disappear at other times in a sky just as pure and clear, as if they had been plunged into the sea.” He also described the times of the day and year at which such observations could best be made, adding that “All these appearances are made on the coast of Genoa by an observer who is always in the same place and at the same height above the surface of the sea.”

He then considers the idea that these variations might be due to the rise and fall of the sea with the tides, but rejects it: “But this explanation is not the most natural, and it is more probable to attribute these appearances to refractions, and to suppose that the visual rays which go from that island to the observer who is on the opposite coasts are refracted differently in the vapors that are between the two . . . .

“This explanation seems all the more probable, as it is equally suited to account for the similar appearances that occur in the middle of the land; as one has noticed for a long time at the Observatory that houses seven or eight leagues to the north, which are hidden during the day by others that are closer, and placed at only half a league distance, often appear in the morning at sunrise, raised above those that are close. The distant houses sink little by little, until they are completely hidden by the nearby ones. This appearance is thus an obvious proof that the rays are curved differently in the air according to the different densities of the air, or the different quantity of vapors through which the rays pass. . . . The appearance of Corsica seen from Genoa, being similar to that which we have reported, can be explained in the same way.”

Then follows a detailed account of the survey of Corsica and other nearby places, including measurements of the varying dip of the sea horizon. The diurnal changes in terrestrial refraction correspond to looming and sinking, and are much more detailed than those described in Mariotte’s 1672 treatise on levelling [13] or Perrault’s “De l’origine des fontaines” (1674) [14].

Similar observations were made by Thomas Shaw [15] in the Sinai desert, some 60 years before Monge went to Egypt: “Where any Part of these Deserts is sandy and level, the *Horizon* is as fit for astronomical Observations as the Sea, and appears, at a small Distance, to be no less a Collection of Water [which] always advances, about a Quarter of a Mile before us, whilst the intermediate Space appears to be in one continued Glow, occasioned by the quivering undulating Motion of that quick Succession of Vapours and Exhalations, which are extracted by the powerful Influence of the Sun.”

A very detailed observation of an inferior mirage was recorded by Boscovich and Maire [16], whose triangulation along the beach near Rimini in July, 1752, was interrupted by the disappearance of a target 8 miles away. They found that the mirage varied with the height of the eye, so that by climbing a ladder, the view of the target could be regained. By raising their quadrant onto a wagon, they were able to finish their measurements. This work was translated into French in 1770 [17].

By this time, several accounts of superior mirages had been published, as well as the inferior mirages of interest here. Among these was Antonio Minasi’s monograph [18] on the Fata Morgana in the Strait of Messina; the accompanying engraving by Willem Fortuyn shows a variety of mirages, including some that look like inferior mirages. Some inferior mirages were also described by Le Gentil [8], a decade before Monge.

The differences among the different types of mirage were not well understood until the middle of the 19th Century. However, inferior mirages were fairly well understood well before Monge, thanks to careful observational and experimental work by Tobias Gruber and by Johann Georg Büsch.

4. Experiments and explanations

4.1. *Gruber*

Tobias Gruber was a minor functionary in an obscure corner of the Habsburg Empire, whose duties took him through the dry bed of the seasonal Lake Zirknitz. In 1781, his "Letters from Carniola" [now part of Slovenia] were published [19]. In a postscript to the Fifth Letter, dated 20 April 1779, Gruber described the essential points: flat, smooth ground; the hiding of objects below a limiting ray; the dependence on season, height of the eye and distance to the object. He specified heights and distances numerically: "At a distance of 1000 to 2000 fathoms [i.e., 2 to 4 km] I saw just the roofs of the village houses, which looked to me like a transparent thicket. … Somewhat higher objects, like trees, structures, towers, etc., appeared twice as high, just because they were reflected as if in a pool of water. … On approach, [the water] vanished, and constantly withdrew. If I stood up from my seat in the carriage, where I still saw them, and raised myself up 3 feet higher, they retreated, or no longer appeared." He was alarmed at the first sight of these phenomena; but "The repeated view in varied circumstances, the appearance and disappearance in relation to rising and descending, and the analogy from optical experiments finally revealed to me the whole mystery." He explained it with a ray diagram, pointing out that the "reflected" rays do not reach the ground. "So one sees nothing of the objects that lie under this line, and the reflected ones will look like water … ."

"But the matter demands a closer definition, for which I have too little time now."

4.2. *Büsch*

Meanwhile, Johann Georg Büsch, a mathematics professor at the academic Gymnasium in Hamburg, had become interested in mirages, inspired by a problem proposed by the Royal Danish Academy of Sciences in 1781. He published his observations in Latin [20] to make them available to the learned societies.

The first 38 pages contain his observations. Unfortunately, he scrupulously gave barometric readings, but no temperatures. Worse, he thought the phenomena have something to do with electricity, and lightning. And, because only distant objects are miraged, he thought the curvature of the ray depends on distance. But he put his finger on a common error of previous workers, saying "Non in vaporibus causa est sita." And he discussed how the mirage varies with the height of the observer, the time, and the weather conditions.

As he was about to publish his observations, he discovered Gruber's letter. He added a Latin translation of Gruber's text, with his own comments, to his manuscript, saying that Gruber is "the first man besides me to have observed the whole phenomenon." But he disagreed with some of Gruber's assertions.

On p. 53, Büsch considered total internal reflection, but rejected it on the grounds that the denser air cannot separate from the lighter to form a visible surface, "like two immiscible liquors, such as terebinth oil [turpentine] and spirit of wine." Finally, he offered advice to those who would investigate further: make observations in all seasons from a fixed place; use an instrument capable of measuring small angles; an achromatic telescope "for avoiding all confusion of the image that deceives the naked eye"; a level to observe "how much objects are raised and lowered for various conditions of the air."

Despite its weaknesses, this is a solid piece of work; together with Gruber's papers, the natural history of inferior mirages was well established, a dozen years before Monge went to Egypt.

4.3. *Gruber again*

After seeing Büsch's comments [20] on his 1781 letter, Gruber continued his investigation with both field observations and experiments, publishing the results in 1786 [21]. There he denied Büsch's suggestion that Gruber had thought the "reflection" in the mirage was due to a definite surface: "It never, not once, occurred to me to have the idea Prof. Büsch expects of me on p. 70, ... through the example of two superimposed liquids, e.g. turpentine oil and wine spirit ...."

He found the mirage easily visible on a sunlit wall, and was able to show it to "anyone who is not short-sighted." His continued observations of mirages outdoors convinced him that their occurrence was not affected by the height of the barometer; that they only appeared when the surface was warmer than the ambient air; and that the heated air that caused these images was not "thicker," [i.e., denser] as he had supposed, but "thinner," as shown by the rising air over a stove. He also noticed that the wave-like motion of the mirage on a hot wall was upward, like the hot air above a candle flame or a coal fire, and the hot smoke from chimneys.

His kitchen experiments, using a heated rectangular iron bar, are similar to Wollaston's [22] later laboratory simulations; but Gruber's observations are scrupulously quantitative, including the exact geometry of the apparatus, and the atmospheric temperature and pressure. Gruber points out that the temperature gradient over a warm surface makes the path of a ray curved, as in his Figure 2. He gives detailed explanations of double images and image elongation where the erect and inverted images meet, with useful ray diagrams.

4.4. *Huddart*

Büsch's wish for someone to measure altitudes in mirages was granted by Joseph Huddart in November, 1796, whose paper [23] contains both measurements of dip (the depression of the apparent horizon below the astronomical one) and drawings of inferior mirages. "I have often observed that low lands and the extremity of headlands or points, forming an acute angle with the horizon of the sea, and viewed from a distance beyond it, appear elevated above it, with an open space between the land and the sea. ... I am convinced that those appearances must arise from refraction, and that instead of the density of the atmosphere increasing to the surface of the sea, it must decrease from some space above it ...." He was aware of the continuous change in density of the air with height, and knew that "the ray of light will form a curve" as a result.

This study provides quantitative measurements of angles, temperatures, and pressures. His measurements of a miraged ship, made from a height of about 40 meters, plainly showed the contraction of the inverted image. He noticed the line "where inversion begins; therefore no land lower than this can be seen;" later this was called the "vanishing line" by Minnaert [24], but today, it is called the "fold line" [25]. Huddart saw that features near this line were "confused, and ill defined." He described the appearance of miraged headlands at this level as "blunted." In addition to drawings of the mirages, he provided ray diagrams to explain what he saw. He knew that two images of the same object require that one set of rays must "pass through one medium which the other has not entered"; this constrained the maximum density to be lower than 10 feet (3 meters) in height. He recorded several observations of how the images change with the height of the eye.

In discussing his dip measurements, he showed how the true dip could be measured from a ship by observing the sun's angular distances from both the northern and southern horizons at noon. This principle was widely used in the following centuries, aided by various specialized instruments designed to make the observation easier. Huddart's measurements provided useful guidance to later workers, despite his conviction that all the effects were caused by "evaporation."

This paper was reprinted [26] in the July, 1797, issue of Nicholson's Journal, and later summarized by Gilbert [27] in 1800.

4.5. *Woltman*

Meanwhile, Reinhard Woltman, a student of Büsch's in Hamburg, became interested in these mirages, and began measuring dip toward the end of the year 1794. He used his position as the Director of the coastal works at Ritzebüttel to get special surveying done to support the investigation, and sent a partial account of his work to Gruber; a 9-page notice of this work appeared in the *Göttingische Anzeigen* in May, 1796 [28]. His full paper [29] appeared in the proceedings of the Bohemian Scientific Society in 1798, and was later summarized by Gilbert [30]. He was well aware of the works of Gruber and Büsch.

He frequently observed inferior mirages, noting that "objects, houses, trees etc. near the horizon often are separated from the visible ground surface by a bright strip of air, and almost seem to stand in the air; or as if a shining empty space were present between the visible horizon and the objects; or if the eye is considerably raised, and sees several distant objects behind one another, as if these houses, mills, churches, trees etc. stood in a calm shining sea, in which the entire landscape were immersed and reflected." He used an achromatic telescope to confirm that the images were inverted. Even in rain the images did not disappear completely, unless the opacity of the air prevented seeing into the distance. They were still seen when the ground and the river ice were covered with snow. He described how the "bright strip" shrinks and vanishes as the miraged object approaches the observer, and also recorded the effects of increasing the height of the eye. "Therefore the bright strip is no object, like a luminous surface, … but is itself an image of a reflected bright object."

[By the way, inferior mirages are "a common winter mirage over frozen lakes," according to Greenler [31], who shows photographs of them in his Plates 7-2 and 7-3, and discusses them on pp. 158–159. Inferior mirages over a frozen lake were also discussed by Forel [32, p. 532]; they were seen over ice or snow by Mathieu [33, p. 697].]

Woltman had two pilings driven into the dikes near his house, and compared the apparent position of another house on the far shore of the Elbe with the sight-line between the tops of the piles. His papers include precise measurements of the positions of the piles and the house, and extensive tables of measured changes in the apparent altitude of the distant house, together with the air and water temperature and other meteorological conditions. He found the terrestrial refraction greatest in the morning and lowest in the evening, with a total range of over 10 arc minutes. The measurements showed that heat has the strongest effect of all the variables: "If the water is 2 or more Fahrenheit degrees warmer than the air, the rays passing over the water are always depressed; but raised, if the air is 2 degrees or more warmer than the water. Among more than 150 observations this rule has not a single exception."

His measurements showed that this mirage is always accompanied by a depression of the apparent horizon, and that the inverted image is always appreciably smaller than the erect one. (This was because his eye was about 6 meters above the water, though he was unable to show that this is an effect of the Earth's curvature.) He was also able to observe some inferior mirages over land: "One must have a whole open plain before him, or a high sight-line over bushes and the like. Then the whole landscape seems to stand in an immobile shining sea, in which all raised objects are reflected." From a low rise, he saw the marshes as if they were covered by shining water. And he found the reflections over land similar to those over water; "the refraction is greater as the difference in heat between the two materials [air and water] is greater." He concluded that "These images cannot arise just from reflection, because no mirror surface exists in the air."

Woltman also noticed the inferior mirages of the Sun and Moon at the horizon. "If one sees the sun come up bright in the morning, its shape gives an infallible indication whether looming or depression will occur. In the latter case, it is not round, but part of the solar disk is reflected downwards; that can amount to a whole eighth or a quarter of the solar diameter." It is as if, at the bottom of the solar disk, even at the part that has risen, the beginning of another disk were placed; he had seen the same effect with the full moon.

In addition to inferior mirages, Woltman saw a few superior ones, accompanied by looming. He found that "… the sea surface, distant shores, coasts, and sandbanks take on an extraordinary appearance. The sea surface becomes concave, nearby ships are lowered, or rather the distant horizon seems almost in front of them, distant low shores look like high coasts; and these, if they are 7 or 8 [German] miles away, and far below our horizon, look like clouds above it; whole landscapes, otherwise hidden by interposed sand hills, appear above them."

Woltman's attempts to explain these phenomena suffered from his inability to allow for the continuous curvature of the rays; however, he did appreciate the importance of considering the Earth's curvature. On the other hand, Gruber's commentary [34] on Woltman's observations took the ray curvature into account, but neglected the Earth's. But Gruber understood that the general ray curvature explains the connection between mirages and dip of the horizon.

5. Monge's memoir

Compared to what was done before, Monge's paper [4] seems very feeble. It has no references to the earlier literature; there are no field measurements, no numbers or calculations. The descriptions of mirages read like a travelogue; then comes a purely verbal description of total internal reflection. This is followed by an assertion that the inverted images are produced by "the surface which separates the lower layer … from the denser layer above it"; later, he again talks about "the reflecting surface which separates the two layers of different densities." But no such separating surface exists. In fact, this notion had already been rejected by Büsch [20], by Gruber [21], and by Woltman [29].

Monge's argument involves the logical error of asserting the consequent: if the air acts like a mirror, an inverted image will be produced; an inverted image is observed; therefore the air is acting like a mirror. But there are other ways of producing inverted images — for example, with a positive lens. (And in fact, the actual formation of the inverted image in the inferior mirage is very much like the inverted real image produced by a convex lens.)

Because Monge was the President of the Institute in Cairo, his mirage paper attracted much attention. An unsigned review [35] of the whole *Memoires* volume is typical: "One will be more satisfied with the description of this phenomenon than with the explanation that M. attempts."

6. Reactions to Monge

6.1. *Reinecke et al.*

As might have been expected, the immediate criticism of Monge's paper fixed on his unjustified assumption of a density discontinuity in the air. The earliest discussion is in *Allgemeine Geographische Ephemeriden*, led by Johann Christoph Matthias Reinecke, a frequent contributor to that journal. Its third issue (March) for 1800 contains an introduction by Reinecke himself, discussing refraction phenomena generally [36], followed by comments from Büsch [37], and Reinecke's reply to Büsch [38]. Then comes a German translation of Monge's paper by Reinecke [39], and finally Reinecke's "translator's supplement" [40] commenting on Monge. The last two are most important here, though the earlier discussions of Minasi's monograph (based entirely on a German translation of Nicholson's partial English translation [41] of Minasi's Italian), and Büsch's "Tractatus" are also relevant, as they contain some discussion of Gruber's work.

The very first sentence of Reinecke's comment on Monge is: "The above theory requires a correction." And of course, that is the unphysical assumption of a discontinuity in air density.

Of Monge's "single definite surface", Reinecke says: "But such a precisely isolated surface does not occur in Nature, as Prof. Büsch remarked on the occasion of Gruber's explanation of the same phenomenon. Instead, the density of the air changes continuously and gradually, and there is no reason why it should reflect more at one place than at another. Instead of a simple reflection, a ray of light experiences a bending at every point of its path through a medium of decreasing density, and therefore describes a curved line ...." He explains that the rays are usually concave toward the Earth; but if we invert the usual decrease of density upward, the rays will be concave upward and turn their convex sides to the Earth — "although no actual reflection results."

"Now this is also the phenomenon treated by Prof. Büsch in his Tractate and in the letter above. His explanation, which *Monge* seems not to have known, is completely different from that given here ...". Reinecke clearly understood that it is the change in ray curvature that produces the inverted images. The different models can be distinguished by the absolute values of the altitudes near the horizon. But "Monge made no angular measurements, and anyway seems to have given no consideration to this point."

Büsch [37] was equally dismissive of Monge's work. He related some of his own mirage observations made while traveling on the Baltic coast. These were clearly inferior mirages, "which had been so remarkable to the French, but from which they learned nothing."

At about the same time, Ludwig Wilhelm Gilbert, Professor of Physics at the University of Leipzig, who had taken on the publication of the "Annalen der Physik" the previous year, began publishing a series of articles on refraction and mirages, which take up most of Part 3 of its third volume. It begins with a German translation of Huddart's paper [23], and contains a German translation of excerpts from Büsch's "Tractatus", followed by summaries [42] of the works of Boscovich, Monge, Gruber, and others. Gilbert complains that Monge's description is much less complete and satisfying than Huddart's, that the inverted image gets "not a word" in Monge's account, and that simple reflection of the rays is in no way sufficient to explain the phenomenon.

6.2. *Wollaston*

Also in 1800, William Hyde Wollaston undertook to explain [43] the phenomena described by Huddart. In this paper, he points out that there are "two opposite states of the atmosphere" that produce double or triple images, and explicitly points out mirages on roads — though without being able to use that word. He also distinguishes simple looming from the multiple-image phenomena. Wollaston re-invents the two-liquid laboratory demonstration of refraction phenomena first described in Hooke's *Micrographia* [44] (1665); these experiments on liquids showed him how "adjacent portions of the converging rays will form a focus, beyond which they will diverge again; and the varied medium will produce effects similar to those caused by a medium of uniform density having a surface similar to the curve of densities ...." He notes that an oblique line appears "bent into different forms" when viewed through such layered media, and that "If it be at the distance of the principal focus, one point of it is bent into a vertical line ...." (This is due to the infinite vertical magnification in mirages at the altitude where erect and inverted images meet, sometimes called the "fold line" [25]) He shows this effect in Figure 10; the appearance is quite different from the sharp angle where an inclined object reflected in a mirror meets its reflection. This paper was translated into German by Gilbert in 1802 [45].

Soon after this, Wollaston found Monge's mirage paper, and commented on it in his Bakerian lecture [22], where he introduced the word *mirage* into English. He began by dismissing Monge's density discontinuity: "The definite reflecting surface which he [Monge] supposes to take place between two strata of air of different density, is by no means consistent with that continued ascent of rarefied air which he himself admits; and the explanation founded on this hypothesis will not apply to other cases, which may all be satisfactorily accounted for, upon the supposition of a gradual change of density, and successive curvature of the rays of light by refraction."

7. Refraction theory

Though Monge never developed a quantitative theory for mirages, others soon did. By 1808, two remarkable theoretical works on refraction existed.

7.1. *Biot*

The physicist Jean-Baptiste Biot had served as Laplace's proofreader on the *Mécanique Céleste*. In 1805, Biot published the first edition of his introductory textbook on astronomy [46], in which he briefly dealt with astronomical refraction, and devoted one page to Monge's mirage observations, saying merely that "Monge has explained this phenomenon according to the laws of optics."

In 1806, Biot was sent with François Arago to finish measuring the southern part of the meridian of Paris to confirm the definition of the meter. In 1808, he continued geodetic measurements near Dunkirk with Claude-Louis Mathieu. This field work made him familiar with the variations of terrestrial refraction, and he prepared a comprehensive treatment of refraction phenomena [47], in which the theory is not only well developed, but is compared with meticulous measurements made during his field observations.

Biot's mirage monograph is illustrated with drawings made by Mathieu during their work near Dunkirk. These engravings were widely reproduced in encyclopedias and other popular works for many years; many of them appeared in Pernter's *Meteorologische Optik* [48] and its revisions by Felix Exner [49], over a century after their first publication.

Its theoretical pages contain many results that were rediscovered many years later, such as the theorem on p. 65 that an inverted image is only possible if one ray is horizontal between the object and the observer, and the well-known fact that erect and inverted images are distinguished by the positions of the extrema of the ray trajectories, which is often attributed to Tait [50].

Biot is clear that the rays are curved, not broken abruptly: "The rays bent in these lower layers of variable densities have trajectories convex toward the surface, and produce inverted images." Of Monge's total internal reflection idea, Biot says, "This comparison makes the thing sensible, and it would be completely correct if the layers of different density were not infinitely thin," and adds: "calculation confirms this explanation by making some very slight modification of it . . . ".

But this is itself not really right. Monge did not merely "compare this inversion" to that produced by internal reflection; he identified these two phenomena, which are in fact quite different. Monge insists on a discontinuous density profile; but in fact heat transfer (which was not understood in his day) forces the profile to be not merely continuous, but smooth — i.e., its *second* derivative must be continuous. The necessary modification turns out to be fundamental, not "very slight."

7.2. *Gergonne*

While Biot was preparing his monograph, Joseph Diez Gergonne was independently developing a theory of mirages at Nîmes. He had seen mirages locally, and the one-page abstract [1] of Monge's talk (which he called [51] "a pathetic explanation" [*une pitoyable explication*] of the mirage); but was unable to obtain a copy of Monge's paper [3]. So he relied primarily on Biot's one-page synopsis of it in his Astronomy textbook [46], without realizing that this was itself taken nearly verbatim from Monge. From just this limited observational information, Gergonne produced an impressively general theory of mirages. While Gergonne's full memoir seems to have been lost, an extended summary of it is available [52], and some of the missing analysis appears in his 1829 papers [51, 53].

Gergonne assumed a medium whose refractive power varies continuously with height. He derived the differential equation for the ray trajectory in a plane-parallel atmosphere, and determined the shapes of possible trajectories (represented as piecewise parabolic splines) from the structure of the differential equation. By considering combinations of atmospheric layers with different refractive profiles, he showed that two rays leaving an object point can intersect elsewhere, so that an eye at that intersection sees two images of the object. He found that an odd number of images can be produced, and determined the positions and orientations of the images.

On applying this theory to the inferior mirages described in Biot's book, Gergonne found that solar heating of the ground causes the density to increase upward in the surface layer, making the rays parabolic and convex toward the ground, and producing inverted images below objects near the horizon, as if they were reflected in water; "and as this double impression can only take place for objects sufficiently distant from the spectator, it happens that as he advances, the limits of the apparent inundation retreat."

"One can thus affirm that, in any place which presents vast, perfectly horizontal plains, quite smooth, strongly heated … by the solar rays, … the phenomenon of mirage should take place daily during the summer at the hottest part of the day." And his summary [52] concludes with a plausible ray diagram.

Gergonne's theory was general enough that he could also predict the possibility of ducted rays, and describe some properties of superior mirages. Unfortunately, it is too long and complicated to summarize here; some of its features appear in the annotated bibliography [54].

8. Reflection or refraction?

8.1. *The smooth bend at the fold line*

Wollaston [22, 43] has fine illustrations of the smooth bend where the erect and inverted images meet. He points out that this feature can be used to distinguish between inverted images produced by reflection and refraction: "… by a moderate share of attention, a very evident difference may be discovered between the inversion occasioned by reflection, and that which is caused by atmospherical refraction. In cases of reflection, the angles between the object and image are sharp, the line of contact between them straight and well defined, but the lower part of the image indefinite and confused, by means of any slight undulation of the water. But, when the images are caused by refraction, the confines of the object and its inverted image are rounded and indistinct, and the lower edge of the image is terminated by a straight line at the surface of the water."

Very similar advice was given by Bravais [55], in the paper that converted Wollaston's "two opposite states" into the modern terminology of *inferior* and *superior* mirages. "The junction of an oblique line with its symmetric [image] below is made, not by an angle with a sharp vertex, but by an arc of a curve having its principal part vertical." In his historical review of the subject, Bravais called Büsch's 1783 treatise "the first truly scientific work that has been published on the mirage." He devoted only a single sentence to Monge's work.

But a few months later, he outlined his plans for further work on mirages to the new Société Météorologique in March, 1853 [56]: "Up to now, the phenomenon of total reflection of objects has played a large part in the theory of the mirage. The reversion of luminous trajectories which, after plunging toward the ground, become horizontal for a short part of their path, and then recede from the ground, appeared to many physicists incapable of explanation by … total reflection. I will show that the conditions necessary to produce this reflection can not exist, and that the mirage is a simple phenomenon of refraction." Unfortunately, this meeting abstract was not printed until 1857.

However, in submitting his publication list to the Académie to support his application for membership, Bravais [57] dealt briefly with Monge's model (though without mentioning his name): "This phenomenon appears to have perplexed the geometers, and has led some to assume total reflection. I show that there is no total reflection, and that the upturn of the trajectory is a natural consequence of applying the wave theory to the path of the ray." Here he cited the short abstract [58] of a paper [59] that was published two years later.

8.2. *The wave structure at the apparent horizon*

Bravais mentioned some additional distinguishing features of the inferior mirage over water: "… if the mirage is strong, [the apparent horizon] will be bordered by a jagged crest whose sinuosities will appear to rise and fall unceasingly; to disappear and reappear." Forel [60] studied these structures further, and published a drawing [32] of the miraged wave crests just beyond the apparent horizon, where Biot's caustic meets the mean sea surface. Perhaps the most vivid description of this "boiling horizon" is that of Ferguson [61]: "Even when neither land nor ships are in sight, the miraging can be determined by the unique moving appearance of the horizon seen through a telescope. The water seems to wave unnaturally and seems to flicker like flames along the upper surface. It does this with a strongly increased dip."

8.3. *The relative smoothness of a mirage on rougher water*

Another characteristic of inferior mirages over water is the remarkable smoothness of the "reflected" mirage image, compared to the fragmented, irregular reflection from the water surface. This was pointed out by Büsch [20, 62]: "I noticed this phenomenon already in my youth on crossings from Hamburg to Harburg, a [German] mile away, where my grandfather lived. If the wind moved the waves in the middle of the river rather briskly about the ship, the water at the shore seemed to be completely calm, like a mirror. The shore inhabitants told me this was caused by the shallow beach; only, if we climbed to a height, and looked at the opposite shore from there, the water was full of waves there too." The effect is well illustrated in his figures 1, 3, 4, and 6. (Cf. Woltman's [28] "immobile shining sea" covering the marshland.)

9. Complications

The 19$^{\text{th}}$ Century theoreticians imagined that all they needed to understand inferior mirages was a correct density profile; but that would not become available until boundary layers were discovered and applied to convectively unstable surface layers [63]. The temperature profile turns out to be determined not by heat conduction, as Gergonne [52] had supposed, but by turbulent heat transfer — one of the most difficult problems in fluid dynamics.

Why is the ocean like the desert? They both support obvious inferior mirages. It turns out that turbulent heat transfer produces a strongly curved temperature profile, and the curvature of the refractivity profile causes the focusing effect that Wollaston [43], Gergonne [52] and Everett [64] identified as producing the ray-crossing needed for multiple images. So these mirages are a side effect of the "law of the wall" — any suitable warm surface will do.

But then why don't other warm surfaces work as well? Because the scaling of the temperature profile depends on the aerodynamic roughness length [65]; and this length depends on the linear size of the surface irregularities with the greatest curvature. In the desert, that's a grain of sand; on the ocean, it's the capillary waves that determine the drag coefficient when the wind interacts with the water surface. In both these cases, the length is about a millimeter [66]. The smaller the roughness length, the more strongly curved is the refractivity profile near the surface, and the more obvious is the mirage. So the inferior mirage is typically found where the surface is smooth: water, sand, and snow work equally well.

Yet another effect is at work here. The lower edge of the inverted image in the inferior mirage (i.e., the apparent horizon) depends on how close to the mean surface we can see. The line of sight can be intercepted by obstructions that stick up enough to block our view of the lowest part of the inverted image. So the total width of this image depends on the height of the tallest irregularities on the surface: rocks and bushes on dry land, or the largest waves on water. The importance of the dip of the sea horizon for celestial navigation has produced a huge literature on this subject; e.g., [67–71], some of which has been incorporated in the literature of refraction [72] and mirages [73, 74].

10. Discussion

Today, explaining these mirages involves much more than optics: boundary-layer meteorology, air-sea interactions, turbulent heat transfer, and other complex topics are involved. Monge's simplistic picture is simply wrong. He was not the first to use the word *mirage*, nor the first to describe mirages in deserts, nor the first to study them scientifically, nor even the first to identify the bright strip at the horizon as an image of the sky. Yet his name is still attached to mirages, after two centuries. Why?

A century ago, Pernter [48] was puzzled that Monge's description had become generally known, while the older works of Gruber and Büsch were forgotten. "On the one hand," says Pernter, that is "thanks to the great name of Monge"; and on the other, to "the circumstance that it was observed during the historical event of Napoleon's Egyptian campaign." (Probably Wollaston's inability to read German played a part as well.) Invoking Monge's great reputation is rather like Merton's [75] "Matthew effect." Yet today, Monge hardly seems famous enough to make this explanation adequate.

However, his work was well publicized. An English translation [4] quickly appeared, but was immediately described [76] as "neither very clear nor scientific." As Gergonne [51] said, Monge's memoir seems "much less written for geometers than for the great number of men who only aspire to acquire a superficial tincture of the causes of various phenomena which the spectacle of Nature can offer to our observation." In other words, it is not a scientific paper, but a shallow popularization. From that point of view, it was a complete success.

There were several descriptions of mirages decades before Monge, a couple of which were in French [8, 17], and one of which [9] even connected an eyewitness description of the phenomenon with the word *mirage*, and correctly attributed it to refraction. So Monge was not the first person to connect the phenomenon to the word in print. Why, then, is he still often given credit for doing so?

Worse yet, some people still believe his explanation was correct. For example, a journalist recently said of Monge, "His hypothesis is remarkably close to the way modern optics explains the phenomenon" [77]. Perhaps its simplicity appeals to writers of elementary textbooks. One finds assertions that "A mirage is formed by total reflection of light" [78] and that "Total internal reflection can cause mirages" [79] in modern textbooks. (Here, Monge has been forgotten; but his incorrect model lives on.)

It is not enough to say, as Minnaert [24] does, that "total reflection" is "wrong because the transition between the different layers is gradual everywhere." The simplistic idea of reflection is attractive to people who know a little optics, but no atmospheric physics; but it is also misleading to those who really want to understand mirages.

The contrast between the simplicity of internal reflection and the actual complexity of mirages is striking. Perhaps the saying [80] that "there is always a well-known solution to every… problem—neat, plausible, and wrong" accounts for the persistence of Monge's story. The only real accomplishment of his mirage papers was to popularize the word *mirage*.

Acknowledgements

I thank Luc Dettwiller, Éric Frappa, Marcella Pace, and Siebren van der Werf for bringing many important references to my attention, and San Diego State University for providing access to them.

References

[1] "Copie des procès-verbaux des séances de l'Institut d'Égypte, envoyés à l'Institut national de France", *Décade Philosophique, An VII, 2^{me} Trimestre* (1798), no. 10, p. 4-5.
[2] G. Monge, "Extrait du mémoire lu à la séance de l'institut du Caire, du II fructidor an 6, par le cit. Monge. Sur le phénomène optique, nommé MIRAGE par les marins", *Ann. Chim.* **29** (1798), p. 207-208.
[3] G. Monge, "Mémoire sur le phénomène d'optique connu sous le nom de MIRAGE", *La Décade égyptienne* **1** (1799), no. 2, p. 37-46.
[4] G. Monge, "Sur le phénomene d'optique, connu sous le nom de Mirage", *Mémoires sur l'Égypte* **1** (1800), p. 64-79.
[5] J. Naylor, *Out of the Blue: A 24-Hour Sky-Watcher's Guide*, Cambridge University Press, Cambridge, 2002, p. 51.
[6] N. Desroches, *Dictionnaire des Termes propres de Marine*, Amable Auroy, Paris, 1687, p. 352.
[7] N. Aubin, *Dictionaire de Marine*, Pierre Brunel, Amsterdam, 1702, p. 565.
[8] G. Le Gentil, "Extrait d'un mémoire sur des observations astronomiques faites sur les réfractions, en 1786, 1787 et 1788", *Mém. Acad. Roy. Sci. année 1789* (1793), p. 224-236.
[9] J.-B. de Chabert, *Voyage fait par ordre du Roi en 1750 et 1751 ...*, Suite des Mémoires de l'Académie des sciences, Imprimerie Royale, Paris, 1753, p. 136.
[10] J.-B. de Chabert, "Voyage fait par ordre du Roi en 1750 et 1751 ...", *Hist. Acad. Roy. Sci. année 1753* (1757), p. 242-256.
[11] J. Picard, *Voyage d'Uranibourg ou observations astronomiques faites en Dannemarck*, Imprimerie Royale, Paris, 1680, p. 8.
[12] J. P. Maraldi, "Détermination géographique de l'Isle de Corse", *Mém. Acad. Roy. Sci. année 1722* (1724), p. 348-355.
[13] E. Mariotte, *Traitté du nivellement*, Jean Cusson, Paris, 1672.
[14] P. Perrault, *On the origin of springs*, Hafner, New York, 1967, trans. by Aurele LaRocque, [new ed. of *De l'origine des fontaines*, Pierre Le Petit, Paris, 1674].
[15] T. Shaw, *Travels, or Observations Relating to Several Parts of Barbary and the Levant*, Theatre, Oxford, 1738, p. 362.
[16] R. J. Boscovich, C. Maire, *De Litteraria Expeditione per Pontificiam Ditionem ...*, Typographo Palladis, Roma, 1755, p. 94-96.
[17] C. Maire, R. J. Boscovich, *Voyage Astronomique et Géographique, dans l'État de l'Église ...*, Tilliard, Paris, 1770, p. 94-96.
[18] A. Minasi, *Dissertazioni sopra diversi fatti meno ovvi della storia naturale*, Benedetto Francesi, Roma, 1773.
[19] T. Gruber, *Herrn Tobias Grubers Briefe hydrographischen und physikalischen Inhalts aus Krain, an Ignaz Edlen von Born*, Johann Paul Krauß, Wien, 1781, p. 54-57.
[20] J. G. Büsch, *Tractatus duo optici argumenti*, apud Carolum Ernestum Bohnium, Hamburgi, 1783.
[21] T. Gruber, "Physikalische Abhandlung über die Stralenbrechung und Abprellung auf erwärmten Flächen", *Abhandlungen der Böhmischen Gesellschaft der Wissenschaften* **2** (1786), p. 298-330.
[22] W. H. Wollaston, "Observations on the Quantity of horizontal Refraction; with a Method of measuring the Dip at Sea", *Phil. Trans. Roy. Soc. Lond.* **93** (1803), p. 1-11.
[23] J. Huddart, "Observations on horizontal Refractions which affect the Appearance of terrestrial Objects, and the Dip, or Depression of the Horizon of the Sea", *Phil. Trans. Roy. Soc. Lond.* **87** (1797), p. 29-42.
[24] M. Minnaert, *The Nature of Light & Colour in the Open Air*, Dover Publications, New York, 1954.
[25] W. Tape, "The topology of mirages", *Scientific American* **252** (1985), no. 6, p. 120-129.
[26] J. Huddart, "Observations on horizontal Refractions which affect the Appearance of terrestrial Objects, and the Dip, or Depression of the Horizon of the Sea", *Journal of Natural Philosophy, Chemistry and the Arts* **1** (1797), p. 145-152.

[27] J. Huddart, "Beobachtungen über die horizontale Strahlenbrechung bey irdischen Gegenständen und über die Vertiefung des Seehorizontes (dip of the sea)", *Gilb. Ann. Phys.* **3** (1800), no. 3, p. 257-280.

[28] "Göttingen", *Göttingische Anzeigen von gelehrten Sachen, 82. St.* (1796), p. 809-818.

[29] R. Woltmann, "Bemerkungen über ein Katoptrisches Phänomen, welches an den Gegenständen nahe am Horizont nicht selten sichtbar ist", *Neuere Abhandlungen der königlichen Böhmischen Gesellschaft der Wissenschaften* **3** (1798), p. 69-97.

[30] R. Woltmann, "Beobachtungen über die Brechung der Lichtstrahlen, die nahe über der Erdfläche hinfahren", *Gilb. Ann. Phys.* **3** (1800), no. 4, p. 397-438.

[31] R. Greenler, *Rainbows, Halos, and Glories*, Cambridge University Press, Cambridge, 1980.

[32] F.-A. Forel, *Le Léman. Monographie limnologique, Vol. 2*, F. Rouge, Lausanne, 1895.

[33] J.-B. Delambre, *Histoire de l'Astronomie au Dix-Huitième Siècle*, Bachelier, Paris, 1827, p. 697.

[34] T. Gruber, "Abbé Gruber's Theorie dieses katoptrischen Phänomens, von Senkung und Hebung der Objekte am Horizonte", *Neuere Abhandlungen der königlichen Böhmischen Gesellschaft der Wissenschaften* **3** (1798), p. 97-107.

[35] "Paris", *Göttingische Anzeigen von gelehrten Sachen, 186. St.* (1800), p. 1849-1852.

[36] "Über die *Fata Morgana*, das *Seegesicht* und die *Erhebung*", *Allgemeine Geographische Ephemeriden* **5** (1800), no. 3, p. 195-222.

[37] J. G. Büsch, "Schreiben des Hrn. Prof. *Büsch* in Hamburg, an d. Herausgeber der A.G.E.", *Allgemeine Geographische Ephemeriden* **6** (1800), no. 1, p. 3-14.

[38] J. M. Reinecke, "Antwort *auf das Schreiben des Hrn. Prof. Büsch*", *Allgemeine Geographische Ephemeriden* **6** (1800), no. 1, p. 14-15.

[39] G. Monge, "Abhandlung über ein optisches Phänomen, die Kimmung genannt, von Gaspard Monge", *Allgemeine Geographische Ephemeriden* **6** (1800), no. 1, p. 15-25.

[40] J. M. Reinecke, "Zusatz des Übersetzers", *Allgemeine Geographische Ephemeriden* **6** (1800), no. 1, p. 25-30.

[41] W. Nicholson, "An Account of the Fata Morgana; or the Optical Appearance of Figures, in the Sea and the Air, in the Faro of Messina. With an Engraving", *Journal of Natural Philosophy, Chemistry and the Arts* **1** (1797), p. 225-227.

[42] L. W. Gilbert, "Beobachtungen besonderer Strahlenbrechung von Boscowich, Monge und Ellicot", *Gilb. Ann. Phys.* **3** (1800), p. 302-308.

[43] W. H. Wollaston, "On double Images caused by atmospherical Refraction", *Phil. Trans. Roy. Soc. Lond.* **90** (1800), p. 239-254.

[44] R. Hooke, *Micrographia*, Dover Publications, New York, 1961, [republication of *Micrographia*, John Maryn and James Allestry, London, 1665].

[45] W. H. Wollaston, "Untersuchungen, wie durch atmosphärische Strahlenbrechung doppelte Bilder von Gegenständen entstehen", *Gilb. Ann. Phys.* **11** (1802), p. 1-65.

[46] J.-B. Biot, *Traité Élémentaire d'Astronomie Physique*, Bernard, Paris, 1805.

[47] J.-B. Biot, "Recherches sur les réfractions extraordinaires qui ont lieu près de l'horizon", in *Mémoires de la classe des sciences mathématiques et physiques de l'Institut de France, Année 1809*, Garnery, Paris, 1810, p.1-266.

[48] J. M. Pernter, *Meteorologische Optik*, Wilhelm Braumüller, Vienna, Leipzig, 1902.

[49] J. M. Pernter, F. M. Exner, *Meteorologische Optik*, second ed., Wilhelm Braumüller, Vienna, Leipzig, 1922.

[50] P. G. Tait, "On mirage", *Trans. Roy. Soc. Edinburgh* **30** (1883), p. 551-578.

[51] J. D. Gergonne, "Du mouvement de la lumière dans un milieu transparent, dont la densité varie, dans tous les sens, suivant une loi mathématique quelconque", *Annales de mathématiques pures et appliquées* **19** (1829), p. 257-285.

[52] J. D. Gergonne, "Recherches sur les réfractions terrestres et particulièrement sur le mirage", *Notice Travaux Acad. Gard année 1808* (1809), p. 219-256.

[53] J. D. Gergonne, "Essai analytique sur le phénomène du mirage", *Annales de mathématiques pures et appliquées* **20** (1829), p. 1-31.

[54] A. T. Young, "Annotated bibliography of mirages, green flashes, atmospheric refraction, etc.", https://aty.sdsu.edu/bibliog/bibliog.html.

[55] A. Bravais, "Notice sur le mirage", *Annuaire Météorologique de la France pour 1852* **4** (1853), p. 227-280.

[56] A. Bravais, "Notice sur le mirage", *Annuaire de la Société Météorologique de France pour 1853 : Bulletin des séances* **1** (1857).

[57] A. Bravais, *Notice des travaux scientifiques de M. A. Bravais*, Mallet-Bachelier, Paris, 1854.

[58] A. Bravais, "Optique. Réfraction", *L'Institut* **21** (1853), p. 193.

[59] A. Bravais, "Explication, par le système des ondes, d'un cas remarquable de la réfraction de la lumière", *Ann. Chim. Phys.* **46** (1856), p. 492-501.

[60] F.-A. Forel, "Mirages d'hiver sur le lac Léman", *Bull. Soc. Vaud.* **25** (1890), p. XII-XIII.

[61] T. Ferguson, "Eenige opmerkingen betreffende de Kimduiking in verband met de temperatuur van zee en lucht", *De Zee* **24** (1902), p. 357-361.

[62] J. G. Büsch, "Beobachtungen über horizontale Strahlenbrechung und die wunderbaren Erscheinungen, welche sie bewirkt", *Gilb. Ann. Phys.* **3** (1800), p. 290-301.

[63] T. Foken, “50 years of the Monin–Obukhov similarity theory”, *Boundary-Layer Meteorology* **119** (2006), p. 431-447.

[64] J. D. Everett, “On the optics of mirage”, *Phil. Mag., Series 4,* **45** (1873), no. 299, p. 161-172.

[65] S. S. Zilitinkevich, A. A. Grachev, C. W. Fairall, “Scaling reasoning and field data on the sea surface roughness lengths for scalars”, *J. Atmos. Sci.* **58** (2001), no. 3, p. 320-325.

[66] A. C. Chamberlain, “Roughness length of sea, sand, and snow”, *Boundary-Layer Meteorology* **25** (1983), p. 405-409.

[67] H. C. Freiesleben, “Geophysikalische Folgerungen aus Kimmtiefenbeobachtungen”, *Deutsche Hydrographische Zeitschrift* **2** (1949), p. 78-82.

[68] H. C. Freiesleben, “Investigations into the dip of the horizon”, *Journal of Navigation (London)* **3** (1950), no. 3, p. 270-279.

[69] L. Hasse, “Über den Zusammenhang der Kimmtiefe mit meteorologischen Größen”, *Deutsche Hydrographische Zeitschrift* **13** (1960), p. 181-197.

[70] L. Hasse, “Temperature-difference corrections for the dip of the horizon”, *Journal of Navigation (London)* **17** (1964), no. 1, p. 50-56.

[71] H. H. Shufeldt, “Precision Celestial Navigation”, *Journal of Navigation (London)* **15** (1962), no. 3, p. 301-324.

[72] A. T. Young, G. W. Kattawar, “Sunset science. II. A useful diagram”, *Appl. Opt.* **37** (1998), no. 18, p. 3785-3792.

[73] A. T. Young, “Inferior mirages: an improved model”, *Appl. Opt.* **54** (2015), no. 4, p. B170-B176.

[74] S. Y. van der Werf, I. Shokaryev, “Wave height and horizon dip”, *Navigation, Journal of The American Institute of Navigation* **62** (2015), no. 2, p. 161-169.

[75] R. K. Merton, “The Matthew Effect in Science”, *Science* **159** (1968), no. 3810, p. 56-63.

[76] “Biographical Anecdotes and Characters: Domestic Literature”, in *The New Annual Register for the Year 1800*, G. and J. Robinson, London, 1801, p. 270.

[77] N. Burleigh, *Mirage: Napoleon’s Scientists and the Unveiling of Egypt*, HarperCollins, New York, 2007, p. 30.

[78] N. Giordano, *College Physics: Reasoning and Relationships. Vol. 2*, second ed., Cengage Learning, Boston, Mass., 2012, p. 832.

[79] D. Reid, G. Groves, C. Price, I. Tennant, *Science for the New Zealand Curriculum Year 11*, Cambridge University Press, Port Melbourne, 2011, p. 87.

[80] H. L. Mencken, *Prejudices: Second Series*, Alfred A. Knopf, New York, 1920, p. 158.