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Turbulence of internal gravity waves in the laboratory

La turbulence d'ondes internes de gravité au laboratoire

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Abstract. In this article, we review the recent efforts of several teams that aimed at observing in the laboratory a turbulence of internal gravity waves in a density stratified fluid in the weakly non-linear regime. The common feature to these studies is that they adopted the same strategy of injecting energy in weakly non-linear waves before increasing the forcing amplitude in order to trigger a transition to a wave turbulence regime. The motivation to these works is twofold. On the one hand, it has long been proposed that the dynamics of small oceanic scales is driven by a regime of weakly non-linear internal wave turbulence, without however a definitive confirmation so far. A better understanding of the weakly non-linear internal wave turbulence thus appears as an important lever for improving the parameterization of small oceanic scales in climate models. On the other hand, the identification of valid solutions to the theory of internal gravity wave turbulence is still an open problem, and the experimental observation of this regime is therefore of great interest to guide future theoretical developments. We conclude that two features should be improved in the experiments in order to access to a genuine weakly non-linear wave turbulence in the laboratory. First, one should mitigate the finite size effects and especially prevent the concentration of the energy in wave eigenmodes of the fluid domain. Second, one should implement a significant increase of the wavelength at which the energy is injected in order to access to larger Reynolds numbers and lower flow Froude numbers and build a turbulence with well developed power law spectra while remaining in the weakly non-linear regime.

Résumé. Dans cet article, nous passons en revue les efforts récents de plusieurs équipes visant à observer au laboratoire une turbulence d'ondes internes de gravité dans un fluide stratifié en densité dans le régime faiblement non-linéaire. Ces études ont en commun d'adopter la même stratégie consistant à injecter l'énergie dans des ondes faiblement non-linéaires avant d'augmenter l'amplitude du forçage afin de déclencher une transition vers un régime de turbulence d'ondes. La motivation à ces travaux est double. D'une part, il est depuis longtemps proposé que la dynamique des petites échelles océaniques soit pilotée par le régime de turbulence d'ondes internes faiblement non-linéaire, sans toutefois qu'une confirmation définitive n'ait pour l'instant pu être apportée. Une meilleure compréhension de la turbulence d'ondes internes faiblement non-linéaire apparaît ainsi comme un levier important pour améliorer la paramétrisation des petites échelles océaniques dans les modèles climatiques. D'autre part, l'identification de solutions valides à la théorie de la turbulence d'ondes internes de gravité reste un problème ouvert et l'observation expérimentale de ce régime

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apparaît donc d'un grand intérêt pour guider les développements théoriques. Nous concluons que deux caractéristiques doivent être améliorées dans les expériences afin d'accéder à un véritable régime de turbulence d'ondes faiblement non-linéaire au laboratoire. Il faut d'une part contrôler les effets de taille finie et en particulier empêcher la concentration de l'énergie dans les modes propres d'ondes du domaine fluide. Il convient d'autre part d'augmenter sensiblement la longueur d'onde à laquelle l'énergie est injectée afin d'accéder à des nombres de Reynolds plus élevés et des nombres de Froude plus faibles pour construire une turbulence avec des spectres en loi de puissance développés tout en restant dans un régime faiblement non linéaire.

Keywords. Internal gravity waves, Stratified fluids, Turbulence, Oceans.

Mots-clés. Ondes internes de gravité, Fluides stratifiés, Turbulence, Océans.

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1. Introduction to internal waves and "stratified" turbulence

The most remarkable feature of fluids that are stably stratified in density is that they support the propagation of a specific class of waves in their bulk [1–3]. These waves are called internal gravity waves and result from the restoring action of the buoyancy force. They are dispersive and anisotropic and possess peculiar features: group and phase velocities normal to each other and a direction of propagation set by the wave frequency. Also, their wavelength is independent of their frequency and is set by boundary conditions, viscous dissipation and non-linearities [4, 5]. In the case of a uniform vertical gradient of density that is generally considered, the inviscid dispersion relation writes [1–3]

$$\omega = \pm N \frac{k_\perp}{\sqrt{k_\perp^2 + k_z^2}} \tag{1}$$

where ω is the wave angular frequency, and $k_\perp$ and k_z are the norm of the components of the wavevector $\mathbf{k}$ normal and parallel to gravity, respectively. The buoyancy frequency $N = \sqrt{-g/\rho_0\, d\bar{\rho}/dz}$ is defined by the amplitude of the density gradient at rest, $d\bar{\rho}/dz < 0$, and the acceleration of gravity g (with the vertical coordinate z oriented opposite to gravity). Equation (1) is obtained from the Euler equation under the approximation of weak density variations (both static and dynamic) with respect to the reference density ρ_0 [2, 3].

It is worth to note that a global rotation at a rate Ω also allows for a specific class of waves to propagate in the volume of a fluid, this time as a result of the Coriolis force [6, 7]. These waves, called inertial waves, possess features similar to those of internal gravity waves but also differ from them in several other respects. Plane inertial waves have for example three non-zero velocity components whereas plane internal gravity waves have only two, which importantly differentiates their behavior regarding solid boundaries [5, 8], resonances in finite size domains [9, 10] and non-linear interactions [11, 12]. When rotation and stratification are both present, the two types of waves merge into gravity-inertia internal waves of dispersion relation

$$\omega = \pm \frac{1}{k} \sqrt{(2\Omega k_z)^2 + (N k_\perp)^2}, \tag{2}$$

with $k = \sqrt{k_\perp^2 + k_z^2}$ the norm of the wavevector (we consider a rotation vector along z).

With the rotation of the Earth, the stratification in density of the oceans and the atmosphere is a key and ubiquitous ingredient of the turbulent dynamics of geophysical flows [13–16]. In this context, an important feature of global oceanic and atmospheric models is the use of parameterizations accounting for the "fine scales" [16–20], which are generally strongly affected by the fluid stratification and the associated internal wave dynamics, and often (but to a lesser extent) by the Earth rotation (N being generally significantly larger than Ω). Progress in the modeling of these small scale dynamics constitutes an important lever to improve oceanic and atmospheric simulations [20] which has long motivated the search for a fundamental understanding of "stratified turbulence".

Turbulence in a stably stratified fluid can develop in several different regimes [21–24], which all are anisotropic and which can coexist over different ranges of scales. To avoid any ambiguity, we specify that in this article we do not consider the effects of the diffusion of mass. Strictly speaking, this amounts to consider fluids with Prandtl number much larger than 1, i.e. with a buoyancy diffusivity negligible with respect to the kinematic viscosity. In practice, when focusing on turbulent dynamics at large Reynolds numbers, having a Prandtl number not lower than 1 is generally enough to ignore the effects of buoyancy diffusion. In this framework, the richness of stratified turbulence is related to the fact that *weakly non-linear* internal waves can coexist and interact with *strongly non-linear* vortex structures. In order to anticipate which regime will be at play, one should consider three independent non-dimensional numbers: the Reynolds number $Re = \tau_\nu/\tau_{\rm nl}$, the Froude number $Fr = 1/\tau_{\rm nl}N$ and the non-dimensional frequency $\omega^* = \omega/N$, where $\tau_{\rm nl}$ and τ_ν are the characteristic non-linear and viscous timescales of the flow structures at scale ℓ, respectively, and $1/\omega$ is their linear timescale, which might be different from $\tau_{\rm nl}$ (with $\omega\tau_{\rm nl} \geq 1$) since we consider a system in which waves can propagate. Moreover, since stratified turbulence is anisotropic, in order to characterize the flow structures "at scale ℓ", one might have to introduce different characteristic values for the horizontal $u_\perp$ and vertical u_z velocity components and for the horizontal $\ell_\perp$ and vertical ℓ_z length scales. In practice, a cautious analysis of the equations of the dynamics is necessary in order to explicit the scaling of the non-linear and viscous times, the result depending on the considered regime of the flow. In this context, theoreticians often consider the strongly anisotropic limit $\ell_z \ll \ell_\perp$ in order to simplify the problem. In this limit, the non-linear timescale follows the scaling $\tau_{\rm nl} \sim \ell_\perp/u_\perp$ [25, 26] and the viscous timescale $\tau_\nu \sim \ell_z^2/\nu$. Besides, the linear timescale simplifies to $1/\omega \simeq k_z/(Nk_\perp) \sim \ell_\perp/(N\ell_z)$ according to the dispersion relation (1). This scaling reveals that this regime, often called "hydrostatic", corresponds to slow dynamics.

Coming back to the general case, two main classes of turbulence can be expected for stratified turbulence depending on the value of the ratio Fr/ω^* of the Froude number and the non-dimensional frequency (see Figure 1). This ratio, also equal to $1/\omega\tau_{\rm nl}$, compares the linear timescale to the non-linear one. When it is of order 1, we expect a regime of strongly-nonlinear turbulence with an energy cascade carried by flow modes with comparable linear and non-linear timescales. This class of stratified turbulence includes the phenomenologies called "saturated wave" turbulence [27], "critical balance" turbulence [28] and "strongly stratified turbulence" [25, 29] which have been introduced to describe the strongly non-linear regime in the "hydrostatic" limit where $Fr \simeq \omega^* \ll 1$.

The other limit case is expected when, at scale ℓ, the Reynolds number Re is large whereas the Froude number Fr is low compared to the non-dimensional frequency ω^* and therefore $\omega\tau_{nl} \gg 1$. This is the "weak turbulence" regime also called "wave turbulence" [30, 31]. In this regime, an energy cascade is expected to result from triadic resonant interactions within a statistical ensemble of weakly non-linear internal gravity waves which exchange energy a rate much slower than their frequency [20, 32]. This framework has often been suggested as a potential explanation for the oceanic dynamics at small scales [20, 33, 34], without however a direct confirmation so far. This question is of great interest in view of the advance that a validation of the weak turbulence theory could bring for the parameterization of the oceanic small scales in climate models [16–20].

In this context, even if significant progresses have been achieved during the last two decades regarding the internal wave turbulence theoretical problem [26, 32, 35–39] and its comparison with oceanic data [20, 32], analytical solutions are still out of reach in the general case [38, 39], in

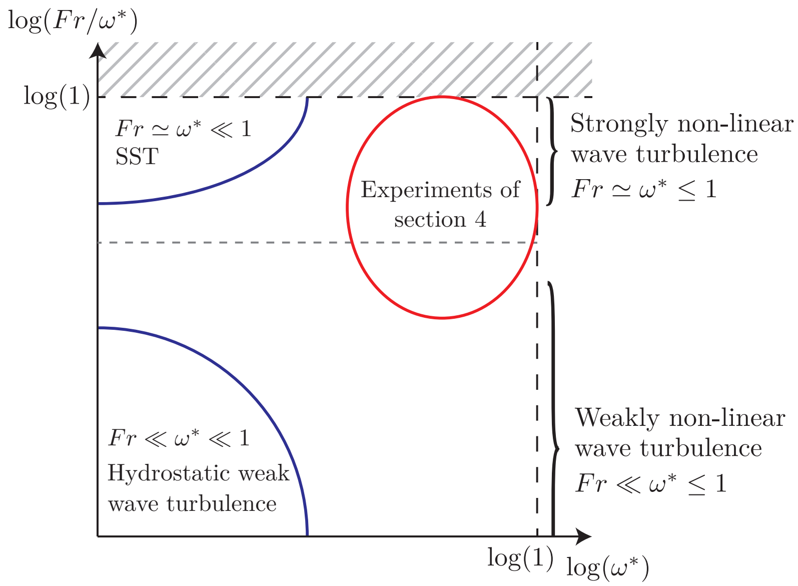

Figure 1. Diagram of the expected regimes of internal gravity wave turbulence as a function of the non-dimensional frequency $\omega^* = \omega/N$ and the ratio Fr/ω^* between the Froude number $Fr = 1/\tau_{\rm nl}N$ and ω^*. We recall that $Fr/\omega^* = 1/\omega\tau_{\rm nl} \le 1$ identifies to the ratio between the flow linear $1/\omega$ and non-linear $\tau_{\rm nl}$ timescales and thus assesses its degree of non-linearity. The Reynolds number $Re = \tau_\nu/\tau_{\rm nl}$ is assumed infinitely large. The location in this diagram of the experiments described in Section 4 is also indicated even if the corresponding Reynolds numbers are not always very large.

particular, without using the hydrostatic hypothesis. As a consequence, conducting experiments to identify empirically what is(are) the fundamental solution(s) to the internal wave turbulence problem is of great interest.

This article focuses on experimental studies that have recently been carried out in order to access to the regime of weakly non-linear internal gravity wave turbulence in the laboratory. Before we present these experimental works in Section 4, we introduce in more detail the theoretical framework of the "strongly stratified turbulence" in Section 2 and of the "internal wave turbulence" in Section 3.

2. The "strongly stratified turbulence" phenomenology

The "strongly stratified turbulence" phenomenology (SST) have been developed to describe stratified turbulence in its strongly non-linear and low-frequency limit $Fr \simeq \omega^* \ll 1$ [21, 23, 25, 29]. This regime is thought to be relevant for the atmospheric dynamics at small and intermediate scales [40]. In this regime, the flow consists in an ensemble of horizontally elongated "pancake" eddies (with $\ell_z \ll \ell_\perp$ and $u_z \ll u_\perp$) which are shearing each other in the vertical direction producing a cascade of energy from large to small scales. Phenomenological predictions have been put forward for the 1D spatial kinetic energy spectra of SST [25, 27–29]. One expects a Kolmogorov scaling for the spectrum as a function of the horizontal wavevector component, $E(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}$, where ϵ is the mean rate of kinetic energy dissipation per unit mass, and a scaling involving the buoyancy frequency N for the spectrum as a function of the vertical wavevector component, $E(k_z) \sim N^2 k_z^{-3}$. This vertical spectrum is often referred to as the "saturation spectrum" [41, 42] in relation with the concept of waves which saturate because of strong non-linearities [27]. This scaling for the vertical spectrum can also be found from the "critical balance" phenomenology [28] (based on the assumption that $\omega\tau_{nl} \simeq 1$) which is close to the concept of turbulence of "saturated waves". A more demonstrative and complete justification of the SST phenomenology has been proposed during the 2000's on the basis of a self-similarity of the system of dynamical equations [25, 29, 43].

Adopting the usual non-dimensional numbers used in the SST literature [25], the SST regime is expected for large "horizontal" Reynolds numbers $Re_\perp = u_\perp \ell_\perp/\nu \gg 1$ and low "horizontal" Froude numbers $Fr_\perp = u_\perp/N\ell_\perp \ll 1$ on the condition that the parameter $\mathscr{R} = Re_\perp Fr_\perp^2$ (sometimes called buoyancy Reynolds number) is large [25]. Besides, it is expected that the largest vertical thickness of the "pancake eddies", i.e. the vertical integral scale of the turbulence, adjusts dynamically to the so-called buoyancy scale $u_\perp/N$ [21, 25, 43, 44]. This process can result for instance from the zig-zag instability [45, 46]. This vertical scale selection eventually leads the "vertical" Froude number $Fr_z = u_\perp/N\ell_z$ to be of order 1 at the integral scale, this result being also true on a scale-by-scale basis. It is important to highlight that the conditions under which the SST regime is expected are actually simpler when expressed in terms of the non-dimensional numbers we introduced in the previous section: these conditions simply are $Re = \tau_\nu/\tau_{\rm nl} \gg 1$ and $Fr = 1/N\tau_{\rm nl} \simeq \omega^* \ll 1$. This can be shown from the fact, in the strongly anisotropic limit $\ell_z \ll \ell_\perp$, one has $\tau_{\rm nl} \sim \ell_\perp/u_\perp$ and $\tau_\nu \sim \ell_z^2/\nu$ such that (in the strongly anisotropic limit) $\mathscr{R}$ is nothing more than the Reynolds number Re, $Fr_\perp$ the Froude number Fr, and Fr_z the ratio Fr/ω^* of the Froude number to the non-dimensional frequency.

To conclude this section, it is important to mention that the predictions of the "strongly stratified turbulence" phenomenology are in line with the results of several direct numerical simulations [25, 29, 42, 47, 48]. These predictions for the 1D spatial energy spectra, i.e. $E(k_\perp) \sim k_\perp^{-5/3}$ and $E(k_z) \sim k_z^{-3}$, also remarkably seem to agree with observations at small and intermediate scales in the atmosphere [25, 27, 40, 49–51] as well as at (very) small scales in the oceans [40].

3. The internal wave turbulence theory

In this section, we still consider flows with large Reynolds numbers $Re = \tau_\nu/\tau_{\rm nl} \gg 1$ and small Froude numbers $Fr = 1/N\tau_{\rm nl} \ll 1$. However, they will be weakly non-linear: This implies a separation between the linear timescale $1/\omega$ and the non-linear timescale $\tau_{\rm nl} \gg 1/\omega$ which is the primary characteristic of a "wave turbulence" [30, 31]. This timescale separation is equivalent to having a Froude number $Fr = 1/\tau_{\rm nl}N$ much smaller than the non-dimensional frequency ω^* which is itself bounded by 1 for internal waves. Since the strongly anisotropic limit, corresponding to $\ell_\perp \gg \ell_z$, is often considered in the theoretical literature, it is worth to mention that in this limit the weak non-linearity condition is equivalent to having a very small vertical Froude number $Fr_z = u_\perp/N\ell_z$ (remembering that in the hydrostatic limit $\tau_{\rm nl} \sim \ell_\perp/u_\perp$). Coming back to the general case, the wave turbulence regime is expected when the condition $Fr = 1/\tau_{\rm nl}N \ll \omega^*$ is fulfilled [30]. In this regime, the energy, carried by an ensemble of weakly non-linear internal waves, is expected to be transferred on statistical average from large to small scales at a rate much lower than the wave frequencies. Also, because of the quadratic non-linearity of the Navier-Stokes equation, this energy cascade is expected to result from triadic resonant interactions between internal waves [26, 30, 32].

The theoretical study of the weakly non-linear wave turbulence regime in stratified fluids is a very active area [26, 32, 33, 35–39, 52]. An underlying motivation is the understanding of the oceanic dynamics and energy spectra at small scales [20, 34, 53]. To be more specific, oceanic data classically reveal one-dimensional (1D) energy spectra, in frequency ω and in vertical wavenumber k_z, following power laws with an exponent of the order of -2 [34]. These scaling laws are proposed to result from a cascade of energy from low to high frequencies (periods typically in the range from the tides period, of about 12 h, to a few tens of minutes) and from large to small vertical scales (typically from a few hundred meters to a few meters). This fine-scale oceanic behavior is often summarized by the two-dimensional (2D) energy spectrum $E(k_z,\omega) \sim k_z^{-2}\omega^{-2}$ introduced by Garrett and Munk (GM) in the 1970s [53–55]. It should be noticed that this extrapolation of a 2D energy spectrum from two 1D effectively measured energy spectra is based on the strong assumption of a decorrelation between ω and k_z which is not necessarily valid for internal wave turbulence [26]. These oceanic spectra have nevertheless long been thought to result from the non-linear dynamics of internal gravity waves [20, 34] producing a forward energy cascade. Oceanic data also suggest that this small-scale internal wave turbulence eventually feeds, at even smaller oceanic scales, a cascade of energy of the "saturated wave/critical balance" type (see, e.g., [40, Fig. 1] and [34, Fig. 21] which illustrate the transition between the two regimes).

From a more fundamental perspective, the Wave Turbulence Theory (WTT) is also attractive because of its analytical nature [30, 31] and because it has already been successfully applied to other wave systems, such as capillary surface waves[1] [56] and inertial waves in rotating fluids [57]. Similarly to the SST regime, we still expect in the wave turbulence regime a direct and anisotropic energy cascade from large to small scales but with different scaling laws for the rate of energy transfer and for the energy spectra [26, 30, 35, 38].

The classical derivation of the wave turbulence theory in stratified fluids leads to an analytical prediction for the 2D (axisymmetric) spatial energy spectrum scaling as $E(k_\perp, k_z) \sim \sqrt{\varepsilon N}\, k_\perp^{-3/2} k_z^{-3/2}$ [35, 36, 52]. This steady solution of the kinetic equation –the central equation in the wave turbulence theory– has been derived in the limit $k_\perp \ll k_z$ corresponding to low frequency internal waves ($\omega^* \simeq k_\perp/k_z$). It also results from the assumption that the non-linear interactions are local in the wavenumber and frequency spaces. Nevertheless, the derivation of this spectrum $E(k_\perp, k_z) \sim \sqrt{\varepsilon N}\, k_\perp^{-3/2} k_z^{-3/2}$ is achieved by ignoring a divergence of the so-called "collision integral" which, at the end, makes this spectrum irrelevant [26, 35, 38]. This divergence suggests that non-local triadic interactions between internal gravity waves of very different frequencies and/or wavenumbers are important in the turbulent dynamics and cannot be ignored.

In practice, the implementation of the WTT in the case of stratified fluids is still an open subject which involves delicate analytical questions regarding the convergence of the collision integral [38]. Over the past 20 years, several theoretical works have been carried out in an attempt to solve this issue by taking into account non-local interactions [26, 32, 38]. These works, all derived in the hydrostatic limit, first suggested that a whole family of stationary solutions with a constant energy flux exists [32, 37] before identifying that the spectrum $E(k_\perp, k_z) \sim k_\perp^{2-a} k_z^{-1}$ with $a \simeq 3.69$ is a remarkable solution because it leads to an exact compensation of two diverging parts of the collision integral [32, 38]. Remarkably, this prediction is relatively close to the Garrett-and-Munk spectrum (which corresponds to the case $a = 4$). This spectrum is nevertheless suspected to be a non-realizable singular stationary state of the kinetic equation in reference [39].

In this context, a stationary solution of a simplified kinetic equation describing the small scales of the internal wave turbulence problem in the hydrostatic limit has recently been derived analytically in reference [26]. This derivation is based on the assumption that energy transfers are driven only by a subset of the non-local resonant triads, the so-called "induced diffusion" triads, which leads to a scale separation in the kinetic equation. Despite it ignores the other types of resonant triads (the so-called "elastic scattering" triads, "parametric subharmonic instability" triads and local triads [58]), it appears that this solution is consistent with the -2 exponents typically observed in the ocean interior for the 1D energy spectra in ω and k_z [34]. In this

[1] It is worth mentioning that the anisotropic nature of the dynamics of stratified fluids makes the implementation of WTT more complex from the outset for internal waves than for isotropic waves with a power-law dispersion relation between their frequency and the norm of their wavevector [30, 31].

theoretical framework, the phenomenology of the "induced diffusion" triads leads to a wave frequency ω proportional to the vertical wavenumber k_z along the turbulent cascade.

In parallel, Dematteis et al. [20] recently succeeded to quantitatively compute the actual small-scale energy dissipation in the ocean interior by injecting oceanic intermediate-scale data as an input in the equations of wave turbulence theory in the hydrostatic limit which are analyzed numerically. This work seems to demonstrate that the internal wave turbulence framework can quantitatively account for the oceanic processes at very small scales from the knowledge of the mesoscale properties of the flow (putting aside regions where internal waves interact with strong mesoscale currents). Considering realistic values for the oceanic buoyancy frequency and the Earth rotation which is here included in the model, Dematteis et al. [20] also propose that both local and non-local triadic interactions are important in oceanic internal wave turbulence. It should nevertheless be highlighted that, in this work, 2D energy spectra are extrapolated from *in situ* measurements of 1D energy spectra using the assumption of the decorrelation of k_z and ω.

It is finally worth to note the recent theoretical work by Labarre et al. [39] in which the kinetic equation of internal wave turbulence (without rotation) is for the first time shown to have a canonical structure, typical of Hamiltonian systems, even in the general non-hydrostatic case. This work paves the way for searching (at least numerically) for solutions of the internal wave turbulence problem in the general case (not restricted to the low-frequency limit).

In this complex theoretical framework, where no definitive solution(s) has(have) been identified to the internal wave turbulence theoretical problem, experiments (possibly numerical) promoting the emergence of a turbulent flow in the regime of weakly non-linear internal wave turbulence are of strong interest to guide and test theoretical developments.

4. Internal wave turbulence experiments

In the following, we review several experiments [59–70], most of which conducted over the last decade, in which the authors aimed at forcing a turbulent flow in a linearly stratified fluid by injecting energy in internal gravity waves. Turbulence experiments in stratified fluids were previously most often carried out with a preferential energy injection in vortical structures. These works consisted in practice in studies of decaying turbulence triggered by a translated grid [71, 72] or a grid in a wind tunnel [73, 74] and of statistically steady turbulence forced by an oscillating grid [75] or vortex dipole generators [76]. In any case, these kinds of forcing naturally lead, if the Reynolds number is large enough, to flows dominated by vortical structures or "saturated waves" in a strongly-non-linear regime of the type of the SST regime described earlier.

4.1. *Preliminary remarks*

Despite a strong interest discussed in the previous section, weakly non-linear stratified turbulence obtained from energy injection in internal gravity waves has only recently been approached in experiments and numerical simulations. This is most probably due to the fact it requires large facilities in the experimental case in order to simultaneously reach large Reynolds numbers and small Froude numbers and large computational resources in the numerical case, especially because of the linear/non-linear timescale separation, characteristic of wave turbulence, which imposes an integration of the dynamics over particularly long durations.

In order to succeed producing a weakly-non-linear internal wave turbulence regime in the lab, the primary condition is indeed to impose distinct linear and non-linear timescales. To do so, one must enforce independently an oscillation period $T = 2\pi/\omega$ and a velocity u to the fluid motion such that the weak non-linearity condition $T \ll \tau_{nl}$ is fulfilled. We already noticed in the previous

Table 1. Parameters of several experiments presented in this article. $Fr = u/N\ell$ and $Re = u\ell/\nu$ are the Froude and Reynolds numbers based on the flow parameters. u is the typical velocity of the turbulent flows in the considered study (for the experiment at the largest forcing amplitude) and ℓ is the typical wavelength of the forced internal waves (u and ℓ are just orders of magnitude). $\omega_0^* = \omega_0/N$ is the forcing frequency non-dimensionalized by the buoyancy frequency N. In the cases of Benielli & Sommeria [59, 60] and Le Reun et al. [65], ω_0 is half the forcing frequency since the primary waves are forced through a parametric instability. We take $\nu = 10^{-6}$ m^2/s for the kinematic viscosity of the fluid. H is the height of the stratified fluid.

	N (rad/s)	ω_0^*	ℓ (cm)	u (cm/s)	Fr	Re	H (m)
Oscillating water tank (Benielli et al. [59, 60])	1.9	0.7	26	15	0.30	39 000	0.25
Tidal forcing simulations (Le Reun et al. [65])		0.25			0.012	720	
		0.67			0.035	3 700	
Attractor experiments (Davis et al. [66])	0.7	0.6	30	1.5	0.07	4 500	0.92
Oscillating wall experiments (Savaro et al. [67])	0.6	0.7	100	1.5	0.03	15 000	1
Oscillating wall experiments (Rodda et al. [68])	0.4	0.16	100	0.6	0.014	6 000	1
	0.4	0.38	100	1.5	0.03	15 000	1
Self-similar wave beam experiments (Lanchon et al. [70])	1.0	0.94	15	0.5	0.03	700	0.62

section that in the low-frequency limit, the actual non-linear time driving the triadic interactions is set by the ratio between the wave horizontal scale and the horizontal velocity, $\tau_{\rm nl} \sim \ell_\perp/u_\perp$, despite the alternative time built with the vertical scale, $\ell_z/u_\perp$, would be much smaller. This illustrates the fact this is not trivial to choose the relevant timescale in the anisotropic context of stratified turbulence. Nevertheless, since in the experimental works we will discuss, mainly waves at finite frequencies are considered (i.e. with $\omega^* = \omega/N$ neither too close to 0 nor too close to 1) we will simplify in the following the weak non-linearity condition to $T \ll \tau_{\rm nl} = \ell/u$, with ℓ the wavelength. We recall that the wavelength of internal waves ℓ is independent of their frequency which provides an additional degree of freedom to the experimentalist. The weak non-linearity condition, essential to wave turbulence, is equivalent to having a Froude number $Fr = u/N\ell$ small compared to the non-dimensional wave frequency $\omega^* = \omega/N$. At the same time, the Reynolds number $Re = \tau_\nu/\tau_{\rm nl} = u\ell/\nu$ should be large in order for a turbulent cascade over large ranges of scales to develop.

Experimentally, stably stratified fluids are made of salt-water mixtures with a decreasing concentration of salt as a function of the vertical coordinate z (opposite to gravity). They are fabricated by means of the so-called *double-bucket* method [77, 78]. Some experimentalists alternatively use water-salt-ethanol mixture to fabricate the stratified fluid [79, 80]. The advantage is that this makes it possible to obtain a fluid stratified in density with a uniform optical index, which becomes important for particle image velocimetry measurements when the fluid displacements do not remain weak [69]. In both cases, the typical largest buoyancy frequencies N that can be reached are of the order of 1 rad/s. One therefore realizes that the only mean to progress toward the limit under which internal wave turbulence is expected (low Froude number $Fr = u/N\ell$ and large Reynolds number $Re = u\ell/\nu$) is to access to large wavelength ℓ of the forced waves and therefore necessarily to build large experiments.

Another interesting point is the frequency ω_0^* at which the energy should be injected in the wave frequency domain $0 \leq \omega_0^* \leq 1$. The canonical solution to the theoretical internal wave turbulence problem $E(k_\perp, k_z) \sim \sqrt{\varepsilon N}\, k_\perp^{-3/2} k_z^{-3/2}$ [35, 36, 52] (which leads to a divergent collision integral) is associated to an energy cascade from large to low frequencies. In parallel, this is not necessarily the case in the alternative theoretical approaches that have later been developed for the internal wave turbulence problem [20, 26, 32, 37–39]. The non-local low-frequency solution proposed in reference [26] is for example associated to an energy flux toward large frequencies. We should also mention that typical fine-scale oceanic spectra [20, 34] involve dominant energy sources at low frequencies of the order of $0.01\,N$ to $0.1\,N$ (due to near-inertial waves and tidal forcing) and an energy spreading up to the buoyancy frequency N (and often even beyond), consistent with an energy cascade from small to large frequencies. In this context, the authors of the experimental studies discussed in the following, not sure about in which frequency direction an energy cascade might emerge, have tried to trigger wave turbulence by forcing waves at frequencies either of the order of N but also at significantly lower frequencies.

In the following, we will not present the measurement techniques specifically used for each of the described experimental works. We just specify here that the studied flows are characterized by measurements of 2D cross sections of the velocity field by particle image velocimetry, vertical profiles and time-series of the fluid density derived from conductivity measurements and/or dye motion visualizations.

4.2. *Internal waves forced by a parametric instability*

4.2.1. *Vertically oscillating water tank experiments*

We start our review by describing the work of Benielli and Sommeria [59, 60] who conducted experiments in which a parallelepipedic tank filled with a stratified fluid is submitted to vertical oscillations. These experiments are not fully in line with the ones described in the sequel in which weakly non-linear waves are forced directly. However, since in this work internal waves eventually lead to a turbulent state, it is worth to present these experiments.

In references [59, 60], the tank horizontal dimensions are of about 10 cm × 26 cm, the fluid height is of 25 cm and the buoyancy frequency of $N \simeq 1.9$ rad/s. The fluid container is oscillated at a frequency lower than $2N$ (several values are studied). No internal gravity waves directly resulting from the forcing are reported. This follows primarily from the fact the forcing consists in an oscillating vertical translation which is equivalent to an apparent oscillating uniform gravity. This is even more natural for the forcing frequency which is discussed the most since it is about $1.4\,N$, i.e. a frequency at which internal waves cannot propagate. Internal waves nevertheless emerge in the fluid for discrete resonance frequencies by a mechanism of parametric instability: above a threshold in the forcing amplitude, internal waves develop at half the forcing frequency, as standing internal wave eigenmodes of the fluid domain [9]. As the forcing amplitude is increased above this threshold, the standing wave eigenmodes rapidly become unstable themselves through the mechanism of the triadic resonance instability also known under the name of parametric subharmonic instability in the inviscid limit [3]. This secondary instability feeds localized propagative internal waves at smaller scales and at frequencies of the order of half the primary wave frequency. After a transient growth, these secondary internal waves rapidly experience wave breaking resulting in three-dimensional instabilities and local overturning of the density field. The flow eventually reaches a turbulent regime with an energy cascade toward small scales.

In these experiments, Benielli and Sommeria [59] report temporal potential energy spectra with a clear power law of exponent -3 over more than a decade of frequencies above the buoyancy frequency N. They interpret it as the consequence of a k_z^{-3} 1D vertical energy spectrum that is transposed in time by an intense vertical advection of the small-scale turbulent structures by the largest scales of the flow, a phenomenon known as the sweeping effect [81]. As discussed for example by Staquet [82] and by Nazarenko and Schekochihin [28], 1D vertical energy spectra scaling as $N^2 k_z^{-3}$ are expected from phenomenological arguments for a turbulent cascade driven by strongly non-linear internal waves, which phenomenology is consistent, in the low frequency limit, with the more complete scenario of the strongly stratified turbulence presented earlier in Section 2. We should nevertheless mention that the experiments of Benielli and Sommeria [59], like those described in the following, are not involving low-frequency strongly-anisotropic waves but modes with $k_\perp$ and k_z of the same order of magnitude. In any case, the observation by Benielli and Sommeria of spectra in line with a strongly non-linear internal wave turbulence is compatible with the values of their flow Froude numbers which tend to be of order 1 whereas their Reynolds numbers are relatively high (see Table 1).

4.2.2. *Simulations driven by the tidal excitation of internal waves*

In this section dedicated to studies in which internal waves are indirectly forced by a parametric instability, it is worth to also discuss the direct numerical simulations of the Boussinesq equations by Le Reun et al. [65]. In this study, the flow forcing mimics a periodic tidal deformation which indirectly injects energy in internal waves at half the forcing frequency through a parametric instability. The flow finally reaches a turbulent state which can be, depending on the simulation parameters, statistically steady or made of cycles between turbulent phases and relaminarisation phases, a situation which is actually also reported in the experiments of Benielli and Sommeria [59, 60].

Le Reun and coworkers [65] show that the energy of the turbulence they observe for their largest Reynolds numbers is carried almost only by internal waves verifying the wave dispersion relation. For these simulations, they report 1D spatial energy spectra compatible with scaling laws in $k_\perp^{-3}$ and k_z^{-3} and a dominant transfer of energy toward frequencies smaller than the primary wave frequency ω_0^*. Such scaling laws are typical of the phenomenology of the "saturated wave" turbulence, in which the internal waves are strongly non-linear, here applied in the case where $k_\perp$ and k_z are comparable in order of magnitude (which is the case in all the experiments presented in this review). Indeed, assuming a "critical balance" between the linear and non-linear timescales of the form $N \sim uk$ dimensionally leads to spatial energy spectra in $N^2 k^{-3}$ where k is the considered wavenumber (either vertical or horizontal) and u the typical velocity at wavenumber k [82]. In addition to these features and at variance with the results of Benielli and Sommeria [59, 60], a power-law behavior of the temporal energy spectrum, compatible with an exponent -2, is observed at frequencies larger than the primary wave frequency ω_0^*, when it is significantly smaller than 1 (the limit frequency for internal waves). This scaling can also be seen as compatible with a "saturated wave/critical balance" type of turbulence of internal waves (through nevertheless modifying the critical balance criterion to $\omega \sim uk$).

To conclude this section, even if the parametric instabilities that feed the primary internal waves in the flow are different, the results of the simulations of Le Reun et al. [65] are similar from many points of view to the ones of the experiments of Benielli and Sommeria [59, 60]: Once the forcing has overcome the threshold of the primary parametric instability, the flow rapidly saturates to a turbulent state which seems compatible with a turbulence of "saturated" internal waves in a "critical balance" regime.

4.3. *Internal wave attractor experiments*

We start now the description of recent efforts to access a weakly non-linear internal wave turbulence regime in the laboratory by discussing a series of experiments conducted at ENS de Lyon [61–64, 66] in which an attractor of internal waves is used as a base flow. In these works, the authors use a wave maker to force a large-scale internal gravity wave at a specific frequency $\omega_0 \sim 0.6\,N$ in the wave frequency domain. These experiments have been conducted in trapezoidal containers which are nearly 2D, i.e. thin in the horizontal direction normal to the vertical trapeze. The water tanks that have been used, with a height of either 42.5 cm or 92 cm, are filled with a water solution of salt, linearly stratified in density. The resulting buoyancy frequencies are of the order of 1 rad/s.

Internal gravity waves follow anomalous reflection laws on inclined solid boundaries and their wavelength is modified during reflection [83, 84]. This is a consequence of their peculiar dispersion relation $\omega^* = \sin\theta$, which relates the wave non-dimensional frequency $\omega^* = \omega/N$ to the angle θ between the wave group velocity, along which the energy propagates, and the horizontal. As a consequence, for certain geometries of the fluid domain (and certain ranges of frequencies depending on the geometry), waves at a given frequency tend to focus on a closed limit cycle called "wave attractor" [85–87]. In particular, it is the case in trapezoidal domains in which the obtained base flow consists in a mono-frequency self-similar internal wave beam resulting from an equilibrium between viscous dissipation and wave focusing at reflection on the tilted surface [5, 87, 88]. The great advantage of this method of forcing internal waves is that, thanks to the wave focusing processes during reflections on the inclined wall, it produces a base flow with relatively high fluid velocities (~cm/s), which would lead to significant mixing of the stratification close to the wave maker if they were directly imposed by the wave maker motion. This system has nevertheless two drawbacks. First, the typical wavelength of the forced wave mode is reduced by the focusing processes building the attractor. Second, the fact that the water tank is thin in one of the horizontal directions results in an enhanced contribution of the viscous dissipation on the vertical walls.

In the experiments of ENS de Lyon, as the forcing amplitude is increased, non-linear effects emerge first through a triadic resonance instability of the primary wave beam (i.e. the attractor) [3, 61, 89]. This instability produces new internal waves at two subharmonic frequencies ω_1 and ω_2 in spatial ($\mathbf{k}_0 \pm \mathbf{k}_1 \pm \mathbf{k}_2 = 0$) and temporal ($\omega_0 = \omega_1 + \omega_2$) resonances with the primary wave beam [61, 63, 64] ($\mathbf{k}_i$ and ω_i are the typical wavenumber and frequency of the wave mode i, respectively). Then, as the forcing amplitude is further increased, secondary triadic resonant interactions between the first three wave modes lead to the emergence, in the temporal energy spectrum, of a series of additional peaks at discrete frequencies in the wave domain [such as $\omega_2 - \omega_1$, $\omega_0 - (\omega_2 - \omega_1)\ldots$] [62, 66]. A whole hierarchy of peaks finally emerges through the triadic resonant interactions between the wave modes. Brouzet et al. [64] further show that some of the energy peaks in these temporal spectra are associated to quasi-standing modes and quasi-resonance frequencies of the experimental vessel (trapezoidal domains do not possess exact standing modes theoretically). Even if not all the peaks in this discrete temporal energy spectrum are associated to standing modes, it is tempting to think that the attraction of the energy by resonance frequencies associated with standing modes of the experimental vessel plays a major role here in the construction of the discrete nature of the energy spectrum.

Increasing further the forcing amplitude, a continuum of energy over the wave frequency domain progressively emerges in addition to the discrete temporal energy spectrum [62, 66]. These large amplitude experiments however seem to start involving some mixing of the stratification in density associated to the presence of vortices [62, 64]: the flow seems to approach a strongly non-linear regime. For these experiments at large forcing amplitude, Davis et al. [66] report a behavior compatible with $E(k) \sim k^{-3}$ for the 1D spatial kinetic energy spectrum averaged over all directions in the vertical plane, a scaling which is again reminiscent of "critical balance/saturated wave" turbulence regimes. We should nevertheless notice that this power-law behavior is not observed over more than half-a-decade of wavenumber k, revealing that the turbulent cascade is still not well developed at the Reynolds numbers reached in these experiments.

4.4. *Experiments forced by oscillating walls*

A second series of experiments aiming at producing a weakly non-linear internal wave turbulence by forcing a periodic flow in a stratified fluid has been conducted recently on the Coriolis Platform in Grenoble [67–69]. This platform consists in a 13-m-diameter water tank which can be driven in rotation (a feature not used in the studies we will discuss) but also filled with a stratified water solution of salt.

In a first article by Savaro et al. [67], the authors report experiments in a stratified fluid (with $N \simeq 0.6$ rad/s), where energy is injected in large-scale internal waves at frequency $\omega_0^* \simeq 0.7$. The effective fluid domain is a parallelepiped of 6 m-side square base filled with 1 m of stratified salt-water mixture. Two of the four vertical walls of the square domain can oscillate around a horizontal axis at mid-depth of the fluid, acting as generators of internal waves of characteristic vertical length of 1 m.

Despite the forced base flow is significantly different from the one of the attractor experiments, the scenario of the emergence of nonlinearities in the flow is very similar when increasing the forcing amplitude. At moderate forcing, the authors indeed report the growth of a set of discrete peaks distributed over the wave frequency domain ($0 \le \omega^* \le 1$) in the temporal kinetic energy spectrum of the flow. The authors further show that these energy peaks (at least several of them) correspond to an ensemble of internal wave eigenmodes of the fluid domain. These wave modes moreover seem to verify temporal and spatial triadic resonance conditions ($\omega_i = \omega_j \pm \omega_p$ and $\mathbf{k}_i = \pm\mathbf{k}_j \pm \mathbf{k}_p$) within themselves and/or with the forced wave mode. The only fundamental difference with the attractor experiments is that parallelepipedic domains theoretically possess exact standing eigenmodes of internal gravity waves [9, 67]. Nevertheless, from a practical viewpoint, this does not seem to significantly modify the observed scenario of the transition to turbulence.

Then, as the forcing amplitude is increased, a continuum of energy at subharmonic frequencies (i.e., below the forcing frequency ω_0) grows and tends to progressively engulf the eigenmode peaks (which are not increasing in amplitude) in the temporal energy spectrum. By computing several physical observables, the authors show that this continuum of energy is compatible with an ensemble of propagating internal waves verifying the wave dispersion relation. The wavelengths of these subharmonic waves are spreading from the scale of the forced mode ($\ell \simeq 1$ m) down to values typically ten times smaller. A remarkable feature is that, among these "turbulent" experiments, the smallest length scales present in the flow are increasing with the forcing amplitude, which might seem counter-intuitive. The authors interpret this feature as the result of the emergence of strong nonlinearities such as overturning and breaking of the subharmonic internal waves, which processes are indeed expected to affect the smallest scales in an internal wave turbulence [26].

In a following paper, Rodda et al. [68] report experiments with a modified version of the setup of Savaro et al. [67] where the shape of the water tank has been changed from square to pentagonal. A second difference is that the forcing injects now the energy in internal waves at a lower non-dimensional frequency ω_0^*, in the range $0.16 \le \omega_0^* \le 0.38$ (and also at $\omega_0^* \simeq 0.7$ for the sake of comparison with the previous work). These experiments led, in the nonlinear regime, to a temporal kinetic energy spectrum dominated by a series of discrete peaks at harmonic frequencies ($2\omega_0^*, 3\omega_0^*, 4\omega_0^* \ldots$) of the forcing frequency. Remarkably, when the forcing Froude number is increased, this discrete spectrum superimposes with a continuum of energy at frequencies larger than the forcing frequency which is compatible with a power law ω^{-2}, an observation reminiscent of the results of Le Reun et al. [65]. This emerging power-law behavior extends here beyond the limit frequency of internal waves, typically up to $\omega \simeq 2N$ (whereas it clearly stops at $\omega = N$ in reference [65]). This extension at frequencies at which internal waves cannot propagate might indicate that the flow is evolving toward a strongly nonlinear regime. It is important also to note the difference with the experiments of Benielli and Sommeria [59, 60] where a ω^{-3} power-law is observed and proposed to result from the conversion of a k_z^{-3} spatial energy spectrum by a turbulent sweeping.

Another remarkable result of Rodda et al. [68] is the presence of so-called *bound waves* revealed by the spatio-temporal scale analysis of the flow energy in the experiments at $\omega_0^* \simeq 0.7$. These wave modes are not verifying the internal wave dispersion relation and are directly forced by the non-linear interaction of two other modes, which can be genuine internal waves, the three modes forming a resonant triad in time and space, i.e., verifying $\omega_{\rm bw} = \omega_1 \pm \omega_2$ and $\mathbf{k}_{\rm bw} = \pm\mathbf{k}_1 \pm \mathbf{k}_2$. In a weakly non-linear regime, such bound waves are expected to be of second order compared to triadic resonances between internal gravity waves. Their presence is therefore another sign that strong non-linearity is emerging in the flow as the forcing amplitude is increased. We can note that such bound waves have also been detected in the attractor experiments where traces of a ω^{-2} power-law have also been reported in the temporal energy spectrum (see [90, Chapter 4], in french). Rodda et al. [68] finally suggest that these bound waves might have a role in the ω^{-2} power-law they observed above ω_0^* in their temporal energy spectrum (for the experiments at $\omega_0^* \simeq 0.7$ but also the ones at lower forcing frequency). This interpretation is motivated by the fact the ω^{-2} power-law extends beyond the frequency limit $\omega = N$ of internal waves. It is nevertheless worth to recall that a temporal energy spectrum in ω^{-2} is also not incompatible with "critical balance/saturated wave" regimes for which the condition $\omega \sim ku$ coupled to the conservation of the energy flux $\epsilon \sim u^3 k$ suggests that $E(\omega) \sim u^2/\omega \sim \epsilon\,\omega^{-2}$ (here, we consider the case of waves with $k_\perp \sim k_z$).

In a subsequent paper by Rodda et al. [69], the team of Grenoble studied the flow at even larger Reynolds number (with $\omega_0^* \simeq 0.7$) for which a turbulent cascade develops over a wide range of scales from the injection scale of about 1 m down to few millimeters. The authors first evidence, through dye visualizations, the intermittent occurrence of intense wave overturning events. They also show that a strong sweeping effect [81] induced by the large-scale wave modes of the flow affects almost all other modes at smaller scales, spreading their frequencies up to values much larger than N (in a way similar to Benielli and Sommeria [59, 60]). The final result of this work is the observation of power laws of exponent -3 for the 1D kinetic energy spectra as a function both of the horizontal $k_\perp$ and the vertical k_z wavenumbers. These power laws are observed from the scale of the forced waves towards smaller scales over typically half-a-decade, which is consistent with the spectral behavior reported in the attractor experiments of Davis et al. [66]. The spectra further show a transition to a second behavior at smaller scales which is interpreted as a small-scale isotropic turbulence driven by the classical Kolmogorov phenomenology. A remarkable result here is the fact the vertical potential energy spectrum (which has been measured in this work) closely follows the vertical kinetic energy spectrum over the half-decade over which the k_z^{-3} behavior is observed. This equipartition between the potential and kinetic energies strongly suggests that the physics of internal waves is at play here (because the kinetic and potential energies in an internal wave are equal) whereas a wealth of strongly non-linear processes are also involved at the same time. Such behavior is (once again) typical of turbulence in a "critical balance" regime [30] driven by strongly non-linear waves.

4.5. *Experiments forced by multiple self-similar wave beams*

In this section, we describe a more recent attempt to observe a weakly non-linear internal wave turbulence in the laboratory, which has been conducted by Lanchon and coworkers at Université Paris-Saclay [70]. These experiments were carried in a glass tank of 105 cm side square-base filled up to a height of 62 cm with a linearly stratified salt water of buoyancy frequency $N \simeq 1$ rad/s. The forcing device consists in twenty-four horizontal cylinders oscillating vertically at frequency $\omega_0 = 0.94\,N$. The cylinders are evenly organized on an 80-cm-diameter virtual sphere centered in the tank. In the linear regime, each oscillating cylinder produces self-similar internal wave beams, which propagate in the four directions normal to the cylinder axis and which make an angle $\cos^{-1}(\omega_0/N) \simeq 20°$ with respect to the vertical. The forcing device eventually produces, in the central region of the experiment, a flow made of an ensemble of internal waves which approaches statistical homogeneity and axisymmetry around the vertical axis (axisymmetry in the Fourier space), which are two central hypotheses under which wave turbulence calculations are conducted [26, 35]. The typical wavelength of the forced mode is $\ell = 15 \pm 5$ cm.

When the forcing amplitude is increased, the flow non-linearities produce, in the statistically steady state, a set of internal wave modes at discrete subharmonic frequencies (i.e. lower than ω_0). These modes, the most energetic of which are shown to be eigenmodes of the fluid domain, are in temporal triadic resonances with the forced waves and/or between themselves. This scenario is very similar to the one observed in the experiments conducted in Lyon and Grenoble and described earlier. The discretization of the energy in frequency and in wavenumber that is observed in all these experiments prevents the flow from approaching the regime described in the wave turbulence theories [26, 32, 33, 35, 37, 38] in which the turbulent cascade is carried by an ensemble of weakly non-linear and propagating waves in an infinite domain which builds an energy continuum in the frequency and wavenumber spaces.

In Lanchon et al. [70], a study of the transient regime of the flow from the start of the wavemakers is also reported. During a first stage after the start of the wavemakers which lasts several hundreds of forcing periods, the time-frequency spectra reveal the growth of two wide subharmonic bumps in temporal resonance with the forced waves. The emergence of these bumps is in line with the classical scenario of the triadic resonance instability observed for inertial waves in rotating fluids [11, 91], a system similar to stratified fluids to some extent but in which eigenmodes of the fluid domain are much less prone to develop compared to stratified fluids [9] (due to differences in the wave interactions with solid boundaries). In a second stage of the experiments of Lanchon et al. [70], the subharmonic wide bumps in the temporal energy spectrum progressively vanish while the eigenmodes associated to the steady state energy peaks slowly establish: the subharmonic energy seems to be slowly drained from the propagating waves produced by the triadic resonance instability towards the eigenmode resonance frequencies.

Again in a way similar to the attractor and oscillating wall experiments, in the steady state of the Paris-Saclay experiments, the eigenmodes coexist with a weakly energetic background which is continuous in frequency and the level of this continuum grows more rapidly than that of the eigenmodes when the forcing amplitude is increased. A natural way forward to try to observe a weakly non-linear wave turbulence is therefore to explore larger forcing amplitudes. However, doing so results, again similarly to the Lyon and Grenoble experiments, in the onset of significant irreversible mixing of the fluid stratification and the emergence of strong non-linearities.

The only way to avoid this transition to a strongly non-linear regime while still promoting the emergence of a turbulent cascade is to increase the flow Reynolds number $Re = u\ell/\nu$ while keeping the Froude number $Fr = u/N\ell$ low. We have seen that one cannot "simply" increase the flow velocity u since this leads to leaving the weakly non-linear regime. Besides, for practical reasons, one cannot increase the buoyancy frequency N significantly above 1 rad/s with usual salt-water(-ethanol) mixtures (the fluid needs to be transparent to allow image velocimetry measurements). The only way forward is therefore to increase the wavelength ℓ at which energy is injected and consequently also the size (and especially the height) of the experiment. At the same time, it is also crucial to inhibit the finite size effects and the associated concentration of the wave energy in eigenmodes of the fluid domain.

Regarding this point, Lanchon et al. [70] identified a slight but crucial modification of their experimental setup which allows to efficiently inhibit the emergence of the wave eigenmodes. It consists in introducing slightly tilted panels in the fluid domain, one at the top and one at the bottom with a 4° tilt in the present case. The change in wavelength induced by these inclined planes when the waves are reflecting on them is small to moderate for most frequencies in the internal wave range ($\omega \le N$) but sufficient to prevent the formation of standing modes in the experimental cavity. With this modified setup, the first non-linear state of the turbulent flow composed of subharmonic propagating internal waves distributed over wide ranges of frequency, which is only transient in the experiments without tilted planes, seems to become the statistically steady state (at least to the time horizon explored experimentally, of 600 forcing periods). For the largest forcing amplitude studied by Lanchon et al. [70], a continuum of energy over typically one decade in the wave frequency range is observed together with a disappearance of the energy peaks associated to eigenmodes of the fluid domain. Moreover, the authors show that this energy continuum is mainly carried by internal gravity waves verifying the wave dispersion relation.

In this configuration, Lanchon et al. [70] report 1D spatial energy spectra, as a function of the horizontal and vertical wavenumbers, which both tend to show a power law with an exponent -3 at the largest explored Reynolds number. Despite the fact wave eigenmodes have been suppressed here, the spatial energy spectra are reproducing the recent observations of Le Reun et al. [65], Davis et al. [66] and Rodda et al. [69]. Here too, the power-law behaviors remain questionable since they are observed over no more than half-a-decade of wavenumbers. Also in line with the previously discussed experiments, the scaling laws with an exponent -3 emerge in the experiments of Lanchon et al. [70] while mixing of the stratification starts to become significant which might reveal a system of internal waves becoming strongly non-linear.

5. Conclusion

In this article, we reviewed recent experimental works that aimed at observing a turbulence of internal gravity waves in a stratified fluid in the weakly non-linear regime. The common feature of these works is that they adopted the same strategy of injecting energy in weakly non-linear waves before increasing the forcing amplitude in order to trigger a transition to a wave turbulence regime.

The motivation to achieve this regime in the laboratory is twofold. On the one hand, it has long been proposed that the dynamics of small oceanic scales is driven by a regime of weakly non-linear internal wave turbulence, without however a definitive confirmation so far. Besides, this dynamics is not resolved in global oceanic models and better understanding the weak internal wave turbulence is a promising lever to improve the parameterization of small oceanic scales in climate models. On the other hand, the identification of valid solutions to the theory of weakly non-linear internal gravity wave turbulence is still not completed and providing experimental data in this regime is of great interest to guide future theoretical developments.

Adopting a synthetic (and probably oversimplifying) point of view, we can say that most of the works described in this review led to comparable scenarios [61–64, 66, 67, 70]. Above a threshold in the forcing amplitude, the waves directly forced by the wave generators at a specific frequency ω_0 (in the range between $0.6\,N$ and the buoyancy frequency N) transfer energy to a couple of subharmonic wave modes (at frequencies $\omega_1 < \omega_0$ and $\omega_2 < \omega_0$) through the mechanism of the triadic resonance instability. Secondary triadic resonant interactions between these wave modes and/or the forced waves then give birth to additional wave modes at other frequencies and wavenumbers. As time goes on and/or as the forcing amplitude is increased, one observes the progressive concentration of the energy of these modes on a set of discrete frequencies in triadic resonance between themselves and/or with the forced wave mode. Many of these discrete modes are actually shown to be internal wave eigenmodes of the experimental water tank. The discretization of the energy on specific frequencies and wavenumbers therefore results from finite size effects and a strong energy attraction toward the tank eigenmodes. In this regime, the flow produced in the experiments reviewed in this article is in some way close to a weakly non-linear wave turbulence. Nevertheless, the energy discretization critically prevents the flow from approaching the regime described by the wave turbulence theory, in which a turbulent cascade forms an energy continuum in the frequency and wavenumber spaces.

We should also highlight the experiments of Rodda et al. [68] in which the frequency ω_0 at which internal waves are forced is significantly lower (in the range between $0.16\,N$ and $0.38\,N$). In this case, the scenario is different but still leading to a discrete energy spectrum in frequency. Non-linearities indeed emerge through the growth of wave modes at harmonic frequencies of the forcing frequency ($2\omega_0, 3\omega_0, \dots$).

In all these experiments, continuing to increase the forcing amplitude leads to the gradual emergence of an energy continuum in the wave frequency domain that tends to ultimately take over the wave eigenmodes (or the harmonic modes in the case of reference [68]). In this regime, finite size effects seem to progressively lose their influence. However, the flow seems in parallel to transition to a strongly non-linear wave turbulence regime of the "critical balance/saturated wave" type. Indeed, different experimental analyses report that the flow is still composed of internal waves verifying the dispersion relation whereas strongly non-linear features (including mixing of the density, wave breaking leading to vortices) seem to emerge at the same time. Also, many of these studies [59, 60, 66, 69, 70] report 1D spatial energy spectra tending to follow power laws with an exponent close to -3 as a function of the vertical and/or the horizontal wavenumber. Such scaling is typical of "saturated wave" turbulence for which a "critical balance" between the linear and non-linear timescales of the form $N \sim uk$ dimensionally leads to spatial energy spectra in $N^2 k^{-3}$ where k is the considered wavenumber and u the typical velocity at k [82]. On the contrary, several power laws are reported for the temporal energy spectrum, with an exponent -3 in reference [59] and an exponent -2 in references [65, 69]. But, both exponents are actually not incompatible with a "critical balance" regime. Indeed, a power law in ω^{-3} for the temporal energy spectrum can result from the conversion of a k^{-3} spatial energy spectrum when a turbulent sweeping induced by energetic large scale modes is dominant [59], whereas a temporal energy spectrum in $\epsilon\omega^{-2}$ might be expected when a "critical balance" of the type $\omega \sim uk$ is at play.

To conclude, we can say that two features should be modified in the experiments in order to access to a genuine weakly non-linear wave turbulence regime in the laboratory. First, one should prevent the concentration of the energy in wave eigenmodes of the fluid domain. In this respect, the work of Lanchon et al. [70] proposed a practical solution consisting in introducing slightly tilted panels at the top and/or at the bottom of the stratified fluid. By slightly modifying the wavelength of the reflected waves, the panels prevented the emergence of standing waves in the flow of Lanchon et al. which was shown to be composed of a continuum of propagating internal waves distributed over nearly one decade of frequencies. The second change that should be implemented is "simply" a significant increase in the wavelength ℓ at which the energy is injected together with an increase in the water tank size. The objective is to access to large Reynolds numbers $Re = u\ell/\nu$, to observe turbulent spectra with developed power laws, while remaining in the weakly non-linear regime which implies to reduce the flow Froude number $Fr = u/N\ell$. In the experiments (and the numerical simulations) discussed in this review, the amplitude of the non-linearities, which are measured by the ratio Fr/ω^* of the Froude number Fr to the non-dimensional wave angular frequency ω^*, is typically of the order of 0.1 for the experiments at the largest Reynolds numbers. This value seems not to be small enough to avoid the transition of the flow to a strongly non-linear regime. It therefore appears crucial to increase the wavelength of the forced waves in order to reduce significantly the non-linearity parameter Fr/ω^*. Regarding this point, we can recall the especially large size of the experiments on the Coriolis platform in Grenoble (6 m in the horizontal direction)[67–69]. Nevertheless, it is also crucial to have a large vertical size of the experimental water tank in order to be able to access vertical scales as large as the horizontal ones. This is especially important if one does not want to focus only on low frequency waves. In conclusion, future experiments aiming at weak internal wave turbulence should be much larger and especially much taller if we want them to fully succeed. This is a real challenge, especially regarding the fabrication of stratified fluids over large heights.

An alternative strategy to access to an internal wave turbulence in the weakly non-linear regime experimentally could be to work in a cryostat filled with liquid helium warming up at liquid-vapor equilibrium, in which system a stable gradient of density can establish. The low viscosity of liquid helium coupled to its important density variations as it approaches its critical point, can lead one to expect higher Reynolds numbers and lower Froude numbers than in most salt water experiments. However, in addition to the intrinsic difficulty of designing a cryogenic experiment, the challenge will be here to perform spatio-temporally resolved measurements of the velocity fields [92] as well as to assess that the Boussinesq approximation is valid.

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