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Volume 26 (2025), p. 7-23

Online since: 8 January 2025

https://doi.org/10.5802/crphys.218



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www.centre-mersenne.org — e-ISSN : 1878-1535

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*Comptes Rendus. Physique*
**2025**, Vol. 26, p. 7-23
https://doi.org/10.5802/crphys.218

Research article / *Article de recherche*

# Geometrical optics methods for moving anisotropic media: a tool for magnetized plasmas

# *Méthodes d'optique géométrique pour les milieux anisotropes en mouvement : un outil pour les plasmas magnétisés*

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**Abstract.** The propagation of a wave in a medium is generally affected when the medium is moving with respect to the observer. Because plasma equilibria often involve plasma flows, for instance in astrophysics or in magnetic confinement nuclear fusion devices, understanding the effect of motion on plasma waves is important. Meanwhile, the presence of a background magnetic field in a plasma makes it anisotropic. To address this problem, we derive here ray tracing equations for the trajectory of rays propagating in a moving anisotropic medium. The proposed approach is to use an effective dispersion relation for the moving medium as seen from the laboratory, obtained by performing a Lorentz transformation of the dispersion relation known for the medium at rest. This formalism is illustrated by considering the low frequency Alfvén waves and the standard ordinary and extraordinary modes in a magnetized plasma at rest.

**Résumé.** La propagation d'une onde dans un milieu est en général modifiée lorsque celui-ci est en mouvement. Les configurations d'équilibre d'un plasma reposant souvent sur un champ de vitesse, comme par exemple en astrophysique ou pour la fusion par confinement magnétique, une compréhension des effets du mouvement sur les ondes plasmas est particulièrement souhaitable. On s'intéresse ici à développer une méthode de lancer de rayon pour étudier la trajectoire des rayons se propageant dans un milieu anisotrope en mouvement dans l'approximation de l'optique géométrique. Une relation de dispersion effective pour le milieu en mouvement vu du laboratoire est identifiée en effectuant une transformation de Lorentz de la relation de dispersion du milieu au repos. Cette relation de dispersion est alors utilisée dans des équations de lancer de rayon, permettant ainsi de modéliser l'effet du mouvement sur la trajectoire des différents modes supportés par le milieu. Le potentiel de cette méthode est pour finir illustré en considérant le cas des ondes d'Alfvén basse fréquence et celui des modes ordinaire et extraordinaire classiques d'un plasma magnétisé.

**Keywords.** geometrical optics, ray tracing, light-dragging, plasma flow, moving dieletric.

**Mots-clés.** optique géométrique, lancer de rayon, entraînement de la lumière, écoulement plasma, diélectrique en mouvement.

**Funding.** This work is supported by the Agence Nationale de la Recherche (ANR) through the WaRP project (ANR-21-CE30-0002). This work has been carried out within the framework of the EUROfusion Consortium, via the Euratom Research and Training Programme (Grant Agreement No 101052200 – EUROfusion). Views

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ISSN (electronic): 1878-1535 https://comptes-rendus.academie-sciences.fr/physique/

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*Manuscript received 19 July 2024, revised 1 October 2024, accepted 16 October 2024.*

## 1. Context and motivations

Waves are used extensively in plasmas. Important applications of plasma waves include plasma control, for instance for plasma heating and current drive in magnetic confinement fusion devices [1], as well as plasma diagnostics, such as estimating interstellar magnetic fields via Faraday rotation [2, 3]. Waves may also offer new means to drive plasma rotation [4–8]. The design of these control systems and the interpretation of these diagnostics rely directly on plasma waves theory to model propagation in these varied environments. Since plasmas are anisotropic when immersed in a background magnetic field, these propagation models classically build on the theory of waves in anisotropic dispersive media, in the presence of possible plasma non-uniformities (density, magnetic field, etc.).

Other than for rare exceptions, these models and the theory of waves in plasmas however neglect the effect of a velocity field. Indeed, even though it has long been established that motion can have an effect on wave propagation [9–11], and that rotation phenomena are encountered and play a key role in a wide range of environments from laboratory plasmas to astrophysics [12] to magnetic confinement fusion [13], wave propagation properties are for the most part determined assuming a plasma at rest. Examining this apparent shortcoming, it has been confirmed in recent years that motion and in particular rotation can lead to a number of peculiar wave manifestations in plasmas [14], including effects on the wave transverse structure [15, 16] and polarization [17, 18], and modifications to wave-particle resonance conditions [19]. Although promising, these analytical studies are limited to simple configurations (e. g. geometry and plasma response). The aim of the work presented here is to present the first elements of an eikonal formalism for electromagnetic waves in a moving anisotropic medium, which could eventually allow simulating wave propagation in moving plasmas in more complex configurations.

This manuscript is organized as follows. First, in Section 2, we recall the basic elements of geometrical optics for static media, pointing how the ray tracing equations (RTEs) for the trajectory can be derived from the dispersion function obtained from a zeroth-order eikonal development, and positioning our work in the broader context of ray tracing studies. Then, in Section 3, we show how a moving medium can be modelled through a medium that is at rest in the lab-frame but bestowed with additional motion-dependent properties, and how the dispersion function for this effective medium can be obtained using covariance properties. Putting these pieces together, Section 4 derives the RTEs for a moving medium, showing how classical results are recovered in the limit of an isotropic medium. Finally, Section 5 applies these findings to the case of a magnetized plasma, demonstrating how the proposed method generalizes previously established results. Lastly, some concluding remarks are given in Section 6.

## 2. Basics of geometrical optics

As an introduction to the discussion in the next sections of the generalization of ray tracing equations to moving media, we first recall here some of the key elements of geometrical optics theory. This discussion follows primarily Tracy's textbook [20], and interested readers are referred to Refs. [20–22] for an in-depth presentation of these concepts.

## 2.1. *Problem statement*

The physics problem of interest here is the propagation of waves in a linear medium. The multicomponent wave field is denoted $\boldsymbol{\Psi}$ and can be for example the electric field. Let us assume that this problem is entirely described by a linear wave equation[1] [20]

$$\underline{\widetilde{\boldsymbol{D}}}\left(\boldsymbol{x}, i\nabla, -i\partial_t\right)\boldsymbol{\Psi} = \mathbf{0} \tag{1}$$

with $\underline{\widetilde{\boldsymbol{D}}}$ a matrix of differential operators that can depend on space. The partial differential equation (PDE) Eq. (1) describes how waves propagate in a given medium.

To make it less abstract, we consider as an illustration an inhomogeneous isotropic non-dispersive medium with space-dependent refractive index $n(\boldsymbol{x})$. The wave equation for the electric field $\boldsymbol{E}$ in this medium is then

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{n^2}{c^2}\partial_t^2\boldsymbol{E} = 0, \tag{2}$$

so that in this case

$$\underline{\widetilde{\boldsymbol{D}}} = \underline{\boldsymbol{\nabla}}\,\underline{\boldsymbol{\nabla}} + \frac{n^2}{c^2}\partial_t^2\underline{\mathbf{1}} \tag{3}$$

with $\underline{\boldsymbol{\nabla}}$ the skew-symmetric matrix associated to the curl operator.

## 2.2. *Geometrical optics and ray tracing equations*

Geometrical optics (GO) is a method to obtain approximate solutions to the wave equation. The central characteristic of GO is the assumption of a scale separation between the characteristic variation length $L$ of the properties of the medium and the wavelength $\lambda$ of the wave[2] [20]. Specifically, the phase is supposed to evolve faster than the envelope of the wave, whose variations are of the same order as those of the properties of the medium. A small parameter $\epsilon = \lambda/L \ll 1$ [23] is then introduced to reflect this ordering and to expand the wave equation. The benefit of this approximation is that the wave equation Eq. (1), which again is a PDE system and is hence in general difficult to solve, can be reduced to a system of ordinary differential equations (ODEs). This system, which is easier to solve, is called the ray tracing equations (RTEs) [20]. On the other hand, a consequence of this simplification is that wave optics effects such as diffraction are no longer accounted for.

Under these assumptions, the wave equation to zeroth order in $\epsilon$ can be written as

$$\underline{\boldsymbol{D}}(\boldsymbol{x}, \boldsymbol{k}, \omega)\boldsymbol{\Psi} = \mathbf{0} \tag{4}$$

where we have defined the zeroth-order wavevector $\boldsymbol{k}$ and the angular frequency $\omega$ as derivatives of the phase $\Phi_0$ of the wave

$$\boldsymbol{k} \doteq -\boldsymbol{\nabla}\Phi_0 \tag{5a}$$

$$\omega \doteq \partial_t\Phi_0, \tag{5b}$$

and where $\underline{\boldsymbol{D}}$ is a matrix called the dispersion matrix [21]. This matrix describes the local behaviour of the wave. Indeed, since it is to zeroth order in $\epsilon$, it does not include any derivative

[1] Rigorously, the general linear wave equation is an integrodifferential equation but can be reduced to the pseudodifferential form (1) using the Weyl symbol calculus. Moreover, for the sake of simplicity, we assume here that it does not depend on time $t$ although the method would be the same. Interested readers is referred to Ref. [20] for a fuller discussion of these two points.

[2] If the medium were not time-stationary, i. e. if the wave equation depended on time, the method would also require for the characteristic time of the variations in time of the medium to be large compared to the period of the wave.

of the medium properties. The eigenvalues of this matrix, which we write $\mathscr{D}_m$, are the zeroth-order dispersion functions, and the corresponding unit eigenvectors $\widehat{\boldsymbol{e}}_m$ are the unit polarization vectors of the modes [20]. The determinant of $\underline{\boldsymbol{D}}$ then writes

$$\mathscr{D} \doteq \det(\underline{\boldsymbol{D}}) = \prod_m \mathscr{D}_m. \tag{6}$$

Eq. (4) has non-trivial solutions if and only if the full dispersion relation $\mathscr{D}(\boldsymbol{x}, \boldsymbol{k}, \omega) = 0$ holds. This requires that $\mathscr{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega) = 0$ is verified for at least one mode $m$, which is then known as the dispersion relation for the mode $m$. If it is satisfied the mode $m$ propagates with a non-zero amplitude. The dispersion relation for a given mode $m$ can be viewed as the Hamilton–Jacobi equation of a particle with position $\boldsymbol{x}$ and momentum $\boldsymbol{k}$, and $\mathscr{D}_m$ the Hamiltonian [21]. Using this analogy, one writes the equations for the trajectory in phase space in the same way as Hamilton's equations [20]

$$\frac{dt}{ds} = \partial_\omega \mathscr{D}_m, \qquad \frac{d\boldsymbol{x}}{ds} = -\boldsymbol{\nabla}_{\boldsymbol{k}} \mathscr{D}_m, \tag{7a}$$

$$\frac{d\omega}{ds} = -\partial_t \mathscr{D}_m, \qquad \frac{d\boldsymbol{k}}{ds} = \boldsymbol{\nabla}_{\boldsymbol{x}} \mathscr{D}_m. \tag{7b}$$

This set of equations are the ray tracing equations (RTEs). Their solutions are the trajectories of the rays in phase space $(t(s), \boldsymbol{x}(s), \omega(s), \boldsymbol{k}(s))$ for the mode $m$ [20]. Note that under the assumption that $\mathscr{D}_m$ does not depend on time, i. e. for a stationary medium, $d\omega/ds = 0$ so the frequency $\omega$ is constant along the ray. These equations are similar to those of the dynamics of a point-particle [24]. The ray in physical space is tangent to the group velocity $\boldsymbol{v}_g \doteq d\boldsymbol{x}/dt$. This can be shown from Eq. (7) through a reparametrization of the ray using the time $t$ instead of $s$.

## 2.3. *Motivation for this work*

The RTEs (7) have long been used to model plasma waves, notably radio-frequency waves in magnetic confinement fusion devices [25] and in space physics [26]. For fusion applications ray tracing is particularly suited and can offer quantitative estimates for high frequency radio-frequency waves such as lower hybrid waves [27, 28] since the wavelength of these waves typically verifies $\lambda \ll L$. It has nevertheless also proved to be a useful tool beyond these regimes, for instance to expose the physics of the lower frequency ion cyclotron waves [29]. These capabilities have led to the development of a number of publicly available codes that are largely used by the fusion community (see, e. g., GENRAY [30, 31], TORAY [32, 33], C3PO [34] or BORAY [35]), with similar efforts by the space physics community [36, 37]. As part of this development, dispersion functions of increasing complexity have been derived. This includes models for cold and hot magnetized plasmas, Maxwellian or non-Maxwellian distributions, relativistic corrections, as well as for a drift of a given species parallel to the magnetic field, which have consequently been implemented in these codes. Yet, the effect of an unspecified global (i. e. all species) motion of the plasma with respect to the observer has to our knowledge remained largely unexplored. One important exception is the work by Walker [38, 39], though in this case as we will show motion is only captured through the Doppler effect. In this work we extend these contributions on the effect of motion by generalizing how the motion modifies the dispersion relation as seen from the laboratory frame, accounting now both for Doppler and aberration. Since, as exposed in Eqs. (7), computing the trajectory of a given mode $m$ only requires to have the dispersion function $\mathscr{D}_m$ describing this mode, our objective in the next section will be to obtain these functions for modes propagating in moving anisotropic media. Our work and Walker's have in common that it provides a systematic means of dealing with motion, as opposed to deriving from scratch lab-frame dispersion functions for a moving medium. Note that we will work here for simplicity with simple rest-frame dispersion functions, but a key advantage of the method proposed here is precisely that it generalizes straightforwardly to more complex forms.

## 3. Dispersion function for a moving medium

In this section we proceed with the derivation of the dispersion function for electromagnetic waves propagating in a moving medium, starting with the simpler case of a medium in uniform linear motion, and then generalizing to arbitrary motions. As we will show the method presented here generalizes in essence Walker's contribution [38] by taking into account the aberration effect and allowing relativistic velocities.

### 3.1. *Electromagnetism in uniformly moving media*

Let us start by showing how to relate the properties of wave propagation in a medium in uniform linear motion to a laboratory frame dispersion function describing the local behaviour of the wave.

#### 3.1.1. *Lorentz transformations*

We begin by classically defining two reference frames : the frame $\Sigma$ attached to the laboratory in which the observer is at rest, called the lab-frame, and the frame $\Sigma'$ attached to the uniformly moving medium, called the rest-frame, in which the medium is at rest [40, 41]. All quantities expressed in the rest-frame $\Sigma'$ will be indicated by a prime. Let us also write $\boldsymbol{v}$ the constant velocity of the medium, and hence of the frame $\Sigma'$, relative to the observer at rest in the frame $\Sigma$.

If we assume the lab-frame $\Sigma$ to be inertial then $\Sigma'$ is also inertial and the Lorentz transformations[3] can be used to transform the four-wavevector $k^\mu = (\omega/c, \boldsymbol{k})$[4] from one frame to the other. The Lorentz transformation for this four-vector writes [42]

$$\omega' = \gamma\left(\omega - c\boldsymbol{\beta}\cdot\boldsymbol{k}\right) \tag{8a}$$

$$\boldsymbol{k}' = \left(\underline{\mathbf{1}} + (\gamma-1)\boldsymbol{\beta}\otimes\boldsymbol{\beta}/\beta^2\right)\boldsymbol{k} - \gamma\boldsymbol{\beta}\omega/c \tag{8b}$$

where the vector $\boldsymbol{\beta} = \boldsymbol{v}/c$ is the dimensionless velocity, $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor and $\otimes$ is the tensor product. The first equation of the Lorentz transformation for the four-wavevector, Eq. (8a), captures the relativistic Doppler effect [42, 43] which causes a shift in the frequency of the wave going from one frame to the other. The second equation of the Lorentz transformation for the four-wavevector, Eq. (8b), describes the relativistic aberration effect, that is the change in the direction of the wavevector going from one frame to the other. Both the Doppler effect and the aberration effect depend on the direction of the wavevector $\boldsymbol{k}$, and notably the angle it makes with the velocity $\boldsymbol{v}$, and on the wave frequency. Anticipating later discussions, we note here that Eq. (8a) is the relativistic generalization of the classical non-relativistic Doppler effect $\omega' = \omega - \boldsymbol{k}\cdot\boldsymbol{v}$ used by Walker [38], both forms yielding the same result to first order in $\beta = v/c$. We also note from Eq. (8b) that the aberration is only negligible if $v\omega/k \ll c^2$, or equivalently $\beta v_\Phi/c \ll 1$ with $v_\Phi$ the lab-frame phase velocity.

[3]To be exact, here we use the Lorentz transformations for a Lorentz boost.

[4]Using a common notation abuse, we write here the contravariant (resp. covariant) four-vectors by the notation of their contravariant (resp. covariant) components where the Greek indices go from 0 to 3.

3.1.2. *Effective medium*

Our goal is to describe the propagation of waves as observed by an observer at rest in the lab-frame. The approach adopted here, as depicted in Figure 1, is to consider the moving medium as an equivalent effective medium at rest in the lab-frame, but with additional properties due to its motion [44, 45]. The velocity is then seen as a vector property of the effective medium, in a way like the magnetic field.

If we denote $\mathscr{D}'_m$ the dispersion functions of the modes propagating in the medium when it is at rest, i. e. in the frame $\Sigma'$, then let $\mathscr{D}_m$ be the dispersion functions of these same propagating modes but seen from the lab-frame $\Sigma$ where the medium is moving. The $\mathscr{D}_m$ are therefore the dispersion functions describing the propagation of the modes in the effective medium [46]. From there, the effective medium can be treated as any classical medium at rest. The effective medium is, however, more complicated than the original medium since the velocity adds a preferred direction. The effective medium is then in general bianisotropic [42, 47], even if the original medium is isotropic, and spatially dispersive [44] even if the original medium is only time dispersive. Yet, it is worth noting that the effective anisotropy due to motion does not necessarily lead to an effective birefringence [42]. For example, in the case of a moving isotropic medium, the two modes seen from the lab-frame are degenerate, i. e. they have the same dispersion relation, although the effective medium is bianisotropic due to motion.

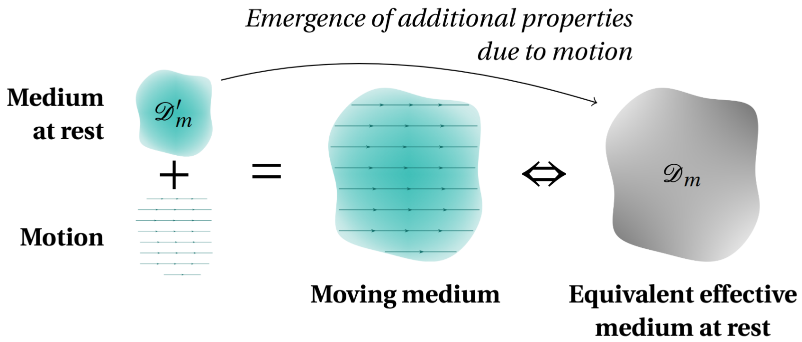



**Figure 1.** The moving medium is modelled in the lab-frame as an equivalent effective medium with additional properties that result from motion. Propagation in this effective medium is described by the dispersion function $\mathscr{D}_m$, which differs from the dispersion function $\mathscr{D}'_m$ known in the original medium at rest.

3.1.3. *Dispersion function in the lab-frame*

As already pointed by a number of authors [42, 44, 48], there are at least two ways to obtain the dispersion functions $\mathscr{D}_m$ of the effective medium that are needed to use the RTEs Eq. (7), and from there to compute the trajectories of the rays.

The first one, used notably in Refs. [17, 22, 41, 49–53], is what we call the *lab-frame approach* since the main part of the calculations are done with a lab-frame point of view. The first step is to derive the constitutive relations of the effective medium as seen in the lab-frame. For this one typically starts with the constitutive relations in the rest-frame before invoking the Lorentz transformations of the electromagnetic fields [54] to rewrite these relations in terms of the lab-frame electromagnetic fields. These relations are finally combined with Maxwell's equations written in the lab-frame to obtain a wave equation in the lab-frame. Proceeding in the framework of GO as explained in Section 2, the lab-frame dispersion matrix $\underline{\boldsymbol{D}}$ is then obtained from the wave equation to zeroth order in the small eikonal parameter $\epsilon$, and the dispersion functions $\mathscr{D}_m$ of the modes are the eigenvalues of this matrix. While conceptually straightforward, this approach has two major drawbacks. First, finding analytical expressions for the eigenvalues of the lab-frame dispersion matrix is not necessarily possible for complex media. Second, the calculations become very complicated when one wants to work beyond the first order in $\beta$.

The second approach, adopted in [38, 55–57] and referred to as the *transformation approach*, consists in transforming the rest-frame dispersion function $\mathscr{D}'_m$ to directly obtain the lab-frame dispersion function of the effective medium $\mathscr{D}_m$. One only needs to know the dispersion function in the rest-frame $\mathscr{D}'_m$, which is often the case, to immediately obtain the dispersion function of the effective medium $\mathscr{D}_m$ in the lab-frame. An advantage of this approach compared to the first one is therefore that it works for very general media [46]. This is notably true for dispersive anisotropic media like magnetized plasmas [55], as we will discuss in Section 5. Another advantage is that it can yield without additional effort the dispersion functions to all orders in $\beta$, hence allowing for relativistic velocities. For these reasons, and because we are primarily interested here in the dispersion function, we choose to work here with this transformation approach rather than the lab-frame approach.

Let us first expose how this approach works for homogeneous media. It at its heart uses the relativistic invariance property for the dispersion relation demonstrated by Censor, first for homogeneous anisotropic media [43] and then more generally for homogeneous bianisotropic media [46] (i.e. for very general media). Specifically, by expressing through the properties of the Fourier and Lorentz transforms the rest-frame dispersion matrix in the lab-frame, Censor demonstrated that the dispersion function is Lorentz covariant [42, 43, 46], in the sense that the dispersion function in one frame can be obtained from the one in the other frame simply by Lorentz transforming the four-wavevector $k^\mu = (\omega/c, \boldsymbol{k})$ using the Lorentz transformation Eqs. (8). Following Refs. [43, 46], we write this property as

$$\mathscr{D}_m(k^\mu) = \mathscr{D}'_m\left(k'^\mu\left(k^\mu\right)\right) \tag{9}$$

where $k'^\mu$ is written as a function of $k^\mu$ through the Lorentz transformation Eqs. (8). This amounts in fact to taking into account the relativistic Doppler effect and the aberration effect seen in the rest-frame due to the medium's motion. As such, this transformation of the rest-frame dispersion function is a direct extension of the method used by Walker [38], in that it now includes the aberration effect in addition to the Doppler effect already considered by Walker, and also captures relativistic corrections.

Although Eq. (9) was established, as introduced above, for homogeneous media, it can in fact be used for ray tracing in inhomogeneous media. The reason for that is that GO relies precisely on the assumption that the medium is locally homogeneous, since it varies slowly compared to the rapid phase of the wave. As such, and in the limit of GO, Censor's covariance property Eq. (9) can be applied at each point in space $\boldsymbol{x}$ (or at each spacetime location $x^\mu$ in case there is a time dependence) as if the medium was homogeneous, with its homogeneous properties set by the local properties at this point. This is actually what is done, albeit again considering only the non-relativistic Doppler effect, by Walker in his ray tracing model for moving inhomogenous media [38]. The generalization of Eq. (9) for inhomogeneous media is thus simply

$$\mathscr{D}_m\left(x^\mu, k^\mu\right) = \mathscr{D}'_m\left(x^\mu, k'^\mu\left(k^\mu\right)\right) \tag{10}$$

where the dependency on the position $x^\mu$ captures that the rest-frame dispersion function $\mathscr{D}'_m$ now depends on position, which carries over to $\mathscr{D}_m$.

### 3.2. *Dispersion function for arbitrarily moving media*

Having shown in the previous paragraph how to obtain lab-frame dispersion function for a medium in uniform linear motion, we would like to extend it to the case of a medium in arbitrary motion. As already underlined in Refs. [22, 38, 58–60], we argue here that this generalization in fact comes again naturally in the GO approximation, assuming weak enough spatial inhomogeneity in the velocity field.

To see this, let us consider a medium with an arbitrary time-stationary[5] velocity field $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})/c$. The idea explored here is to take advantage of the scale separation that is as shown in Section 2 characteristic of the GO framework. Following this direction, we now require for the characteristic variation length of the velocity field to be large compared to the wavelength [58, 61]. As a result $\boldsymbol{\beta}$ does not change significantly over a wavelength, and the wave thus sees locally a medium in uniform linear motion. In this limit, and assuming that "special relativity theory is valid locally and instantaneously" [61], the local behaviour of the wave is then found to be described by $\mathscr{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ [62] as obtained using the methods for a uniform linear motion introduced in Subsection 3.1, considering crucially the local velocity $\boldsymbol{\beta}(\boldsymbol{x})$ as a constant [58]. This amounts to taking into account the local relativistic Doppler and aberration effects. Following this route, we get $\mathscr{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ at each position $\boldsymbol{x}$. This lab-frame dispersion function $\mathscr{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ does not contain derivatives of the velocity field, as expected to zeroth order in the small eikonal parameter $\epsilon$.

Summing up our findings, we showed how the lab-frame dispersion function $\mathscr{D}_m$ describing waves in an inhomogeneous medium with an arbitrary motion can, in the limit of GO, be simply obtained by performing Lorentz transformations of the four-wavevector in the rest-frame dispersion function $\mathscr{D}'_m$. The so obtained relation between the two functions can be written using four-vector notation in the compact form

$$\mathscr{D}_m(x^\mu, k^\mu) = \mathscr{D}'_m\left(x^\mu, k'^\mu\left(k^\mu, x^\mu\right)\right) \tag{11}$$

where $k'^\mu(k^\mu, x^\mu)$ is to be understood as the Lorentz transformation of $k'^\mu$ at the spacetime location $x^\mu$ with the local and instantaneous velocity $\boldsymbol{\beta}(x^\mu)$. This result generalizes the method proposed by Walker [38], by allowing for relativistic velocities, but also and more crucially by taking into account the aberration effect.

For completeness, it should be noted here that Eq. (11) requires the properties of the medium to be known functions of the laboratory space and time coordinates. As an example for the inhomogeneous isotropic non-dispersive medium one must know the scalar field $n'(t, \boldsymbol{x})$. Doing so poses no problem for most cases, including non-relativistic velocities and configurations where the medium properties are naturally expressed from the lab-frame. On the other hand it could prove difficult in case one only knows those properties in the rest-frame, in certain configurations involving relativistic velocities and complex inhomogeneities. In these more exotic situations, spacetime deformation such as length contraction or time dilation could lead to non-trivial modifications in the rewriting of the medium properties in the lab-frame. For those rare cases one may resort to the use of transformations between general coordinates systems.

Note also finally that while, thanks to the GO framework used here, a non-uniform motion can be modelled via the effects known for a uniform motion, a non-uniform motion adds new difficulty in that determining the rest-frame dispersion function may be more intricate. This is because the constitutive relations in the rest-frame of an accelerated media differ from those in this same media at rest. This behaviour has long been known for rotations [63–65], and has recently been clarified in rotating plasmas [66].

[5] The stationary assumption is only here to simplify the discussion by having a wave equation independent in time, as indicated in the general introduction of GO at the beginning of Section 2.

### 3.3. *Validation and discussion for an inhomogeneous isotropic medium*

Let us now illustrate and validate this transformation approach by going back to our example of an inhomogeneous isotropic non-dispersive medium, which we now consider to be moving. The refractive index of this medium at rest is denoted $n'$. Because the medium is inhomogeneous it is a function of space $n' = n'(\boldsymbol{x})$. The rest-frame dispersion function of the modes propagating in such a medium is well-known, namely

$$\mathscr{D}'_{\text{iso}}\left(\boldsymbol{x},\boldsymbol{k}',\omega'\right) = n'(\boldsymbol{x})^2\frac{\omega'^2}{c^2} - k'^2. \tag{12}$$

Using the Lorentz transformations Eqs. (8) locally, we then get from covariance the dispersion function in the lab-frame

$$\mathscr{D}_{\text{iso}}\left(\boldsymbol{x},\boldsymbol{k},\omega\right) = \frac{\omega^2}{c^2} - k^2 + \left(n'(\boldsymbol{x})^2 - 1\right)\gamma(\boldsymbol{x})^2\left(\frac{\omega}{c} - \boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{k}\right)^2. \tag{13}$$

Analysing this result, we see that $\mathscr{D}_{\text{iso}}$ now depends on the direction of the wavector $\boldsymbol{k}$ through the term $\boldsymbol{\beta}\cdot\boldsymbol{k}$, whereas it was not the case for $\mathscr{D}'_{\text{iso}}$. Note also that for the vacuum ($n' = 1$) the dispersion functions in the two frames are the same, which is consistent with the fact that the vacuum keeps the same properties whatever the observer's reference frame. This comes from the fact that the quantity $\omega'^2/c^2 - k'^2 = \omega^2/c^2 - k^2$ is a Lorentz scalar, and is thus invariant under Lorentz transformations. Physically, this corresponds to the fact that the Doppler effect precisely cancels out the aberration effect. Importantly, this quantity is, however, not invariant under the transformation used by Walker [38] as it misses the aberration effect. For this reason Walker's model would fail to describe the invariance of rays in vacuum from different inertial observer's reference frames.

Comparing Eq. (13) with previous work, we find first that it reduces to the prediction for a homogeneous medium (i.e. for $n'(\boldsymbol{x}) = n' = \text{cst}$) in non-uniform motion obtained by Leonhardt and Piwnicki [58]. More generally, we verify that Eq. (13) is the flat spacetime limit of the more general dispersion relation

$$\mathscr{D}_{\text{Gordon}} = \bar{g}^{\mu\nu}k_\mu k_\nu \tag{14}$$

that is obtained (see, e. g. Refs. [58, 60, 67, 68]) from the Gordon optical metric [67]

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \left(n'^2 - 1\right)\beta^\mu\beta^\nu \tag{15}$$

where $g^{\mu\nu}$ is the metric of the physical space and $\beta^\mu = \gamma(1,\boldsymbol{\beta})$ is the dimensionless four-velocity of the medium. Specifically, one finds Eq. (13) when using the flat spacetime metric namely Minkowski's metric $\eta^{\mu\nu} = \text{diag}(+,-,-,-)$ [54, 69] as the background metric $g^{\mu\nu}$. We thus confirm here that the transformation approach proposed in this study is consistent with results derived by considering, as famously proposed by Gordon [67], the effect of motion on rays in a moving isotropic medium as equivalent to that of an empty curved spacetime with metric $\bar{g}^{\mu\nu}$. An advantage of the transformation approach, as we will illustrate in Section 5, is that it applies to moving anisotropic media. Note finally that the observed consistency with $\mathscr{D}_{\text{Gordon}}$ would not be possible if using the transformation adopted by Walker [38], as it misses the aberration effect.

Summing our findings, we have identified a way to obtain the dispersion function $\mathscr{D}_m$ necessary to use the RTEs in the lab-frame, under the requirement that the rest-frame dispersion function $\mathscr{D}'_m$ is known.

## 4. Ray tracing in arbitrarily moving media

Having identified in Section 3 a framework to derive the lab-frame dispersion functions $\mathscr{D}_m$ describing the modes in a medium in arbitrary motion, we can now use them in the RTEs (7) derived in Section 2 to compute the trajectory of the rays, and identify the effects of motion.

In the remaining of this study, the media considered are stationary hence $\omega$ is constant along the ray. Furthermore we will focus on the RTEs for the position $\boldsymbol{x}$ and the wavevector $\boldsymbol{k}$, leaving aside the RTE for the time coordinate $t$.

Here we illustrate and validate the effect of motion on rays by considering again the example of a moving inhomogeneous isotropic non-dispersive medium. Using the lab-frame dispersion function Eq. (13) and limiting ourselves here to first order in $\beta$ for the sake of simplicity, we find the RTEs

$$\frac{d\boldsymbol{x}}{ds} = 2\boldsymbol{k} + 2\left(n'^2 - 1\right)\frac{\omega}{c}\boldsymbol{\beta}, \tag{16a}$$

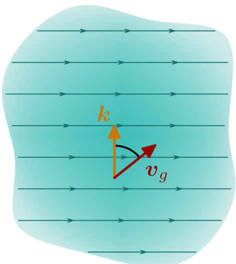
$$\frac{d\boldsymbol{k}}{ds} = 2\left(\boldsymbol{\nabla} n'\right) n' \frac{\omega}{c}\left(\frac{\omega}{c} - 2\boldsymbol{\beta}\cdot\boldsymbol{k}\right) + 2\left(n'^2 - 1\right)\frac{\omega}{c}\left[\left(\boldsymbol{\nabla}\times\boldsymbol{\beta}\right)\times\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{\nabla})\boldsymbol{\beta}\right]. \tag{16b}$$

We verify that these RTEs are consistent to first order in $\beta$ with the covariant RTEs derived by Bourgoin et al. [60]. Eqs. (16) are also consistent with Leonhardt and Piwnicki's results [58] in the limit of a homogeneous medium (constant $n'$), and with those derived by Rozanov [22, 70] for a uniform field velocity (constant $\boldsymbol{\beta}$). We should note here though that Rozanov's derivation [22, 70] also considered the effect of a non-uniform velocity field, obtaining the same term as the second term on the right hand side of Eq. (16a), but with an opposite sign. Because the sign we obtained in Eq. (16b) is consistent with both Bourgoin's [60] and Leonhardt and Piwnicki's [58], we surmise that this might be a sign error.

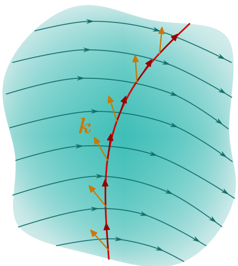
Examining Eqs. (16), we first note that the velocity appears in Eq. (16a) in the form of a term proportional to $\boldsymbol{\beta}$ on the right hand side, which supplements the term $2\boldsymbol{k}$ classically expected without motion. This term modifies the group velocity $\boldsymbol{v}_g \propto d\boldsymbol{x}/ds$ of the wave as seen in the lab-frame. Specifically, there is now as illustrated in Figure 2(a) a finite angle between the wavevector $\boldsymbol{k}$ and the group velocity $\boldsymbol{v}_g$, in contrast with what is found in this isotropic medium at rest where those two vectors are aligned. Considering the particular case $\boldsymbol{k} \perp \boldsymbol{\beta}$ for which from Eq. (13) $kc/\omega = n'$, one finds from Eq. (16a) that the lateral shift $\delta$ per unit length along $\boldsymbol{k}$ induced by the motion writes

$$\delta = \left(n' - \frac{1}{n'}\right)\frac{v}{c}, \tag{17}$$

which is precisely the lateral shift predicted by Player [71] and observed by Jones [11, 72] when neglecting dispersion.




**(a)** Illustration of the angle between the wavevector $\boldsymbol{k}$ and the group velocity $\boldsymbol{v}_g$ revealing the drag undergone by the wave due to the motion of the medium.




**(b)** Illustration of the bending of the ray due to the variation of the wavevector $\boldsymbol{k}$ caused by the non-uniformity of the velocity field $\boldsymbol{v}(\boldsymbol{x})$ of the medium.

**Figure 2.** Illustrations of some effects of the motion on the trajectory of the wave.

Besides the light dragging seen in Eq. (16a), we see that the motion also appears in the second equation Eq. (16b), though it now appears through the non-uniformity of the velocity field. This was to be expected in that the velocity field non-uniformity leads to inhomogeneous properties for the effective medium. Similarly to a refractive index non-uniformity, the velocity field non-uniformity now causes a bending of the rays, affecting the evolution of $\boldsymbol{k}$ along the ray. This second manifestation of motion is illustrated in Figure 2(b).

Taking a step back and summing up our work, we have verified that the RTEs for a moving medium derived here are consistent with known results previously obtained in the limit of isotropic dielectrics. Compared to these earlier contributions though, a strong advantage of the proposed approach, and principal motivation for its development, is that it can easily accommodate for dispersive and anisotropic media, such as plasmas as considered in the next section.

## 5. Ray tracing in moving plasmas

As an illustration of the method, the theory established in Sections 3 and 4 is now applied in this Section to a moving cold plasma. The emphasis is put on deriving the lab-frame dispersion functions $\mathscr{D}_m$, as needed to compute the RTEs. Note that the problem of finding the lab-frame dispersion relations for a plasma in uniform linear motion has already received attention, both using the lab-frame approach [41, 51, 52, 73] and the transformation approach [38, 55–57]. These earlier contributions did however focus primarily either on the high-frequency electronic response [41, 51, 55–57, 73] or conversely on the low frequency response [38], with only a very few studies of more complete plasma responses [52, 74]. Generalization to other plasma models, if possible, will require specific treatments. In contrast the approach presented in Section 3 allows one to seamlessly add motion effects to any model of plasma at rest, i.e. to any plasma model in which the mean velocities of the species are zero. The local velocity $\boldsymbol{v}(\boldsymbol{x})$ of the plasma is then the local mean velocity of the species in the lab-frame.

### 5.1. *Moving unmagnetized cold plasma*

As a first step we consider the simple case of an unmagnetized plasma. There are two different modes in an unmagnetized plasma at rest: the propagating electromagnetic mode and the oscillating electrostatic mode. The latter does not propagate in the rest-frame.

#### 5.1.1. *The propagating electromagnetic mode : the O mode*

The dispersion function of the propagating mode (ordinary or O mode) is [75]

$$\mathscr{D}'_O(\boldsymbol{k}',\omega') = \omega'^2/c^2 - k'^2 - \omega_p'^2/c^2 \tag{18}$$

where $\omega_p'^2 = \omega_{pe}'^2 + \omega_{pi}'^2$ with $\omega'_{ps} = [n'_s e^2/(m'_s\epsilon_0)]^{1/2}$ the plasma frequency of the species $s$ as measured in the rest-frame $\Sigma'$. Recalling that the quantity $\omega'^2/c^2 - k'^2$ is a Lorentz scalar, the lab-frame dispersion function writes straightforwardly

$$\mathscr{D}_O(\boldsymbol{k},\omega) = \omega^2/c^2 - k^2 - \omega_p'^2/c^2. \tag{19}$$

The dispersion function of the O mode is then Lorentz invariant : it has the same form whether it is expressed in $\Sigma'$ or $\Sigma$ [76]. This property can be understood as the canceling of the Doppler effect by the aberration, as already underlined for the vacuum. Importantly, in failing to account for the aberration effect, Walker's method [38] does not recover this fundamental result.

Analysing this result, the dispersion function from the lab-frame $\mathscr{D}_O$ has no dependence on the velocity $\boldsymbol{\beta}$. This is a notable feature since it means that the O mode is not affected by the motion of the plasma, as already noted by Mukherjee [77] and Ko and Chuang [76]. In other words, it propagates identically whether the plasma is moving or not. This result suggests that diagnostics using O mode will likely not be affected by motion, at least in the cold plasma limit and as long as one is focusing on trajectory[6]. Similarly, this results a priori prohibits using an O mode to probe the motion of a cold plasma.

#### 5.1.2. *The oscillating electrostatic mode: Langmuir waves*

The dispersion function of the oscillating mode, also known as Langmuir waves, is [75]

$$\mathscr{D}'_{\mathrm{osc}}(\omega') = \frac{1}{c^2}\left(\omega'^2 - \omega_p'^2\right). \tag{20}$$

Using the Lorentz transformations Eqs. (8) to write $\omega'$ as a function of $\omega$ and $\boldsymbol{k}$, we then obtain the lab-frame dispersion function [43]

$$\mathscr{D}_{\mathrm{osc}}(\boldsymbol{x},\boldsymbol{k},\omega) = \gamma^2\left(\frac{\omega}{c} - \boldsymbol{k}\cdot\boldsymbol{\beta}(\boldsymbol{x})\right)^2 - \frac{\omega_p'^2}{c^2}. \tag{21}$$

Unlike $\mathscr{D}'_{\mathrm{osc}}$, $\mathscr{D}_{\mathrm{osc}}$ contains both $\omega$ and $\boldsymbol{k}$. This implies that this mode is propagative. The oscillating mode, which does not propagate in the rest-frame, does propagate in the lab-frame [43]. This behaviour is a consequence of the loss of simultaneity from the rest-frame $\Sigma'$ to the lab-frame $\Sigma$. Specifically, the phase of electrostatic oscillations is the same at all points in space in $\Sigma'$, that is to say that there is simultaneity. This is however no longer the case in $\Sigma$ due to special relativity, hence the emergence of a propagating wave. This example illustrates the general fact that to study all the propagating modes in a moving medium, one has to consider also modes that are purely oscillatory or evanescent in the rest-frame, as those can become propagative in the lab-frame [43].

### 5.2. *Moving magnetized cold plasma*

Moving to magnetized plasmas, the method presented in Section 3 can be applied to obtain the dispersion functions in the lab-frame for the general case of modes propagating at an arbitrary angle with respect to the background magnetic field $\boldsymbol{B}_0$ in a moving magnetized cold plasma. This essentially boils down to substituting to $\omega'$ and $\boldsymbol{k}'$ in the classic Appleton–Hartree equation [79] – known to hold in the plasma rest-frame – their expressions in terms of the lab-frame variables $\omega$ and $\boldsymbol{k}$, as prescribed by Eq. (8). Although lengthy, this task poses no difficulty. Rather than dealing with these general formulas though, we will focus here on two particular cases : the low frequency Alfvén waves and the O and X modes in perpendicular propagation ($\boldsymbol{k}' \perp \boldsymbol{B}_0'$).

#### 5.2.1. *Low frequency Alfvén waves*

The dispersion function for low frequency Alfvén waves in an inhomogeneous magnetized plasma can be written [75]

$$\mathscr{D}'_{\mathrm{Alfvén}}\left(\boldsymbol{x},\boldsymbol{k}',\omega'\right) = \omega'^2/c^2 - \left(\boldsymbol{k}'\cdot\boldsymbol{v}'_A(\boldsymbol{x})/c\right)^2 \tag{22}$$

where

$$\boldsymbol{v}'_A(\boldsymbol{x}) = \frac{\boldsymbol{B}'_0(\boldsymbol{x})}{\sqrt{\mu_0 n'_i(\boldsymbol{x}) m'_i(\boldsymbol{x})}} \tag{23}$$

is the Alfvén velocity expressed in the instantaneous rest-frame at each point of space $\boldsymbol{x}$. Carrying out the Lorentz transformations Eqs. (8) in Eq. (22), we then obtain the lab-frame dispersion

[6]There could still be first-order corrections to the RTEs, as shown for an inhomogeneous plasma in Ref. [78], though in this case the corrections would stem from motion.

function describing these waves in a plasma moving with the velocity field $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})/c$ in the laboratory frame

$$\mathscr{D}_{\text{Alfvén}}(\boldsymbol{x},\boldsymbol{k},\omega) = \gamma^2\left(\frac{\omega}{c} - \boldsymbol{k}\cdot\boldsymbol{\beta}\right)^2 - \frac{1}{c^2}\left(\boldsymbol{k}\cdot\boldsymbol{v}'_A + (\gamma-1)/\beta^2\left(\boldsymbol{\beta}\cdot\boldsymbol{k}\right)\left(\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right) - \gamma\frac{\omega}{c}\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right)^2. \tag{24}$$

Here $\boldsymbol{v}'_A$, $\boldsymbol{\beta}$ and $\gamma$ are all functions of the space coordinates $\boldsymbol{x}$. Neglecting terms that are second order or higher in $\beta$, Eq. (24) reduces to

$$\mathscr{D}_{\text{Alfvén}}(\boldsymbol{x},\boldsymbol{k},\omega) = \frac{\omega^2}{c^2} - \left(\boldsymbol{k}\cdot\boldsymbol{v}'_A/c\right)^2 - 2\frac{\omega}{c}\boldsymbol{k}\cdot\boldsymbol{\beta} + 2\frac{\omega}{c^3}\left(\boldsymbol{k}\cdot\boldsymbol{v}'_A\right)\left(\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right). \tag{25}$$

Plugging this result into Eqs. (7), the RTEs for the low frequency Alfvén waves in an arbitrarily moving inhomogeneous plasma then write

$$\frac{d\boldsymbol{x}}{ds} = \frac{2}{c^2}\left(\boldsymbol{k}\cdot\boldsymbol{v}'_A\right)\boldsymbol{v}'_A + 2\frac{\omega}{c}\boldsymbol{\beta} - 2\frac{\omega}{c^3}\left(\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right)\boldsymbol{v}'_A, \tag{26a}$$

$$\begin{aligned}\frac{d\boldsymbol{k}}{ds} = &-\frac{2}{c^2}\left(\boldsymbol{k}\cdot\boldsymbol{v}'_A\right)\left(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{k}\right) - 2\frac{\omega}{c}(\nabla\boldsymbol{\beta}\cdot\boldsymbol{k}) \\ &+ 2\frac{\omega}{c^3}\left[\left(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{k}\right)\left(\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right) + \left(\boldsymbol{k}\cdot\boldsymbol{v}'_A\right)\left(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{\beta}\right) + \left(\boldsymbol{k}\cdot\boldsymbol{v}'_A\right)\left(\nabla\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right)\right]\end{aligned} \tag{26b}$$

where $\nabla\boldsymbol{w}$ is the transposed Jacobian matrix of the vector field $\boldsymbol{w}$ with coefficients $(\nabla\boldsymbol{w})_{ij} = (\nabla\otimes\boldsymbol{w})_{ij} = \partial_i w_j$. Equivalently, using instead the ray parameter $t$, one gets

$$\frac{d\boldsymbol{x}}{dt} = \pm\boldsymbol{v}'_A + \boldsymbol{v} \mp \frac{c}{\omega}(\boldsymbol{k}\cdot\boldsymbol{\beta})\boldsymbol{v}'_A, \tag{27a}$$

$$\frac{d\boldsymbol{k}}{dt} = \mp\left(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{k}\right) - (\nabla\boldsymbol{v}\cdot\boldsymbol{k}) \pm \frac{c}{\omega}\left(\boldsymbol{k}\cdot\boldsymbol{\beta}\right)(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{k}) \pm \frac{\omega}{c}\left[\left(\nabla\boldsymbol{v}'_A\cdot\boldsymbol{\beta}\right) + \left(\nabla\boldsymbol{\beta}\cdot\boldsymbol{v}'_A\right)\right]. \tag{27b}$$

We recognize in Eqs. (27) the RTEs found by Walker [38], but with additional terms. Specifically, the last term in Eq. (27a) and the last two terms in Eq. (27b) are new compared to Walker's result [38]. These additional terms stem from the aberration effect, which was again neglected in Walker's transformation of the dispersion function. As already noted in the discussion of Eq. (8b) these terms are negligible if $v\omega/k \ll c^2$. This assumption is generally largely satisfied for Alfvén waves, supporting Walker's analysis [38], but may not hold for other plasma waves. We further note here that Walker's results [38] appear to be consistent with those previously obtained by Bazer and Hurley [49], though interestingly they were then obtained using the lab-frame approach. As such Eqs. (27) also reduce to Bazer and Hurley's results [49] in the same limit.

#### 5.2.2. *Perpendicular propagation : O and X modes*

We focus now on the case where the wavevector is perpendicular to the magnetic field ($\boldsymbol{k}' \perp \boldsymbol{B}'_0$), considering both ionic and electronic responses. In this configuration, that is for perpendicular propagation, the modes propagating in the rest-frame are the well known O and X (for extraordinary) modes. The O mode is the same as in an unmagnetized plasma. From the discussion above we already have its dispersion function from the lab-frame, that is Eq. (19).

Focusing on the X mode, the dispersion function in the rest-frame is [75]

$$\mathscr{D}'_X\left(\boldsymbol{k}',\omega'\right) = \frac{\omega'^2}{c^2} - k'^2 + \frac{\omega'^2}{c^2}\left[\chi'_\perp(\omega') - \frac{\chi'^2_\times(\omega')}{1+\chi'_\perp(\omega')}\right] \tag{28}$$

with

$$\chi'_\perp(\omega') = \sum_s \frac{\omega'^2_{ps}}{\Omega'^2_{cs} - \omega'^2} \tag{29a}$$

$$\chi'_\times(\omega') = \sum_s \epsilon_s \frac{\Omega'_{cs}}{\omega'}\frac{\omega'^2_{ps}}{\omega'^2 - \Omega'^2_{cs}} \tag{29b}$$

the classical perpendicular and cross-field components of the susceptibility tensor $\underline{\boldsymbol{\chi}}'$ of a magnetized cold plasma. Here $\Omega'_{cs} = |q_s| B'_0 / m'_s$ is the unsigned cyclotron frequency of species $s$ written in the rest-frame, and $\epsilon_s = q_s / |q_s|$. Using the Lorentz transformations Eqs. (8) in Eq. (28) directly yields the lab-frame dispersion function

$$\mathscr{D}_X(\boldsymbol{x}, \boldsymbol{k}, \omega) = \frac{\omega^2}{c^2} - k^2 + \gamma^2 \left(\frac{\omega}{c} - \boldsymbol{k}\cdot\boldsymbol{\beta}(\boldsymbol{x})\right)^2 \left[\chi'_\perp\left(\gamma(\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta}(\boldsymbol{x}))\right) - \frac{\chi'^2_\times\left(\gamma(\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta}(\boldsymbol{x}))\right)}{1+\chi'_\perp\left(\gamma(\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta}(\boldsymbol{x}))\right)}\right]. \quad (30)$$

We immediately see that, in contrast with the O mode, $\mathscr{D}_X$ depends on $\boldsymbol{\beta}$, which implies that the propagation of the X mode is modified by the plasma motion. We also verify that Eq. (30) is consistent, when considering only the electronic response, with the result obtained by Unz [55] in the particular configuration $\boldsymbol{k}' \perp \boldsymbol{B}'_0$ considered here and neglecting collisions. In the limit $\boldsymbol{v} \perp \boldsymbol{B}'_0$ it also agrees with the result derived by Mukherjee [77].

The dispersion function Eq. (30) can then be plugged in the RTEs Eq. (7) to compute the ray trajectory of an X mode in an arbitrarily moving magnetized plasma. Following this route and examining more specifically the lab-frame group velocity, it was recently shown that the X mode is dragged in the direction of the plasma motion, and that this drag may be significant near resonances and cutoffs, and also for wave frequencies below the lower hybrid frequency [80].

Note finally that Eq. (30) is used here for illustration purposes. Indeed, as it requires $\boldsymbol{k}' \perp \boldsymbol{B}'_0$ whereas $\boldsymbol{k}'$ may evolve in space for complex velocity fields or complex geometries, it does not in general hold all along a trajectory. For such applications, one should use instead the transformation of the full Appleton–Hartree equation as mentioned at the beginning of Subsection 5.2.

## 6. Conclusion

To conclude, after recalling some basic elements of classical geometrical optics, we showed how this method can be used to study the propagation of rays in arbitrarily moving inhomogeneous bianisotropic media, in the limit that the velocity field and the medium properties vary slowly in space and time compared to the wavelength and the period of the wave considered. Under this assumption, the lab-frame dispersion function that is required in the RTEs is obtained by Lorentz transforming the dispersion function assumed to be known in the rest-frame. This method is shown to be an extension of Walker's method [38], in that it takes into account the aberration effect in addition to the Doppler effect, and also captures relativistic corrections. These corrections are demonstrated to be important when $\beta v_\Phi / c$ is comparable or larger than 1, where $\boldsymbol{v}_\Phi$ is the phase velocity. The RTEs can then be used to compute the trajectory of the wave similarly to a point-particle.

The RTEs give in particular access to the group velocity in the lab-frame. We recovered through this method that, as a result of motion, an additional term depending on $\boldsymbol{\beta}$ appears in the lab-frame group velocity. This term captures the drag of the wave induced by the moving medium. Applying specifically these results to a moving magnetized cold plasma, we further derived the RTEs for low frequency Alfvén waves, and showed that these results are consistent with Walker's result [38], but with additional terms stemming from the aberration effect. Considering more generally perpendicular propagation, we showed that the O mode is unaffected by the motion, while the X mode undergoes a drag in the direction of the velocity.

Looking ahead, the work presented here focuses only on the ray trajectory. However it seems desirable for a number of applications to further have information on the possible effect of the motion on the polarization [81, 82] and the amplitude along the rays. A full, gauge-independent, treatment of this problem requires a first-order eikonal expansion [21, 24, 83, 84]. Although the transformation approach specifically described and used to obtain RTEs here, in focusing only on the dispersion function, does not lend itself to it, an extension of this work to obtain first-order RTEs is currently underway.

## Acknowledgments

JL acknowledges the support of ENS Paris-Saclay through its Doctoral Grant Program. The authors would like to thank Adrien Bourgoin for valuable discussions.

## Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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