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Intervention in a conference / *Intervention en colloque*

Interlayer phase coherence and composite fermions

Cohérence de phase entre bicouche et fermions composites

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Abstract. The fractional quantum Hall effect (FQHE) realized in two-dimensional electron systems is explained by the emergent composite fermions (CF) out of ordinary electrons. It is possible to write down explicit wavefunctions explaining many if not all FQHE states. In bilayer systems there is a regime at integer filling of the lowest Landau level that displays a spontaneous breakdown of the $U(1)$ relative phase between the two layers. This can be seen as interlayer phase coherence (ILC) in terms of *electrons*. Recent experiments in double layer samples of graphene have revealed the appearance of many FQHE states unique to the bilayer case. We discuss extensions of the CF idea in this situation as well as the possible existence of ILC of CFs.

Résumé. L'effet Hall quantique fractionnaire (FQHE) réalisé dans les systèmes électroniques bidimensionnels s'explique par l'émergence de fermions composites (CF) à partir d'électrons ordinaires. Il est possible d'écrire des fonctions d'onde explicites expliquant de nombreux, voire tous les états de l'effet Hall quantique fractionnaire. Dans les systèmes formés de bicouches, il existe un régime de remplissage entier du niveau de Landau le plus bas qui présente une brisure spontanée de la phase relative $U(1)$ entre les deux couches. Cela peut être considéré comme une cohérence de phase entre les couches (ILC) en termes d'*électrons*. Des expériences récentes dans des échantillons à double couche de graphène ont révélé l'apparition de nombreux états FQHE propres au cas bicouche. Nous discutons des extensions de l'idée de CF dans cette situation ainsi que de l'existence possible d'ILC de CFs.

Keywords. Quantum Hall effect, Composite fermions, Interlayer phase coherence, Bilayer graphene.

Mots-clés. Effet Hall quantique, Fermions composites, Coherence entre couches, Bicouche de graphene.

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1. Introduction

In the low-temperature regime quantum fluids may display physical properties governed by spontaneous symmetry breaking. This is the case of superfluids with the symmetry associated to the number of neutral particles. This also happens in superconductors with breakdown of the gauge symmetry of electrodynamics due to the condensation of Cooper pairs. Another set of phenomenon is due to the development of strong topological order known to happen in two-dimensional electrons gases in a magnetic field. This is the realm of the fractional quantum Hall effect (FQHE) [1, 2]. In this short review we will discuss the interplay of both phenomena in the context of graphene bilayers. Indeed it has been possible recently to fabricate devices

by stacking monolayers of graphene separated by a barrier of hexagonal boron nitride leading to an atomic scale physical separation between the layers and essentially no tunneling. Such devices have revealed a complex pattern of FQHE states and some of them may be described by spontaneous interlayer phase coherence of emerging entities [3] called "composite fermions". These composite fermions are in this case even more complex objects [4] than the ones appearing in single layer devices.

We will focus only on the approaches based on explicit wavefunctions written in terms of electron coordinates that have been shown to be extremely useful [3, 5, 6].

2. Fractional quantum Hall effect

To set the stage for the wavefunctions describing FQHE we first give a short introduction to Landau level physics and the integer quantum Hall effect which does not involve in a crucial way electron–electron interactions. The emerging dressed electrons called "composite fermions" that describe the FQHE will essentially be ruled by this regime. Numerous books are available, giving all necessary details [3, 7–9].

2.1. *Integer quantum Hall effect*

We consider a system of N_e electrons confined to two-dimensional plane under a perpendicular uniform magnetic field. This is a physical situation describing some electron gases in semiconductors, heterostructures or quantum wells. A simple description of this situation is given by the following Hamiltonian:

$$\mathscr{H} = \frac{1}{2m^*}(p + e\mathscr{A})^2, \tag{1}$$

where the vector potential can be taken in the so-called symmetric gauge $\mathscr{A} = (\mathscr{B} \times r)/2$. The effective mass m^* is a material dependent parameter. Solving for eigenvalues of the one-body Hamiltonian Equation (1) gives the Landau levels that have energies $E_N = (N + 1/2)\hbar\omega_\mathrm{c}$ with N a positive integer and $\omega_\mathrm{c} = eB/m^*$ the cyclotron frequency. There a macroscopic degeneracy in this situation: for each value of N there are exactly N_ϕ states that have this peculiar energy where N_ϕ is the number of flux quanta through the system $N_\phi = B \times A/(h/e)$. The FQHE is strongest when all electrons lie in the lowest Landau level (LLL) whose one-body eigenstates are given by the following formula:

$$\phi_M(r) = z^M \exp(-|z|^2/4\ell^2), \tag{2}$$

where we use the complex coordinate in the plane $z = x - \mathrm{i}y$ and we have defined the magnetic length $\ell = \sqrt{\hbar/(eB)}$. The power M is a positive or zero integer (we omit the normalization). With the special Landau level spectrum it is clear that even without considering interactions there are special cases when one has exact filling of an *integer* number of Landau levels, say p. With the formula for the degeneracy this happens when:

$$\nu = \frac{N_\mathrm{e}}{N_\phi} = \frac{nh}{eB} = p, \tag{3}$$

where we have defined the filling factor ν which is the fraction of occupied states, and n is the areal density of electrons. A naive thought is that in this integer case one has a magnetic-field induced insulator due to the presence of cyclotron gap $\hbar\omega_\mathrm{c}$. This gap lies between the highest occupied pth Landau level and the lowest empty $p + 1$th level at zero temperature. This is the physical picture for the bulk sample. In a real finite-size sample the Landau levels are bent when approaching the boundaries and as a consequence the Landau levels will cross the Fermi

energy at some points that we may call "Fermi points" at the real-space edges of the system. As a consequence there is electric conduction with appearance of a Hall voltage V_{H} perpendicular to an imposed current I and its value is given by:

$$V_{\mathrm{H}} = \frac{1}{p}\frac{h}{e^2} I = \frac{1}{\nu}\frac{h}{e^2} I, \tag{4}$$

so that the Hall resistance defined by $R_{\mathrm{H}} = V_{\mathrm{H}}/I$ is quantized in units of h/e^2 and I is the current intensity in the sample. The presence of disorder in the sample leads to the appearance of plateaus as as a function of the magnetic field where the Hall resistance stays constant at the value Equation (4). This phenomenon, the so-called integer quantum Hall effect [10], does not involve in any crucial way the electron–electron interactions.

The closed-shell argument above suggests that for fractional filling no similar quantization should develop since partial filling of a flat band enlarged by disorder may host a Fermi liquid albeit with a large density of states. But striking experiments found evidence for the existence of a fractional quantum Hall effect when the filling factor ν is some rational fraction and the phenomenology of current transport is the same as in the integer case. The original discovery involved a plateau in the Hall resistance $R_{\mathrm{H}} = 3h/e^2$ corresponding to filling factor $\nu = 1/3$ in the LLL. To understand the fractional case one has to dig deep into the many-body problem of interacting electrons in the LLL. A generic many-body wavefunction is a sum of products of one-body states Equation (2) and thus can be written as a polynomial $\mathscr{P}$ in complex coordinates z_i times a universal (state-independent) Gaussian factor:

$$\Psi(z_1,\ldots,z_N) = \mathscr{P}(z_1,\ldots,z_N)\exp\left(-\sum_i |z_i|^2/4\ell^2\right). \tag{5}$$

In the spin-polarized case the polynomial $\mathscr{P}$ is antisymmetric to comply with the Pauli principle. In the general case it is hard to guess what are the relevant polynomials $\mathscr{P}$ but in the realm of the FQHE the successfull approach leads to explicit guess of the polynomials, the most famous case being the Laughlin wavefunction. We conclude this section by the giving the expression of $\mathscr{P}$ for the completely filled LLL. We first observe that the one-body states Equation (2) has a probability distribution which is a ring centered at $z = 0$ (because of the choice of gauge) whose radius grows with the exponent M. So fully filled LLL means that we occupy all one-body states from $M = 0$ up to a maximal value given by $M = N-1$ without any vacancy $z^0, z^1, z^2, \ldots, z^{N-1}$. This state is exactly a Slater determinant:

$$\Psi_{\nu=1} = \det(z_i^{j-1}) = \prod_{i<j}(z_i - z_j), \quad i,j = 1,\ldots,N, \tag{6}$$

where we have used the fact that the determinant is the so-called Vandermonde determinant. From now on we will omit the Gaussian factor from the one-body states since it is independent of the state under consideration. If we draw the charge density of state $\Psi_{\nu=1}$ in real space we find that it has the shape of a very flat pancake with uniform density in its interior and going to zero very quickly at the edge with a characteristic length ℓ.

2.2. *Laughlin wavefunction*

To explain the properties of the quantum state of electrons at $\nu = 1/3$ Laughlin proposed the following candidate wavefunction:

$$\Psi^{(3)} = \prod_{i<j}(z_i - z_j)^3 \exp\left(-\sum_i |z_i|^2/4\ell^2\right). \tag{7}$$

If we compute the charge density by some means we find also a very flat pancake as in state Equation (6) but with a mean density three times lower, corresponding to a uniform electronic

state with filling factor 1/3 (this would be $1/m$ if we were to put power m to the Vandermonde factor). This wavefunction has very good trial energy but is not an exact eigenstate of the Coulomb interacting electrons. So its theoretical status is not immediately clear. Another ansatz was the idea of starting from the Slater determinant of $\nu = 1$ and fill only one out of three orbitals: $z^0, z^3, z^6, \ldots$. Since this is still a determinant it can be easily computed giving $\prod_{i<j}(z_i^3 - z_j^3)$. This is not a good candidate for the FQHE state at $\nu = 1/3$. While it has the correct filling factor by construction it has higher energy than the Laughlin state because it lacks the big correlation hole around each electron due to the $(z_i - z_j)^3$ factor in the Laughlin wavefunction. Also this state called the Tao-Thouless state does not have a uniform density in real space and does not lead to the correct phenomenology of the FQHE at $\nu = 1/3$, contrary to the Laughlin state.

To clarify the status of the Laughlin state we now show to solve by elementary means any two-body problem in the LLL. The interacting Hamiltonian is:

$$\mathscr{H}_2 = \frac{1}{2m^*}(p_1 + e\mathscr{A}_{r_1})^2 + \frac{1}{2m^*}(p_2 + e\mathscr{A}_{r_2})^2 + V(r_1 - r_2), \tag{8}$$

where the positions of the electrons are r_1, r_2 and V is the interaction potential. In the context of the FQHE it is given by the Coulomb interaction $V(r) = e^2/r$. We now introduce the center of mass and relative coordinates $R = (r_1 + r_2)/2$, $r = r_1 - r_2$. The fact that the vector potential is linear in the coordinates leads to a simplification:

$$\mathscr{H}_2 = \frac{1}{2M}(\mathscr{P} + 2e\mathscr{A}_R)^2 + \frac{1}{2\mu}(p_r + (e/2)\mathscr{A}_r)^2 + V(r), \tag{9}$$

so the center of mass as well as the relative particle will live in their own separate Landau levels. If we focus on the relative particle we can take as a basis the one-body states of Equation (2) that are eigenstates of the angular momentum. If we consider a potential $V(r)$ which is rotationally invariant it will be automatically diagonal in such a basis. The eigenenergies are thus given by $V_m = \langle \phi_m | V(r) | \phi_m \rangle$. The numbers V_m are often called pseudopotentials and are given by a simple integral. In the Coulomb case the pseudopotentials decrease as $1/\sqrt{m}$. For spinless electrons only the odd values of m matter due to wavefunction antisymmetry. As a consequence, the full many-body interacting Hamiltonian can be written as:

$$\mathscr{H} = \sum_{i<j} \sum_{m=0}^{\infty} V_m \mathscr{P}_{ij}^{(m)} \tag{10}$$

where we have defined $\mathscr{P}_{ij}^{(m)}$ the projector onto the state of relative angular momentum m for the pair i, j of particles.

If now we retain only the $m = 1$ projector in the Hamiltonian Equation (10) then we note that the Laughlin state is an exact zero-energy eigenstate. Indeed since for all pairs of particles there is an overall factor z^3 with z the relative particle of pair i, j it means that it has no weight onto $m = 1$. For the experimentally relevant case of Coulomb interaction the $V_{m=1}$ pseudopotential is indeed the strongest which gives some weight to the relevance of the Laughlin state beyond the fact of having a good energy. The parametrization of Equation (10) suggests a way to interpolate between the Coulomb case and the hard-core limit involving only the projector $\mathscr{P}_1$ by varying the set of discrete pseudopotentials V_m. This idea was successfully implemented by F. D. M. Haldane (see his contribution in book [9]) showing that the Laughlin state is smoothly connected to the Coulomb ground state and thus these two states share the same physics.

As a polynomial in the electronic coordinates the Laughlin ansatz gives us a state which is already fully factorized. This is a mathematical statement concerning the polynomial appearing in the many-body wavefunction. It does not mean that the quantum state is factorizable, the Laughlin state is *not* a Slater determinant. Indeed it is known to be a case of strong entanglement, a property which does not exist in Slater determinant states. The zeros of this polynomial are

located exactly at the positions of the electrons, a very special feature which is not true in general. Let us rewrite the Laughlin polynomial in the following way:

$$\prod_{i<j}(z_i-z_j)^3=\prod_{i<j}(z_i-z_j)\times\prod_{i<j}(z_i-z_j)^2=\det(z_i^{j-1})\times\prod_{i<j}(z_i-z_j)^2. \tag{11}$$

Here we have explcitly factored out one power of the Vandermonde determinant Equation (6). We observe that if we pick one electron and drag it around the closest neighbor we see that the phase of the wavefunction changes by 6π while antisymmetry due to the Pauli principle requires only a 2π turn. We interpret this property by saying that there are exactly two vortices bound to each electron in this state of matter. The vortex attachment is directly due to the extra factor $\prod_{i<j}(z_i-z_j)^2$. While this reveals a fundamental property of the FQHE state, it does not lead immediately to a recipe to construct other candidate wavefunctions for filling factors beyond $\nu=1/3$. We now show in the next section how another rewriting of the Laughlin polynomial leads naturally to the so-called composite fermion construction.

2.3. *Composite fermion wavefunctions*

Experiments revealed also the appearance of the FQHE for fractions other than 1/3. The most prominent set of such states appears for filling factors $\nu=p/(2p\pm1)$ with p integer, asking for an explanation in terms of wavefunctions beyond the Laughlin state. One such state is given by the so-called composite fermion construction that we describe now. Much of our intuition of electron systems is based on Slater determinants and occupied/empty orbitals so the first step is to rephrase the successfull Laughlin state in this language. We write the correlation factor of the Laughlin state in the following way:

$$\prod_{i<j}(z_i-z_j)^3=\prod_{i<j}(z_i-z_j)\times\prod_{i\neq j}(z_i-z_j)=\det(z_i^{j-1})\times\prod_{i\neq j}(z_i-z_j). \tag{12}$$

Distributing the last among the columns of the determinant we arrive at the following identity:

$$\prod_{i<j}(z_i-z_j)^3=\det\left(z_i^{j-1}\prod_{k\neq i}(z_i-z_k)\right). \tag{13}$$

This can be interpreted as a Slater determinant for a filled Landau level provided one replaces the one-body wavefunctions by correlated one-body wavefunctions:

$$z^m\to z^m\prod_j(z-z_j), \tag{14}$$

where the product over j is over all other particles. The extra correlation factor is called a Jastrow factor in many-body physics. If we decide to adopt such correlated orbitals instead of the one-body states we can view the Laughlin state at $\nu=1/3$ as a completely filled Landau level with $\nu^*=1$. We guess then that excited states may involve higher-lying Landau levels provided one uses modified orbitals including the correlation factor Equation (14). In Landau levels other than the lowest the one-body eigenstates involve the complex conjugate z^* in addition of the complex coordinate z. Let us call $\phi_{N,m}(z,z^*)$ such a state in the Nth Landau level. We add Laughlin-style correlations by making the product as above:

$$\phi_{N,m}(z,z^*)\to\phi_{N,m}(z,z^*)\prod_j(z-z_j). \tag{15}$$

Such a state however does not live in the LLL due to the appearance of z^* factors. Since there is overwhelming evidence for FQHE in the LLL only we want to project such states into the LLL and then play the game of making Slater determinants. The operation of projection onto the LLL amounts to putting all z^* to the left-hand side of the formula and next replacing them by the

operator $2\partial/\partial z$. The result is then entirely in the LLL. In the $N = 1$ Landau level this manipulation amounts to the substitution:

$$z^m \to z^m \frac{\partial}{\partial z} \prod_j (z - z_j), \tag{16}$$

and going to even higher Landau levels simply add more derivatives acting onto the correlation factor. Since the correlation factors map $\nu = 1/3$ onto $\nu^* = 1$ one can say that the composite fermions feel a reduced magnetic field $B^* = B - 2n\phi_0$ by using the definition Equation (3) of the filling factor. This immediately suggests that there will be an IQHE for the composite fermions when $\nu^* = p$ with p filled Landau levels of the CFs. This translates in filling factors for electrons as $\nu = p/(2p+1)$. One can then write down Slater determinants made of correlated orbitals to describe such states. This procedure gives us explicit trial wavefunctions whose energies can be computed by a simple Metropolis sampling. Many if not all properties extracted from these CF wavefunctions are in excellent agreement with the known experimental data and also in agreement with numbers obtained from exact diagonalization of systems with a (very) small number of electrons. The series of FQHE states at $\nu = p/(2p+1)$ is prominent in high-quality samples and is called the Jain series of states. It is observed from $p = 1$ (the Laughlin state) up to at least $p = 10$. It is important to note that these states have *no* variational parameters and nevertheless offer a very accurate description of FQHE states (for detailed comparisons see e.g. [3]). The heuristic mapping $B \to B^*$ also correctly predict that when $B^* = 0$ the CFs form some kind of Fermi sea which is gapless as is observed in electron gases at $\nu = 1/2$.

With the notion of effective magnetic field B^* we note that it may be negative, leading to states with $\nu^* = -p$ hence $\nu = p/(2p-1)$ as observed in experiments. It is also easy to generalize the CF construction to fractions descending from the parent state $\nu = 1/5$. Indeed the Laughlin state can accommodate any odd power of the Jastrow factor for spin polarized fermions (even power for bosons). The same line of reasoning leads to series of states with $\nu = p/(4p+1)$ and $\nu = p/(4p-1)$, again many of such states are observed in nature. When the filling factors becomes low the FQHE are competing with a crystal state made out of electrons, called the Wigner crystal which does not have the same striking properties as the FQHE states.

The CF construction also gives a very simple picture of excited states. Since the CF are filling an integer of pseudo LLs a first type of excitations consists of promoting a CF from the topmost filled pseudo LL to the lowest empty pseudo LL. This is a neutral excitation with no change of the number of electrons or the number of flux quanta. One may expect that such an excited state has an energy cost given by the effective cyclotron energy for the CFs. There are also charged excited states obtained by making a hole in the topmost filled pseudo LL: such a state is called a quasihole. By reducing the applied magnetic flux one can also create a situation with only one electron promoted to the next pseudo LL. This is then the quasielectron state.

In the CF picture one still has to explain why the Hall resistance is quantized as $R_{xy} = (1/\nu)(h/e^2)$ with the ν the *electron* filling while there are p filled pseudo Landau levels of CF in the series $\nu = p/(2p+1)$. The explanation is that while the CF contains a charged electron it also binds two vortices. Indeed in the construction of the CF states the Jastrow factor squared is always present and in line with the case of the Laughlin state we interpret this factor by saying that there are two vortices bound to each eletron. The CF vortex carries two units of flux ϕ_0 and if such a vortex crosses a Hall bar it will induce a voltage drop $e = -\mathrm{d}\Phi/\mathrm{d}t = 2(h/e)(I/e)$ where I/e is the number of CF per unit time crossing the Hall bar. So there is an additional contribution to the Hall voltage: $V_{\mathrm{H}} = (1/p + 2)h/e^2 I$. This completes the explanation of transport phenomenology in the FQHE regime which is mapped onto that of the IQHE.

Finally we mention that the composite fermions may undergo a pairing instability. This was proposed by Moore and Read [11] who introduced yet another intriguing explicit wavefunction called the Pfaffian:

$$\Psi_{\rm Pf} = {\rm Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2, \tag{17}$$

where the symbol Pf stands for the Pfaffian of the square matrix $1/(z_i - z_j)$. If we compute the determinant of an antisymmetric matrix we find that it is the square of a polynomial of the matrix elements. This polynomial is called the Pfaffian of the matrix. It appears when one projects a paired state like the BCS wavefunction onto a state with fixed number of particles. So the appearance of this peculiar factor is indicative of the paired nature of the Pfaffian state. Indeed the Pfaffian state is a p-wave paired state of composite fermions. The filling factor of the state can be easily computed and is $\nu = 1/2$. This state is not an exact eigenstate of the Coulomb interaction and is a trial state competing with the Fermi sea of composite fermions that exists at the same filling factor. In the LLL in semiconductors the Fermi sea has lower energy and the $\nu = 1/2$ state is compressible but it may be that this state becomes the ground state in the next Landau level where the effective Coulomb interaction is different from the LLL so it is a candidate for the FQHE state observed for $\nu = 2 + 1/2 = 5/2$ in some very clean devices. This state supports excitations with non-Abelian statistics which are very interesting quasiparticles. We will not discuss in more details this important FQHE topic.

3. Interlayer phase coherence

3.1. *The role of spin*

We now discuss the modifications of previous ideas when we consider the spin degree of freedom of charge carriers. The first remark is that the $\nu = 1$ state should be written as:

$$\Psi^{(1)} = \prod_{i<j}(z_i - z_j)|\uparrow\cdots\uparrow\rangle \tag{18}$$

since by construction it is fully polarized. More general states will not be a simple product of a spin part times an orbital part. If we consider the many-body problem of electrons interacting by the Coulomb potential we note that it has full $SU(2)$ spin rotation symmetry. This rotation symmetry will be broken down to $U(1)$ by the Zeeman coupling of the external field to the total spin. The state (18) has a spin projection onto the z axis equal to $S^z_{\rm tot} = +N/2$ and by rotational symmetry of the Hamiltonian it is also member of a multiplet of total spin $S_{\rm tot} = S^z_{\rm tot}$. This multiplet is exactly degenerate without Zeeman effect.

$$\Psi^{(1)} = \prod_k c^\dagger_{k\uparrow}|0\rangle \tag{19}$$

The $S^z_{\rm tot} = 0$ member of this spin multiplet is obtained by acting repeatedly with the spin lowering operator:

$$|S^z = 0\rangle = (S^-_{\rm tot})^{N/2}\prod_k c^\dagger_{k\uparrow}|0\rangle\,. \tag{20}$$

In this formula the spin operator acts only on the spin degrees of freedom but does not change the orbital part of the state. If we write the state in first quantization we decide to call z_i the coordinates of $\uparrow$ spins and w_k of $\downarrow$ spins and the orbital part of the $S^z_{\rm tot} = 0$ state is then:

$$\Psi^{(1)} = \prod_{i<j}(z_i - z_j)\prod_{k<l}(w_k - w_l)\prod_{i,k}(z_i - w_k). \tag{21}$$

The full wavefunction with the spin part is the antisymmetrized product of this orbital factor and of the zero spin state $|\uparrow\cdots\uparrow\downarrow\cdots\downarrow\rangle$. Halperin has proposed a generalization of the Laughlin wavefunction appropriate to states involving spin:

$$\Psi^{(mmn)} = \prod_{i<j}(z_i - z_j)^m \prod_{k<l}(w_k - w_l)^m \prod_{i,k}(z_i - w_k)^n. \tag{22}$$

Again this only the orbital part—it has to be supplemented by the spin part and antisymmetrized. Evaluation of spin-independent observables like the Coulomb energy only involve the orbital part so we simply omit the spin part for clarity. The filling factor of the trial state mmn has to be computed and is found to be $\nu = 2/(m+n)$ where the filling factor refers to the total filling including both species. In general it is not an eigenstate of total spin. Special cases include $m = n+1$ which is a singlet $S_{\rm tot} = 0$ and $m = n$ states that are ferromagnetic states as the state Equation (18). For example the (332) Halperin state describes the singlet state $\nu = 2/5$ that is a FQHE state appearing in samples with small Zeeman effect.

We note that at a given filling factor there are several Halperin state that are competing. For example at $\nu_{\rm tot} = 1/3$ one may construct the (333) state which has Laughlin correlations irrespective of the particle index and one can also build the (551) state which may become relevant when repulsion is weaker between the two components.

3.2. *Quantum Hall bilayers*

There are several physical situations where the electrons have a pseudospin index. This happens notably in many-valleys semiconductors. Monolayer graphene has two valleys and thus an extra pseudospin with two values in addition to the real spin. Some semiconductors like Si have up to six valleys. It may happens that the Coulomb interaction is independent of these extra degrees of freedom. This is approximately the case of monolayer graphene. We discuss now the case of engineered systems where two spatially separated layers of two-dimensional electron gases are close enough so that there are sizable Coulomb interactions. The layer index is then a pseudospin and the Coulomb interaction is different inside a given layer and between layers:

$$V_{\uparrow\uparrow} = V_{\downarrow\downarrow} = e^2/r, \quad V_{\uparrow\downarrow} = e^2/\sqrt{r^2+d^2}, \tag{23}$$

where d is the distance between layers. With this interaction the problem does not have the full $SU(2)$ rotation symmetry in pseudospin space but only the $U(1)$ rotation around the z axis which is the conservation law of the difference of particle numbers in the two layers. Let us concentrate on the case $\nu = 1$ first. In the limit $d \to 0$ we are back to the symmetric situation discusses in the previous section and the ground state is a ferromagnetic spin multiplet. If we now tune d small the members of the multiplet will no longer be degenerate. Since electron–electron interactions are weaker when they are in separate layers it means that the $S^z_{\rm tot} = 0$ state will have lower energy than all other states in the multiplet.

3.3. *Symmetry breaking and the phase*

If we increase the number of electrons at fixed interlayer distance d we discover that the multiplet of states $S^z_{\rm tot} = -N/2,\ldots,+N/2$ becomes degenerate as $N \to \infty$. This emergent degeneracy is the hallmark of broken symmetry. Since these states differ by the transfer of electron between layer it is the XY symmetry associated to the relative phase between layers which is broken. As in situations involving broken symmetry the physics become transparent once we use a function which breaks explicitly the symmetry. A simple choice is:

$$\Psi_x = \prod_k c^\dagger_{k,x}|0\rangle = \prod_k (c^\dagger_{k,\uparrow} + c^\dagger_{k,\downarrow})|0\rangle, \tag{24}$$

where k labels the one-body states of the LLL. This state has $\langle S^z_{\rm tot}\rangle = 0$ even though it is not an eigenstate of $S^z_{\rm tot}$. Expansion of the second formula reveals that it has weight over all members of the multiplet. This state is the analog of the BCS wavefunction for broken particle number. Its generalization to an arbitrary relative phase ϕ between layers is then:

$$\Psi(\phi) = \prod_k (c^\dagger_{k,\uparrow} + {\rm e}^{{\rm i}\phi} c^\dagger_{k,\downarrow})|0\rangle \tag{25}$$

This state gives equal weight to the two layers and this broken symmetry is aptly called interlayer phase coherence. In the magnetic language we are dealing with XY symmetry breaking and the relative phase ϕ is the XY order parameter. This phenomenon has been observed in semiconductor devices [12] for total filling factor unity and $d \lesssim \ell$. If the layer separation is too large then we have two essentially decoupled layers each hosting a $\nu = 1/2$ CF Fermi sea. There is at least one transition between the fully decoupled regime at large separation and the ILC phase at small separation. As expected in a system with a phase associated with symmetry breaking there should supercurrents in states with a gradient in space of this phase. However since the phase corresponds to the relative phase between the layers it implies that the supercurrent consists of opposite flows of charge carriers in opposite layers so with zero total current.

Let us now perform a particle-hole transformation on only one spin species of the state Equation (25) and call $d_{k,\uparrow}, d^\dagger_{k,\uparrow}$ the associated creation/annihilation operators. Then the state with definite phase Equation (25) can be written as:

$$\Psi(\phi) = \prod_k (1 + {\rm e}^{{\rm i}\phi} c^\dagger_{k,\downarrow} d_{k,\uparrow})|\bar{0}\rangle \tag{26}$$

where we have defined the new vacuum $|\bar{0}\rangle = \prod_k c^\dagger_{k,\uparrow}|0\rangle$. This new writing show that the ILC state at total filling factor $\nu = 1$ can be aptly called an exciton condensate where the two members of the exciton pairs reside in different layers.

There are two configurations of currents that can be used to reveal the ILC. The first one is the drag configuration in which a current is imposed only one layer (the "drive" layer) and may then measure the Hall voltage across the drive layer giving then a measurement of $R^{\rm drive}_{xy}$ and measure the Hall voltage across the other layer with no drive current (the "drag" layer) giving access to $R^{\rm drag}_{xy}$. If we consider the Halperin (111) wavefunction we note that driving an electron in one layer is accompanied by one vortex in the other layer so ILC has the special value:

$$R^{\rm drive}_{xy} = R^{\rm drag}_{xy} = \frac{h}{e^2} \tag{27}$$

The other configuration is the counterflow set-up in which the current of the drive layer is injected at the end of the Hall bar backwards in the top layer inducing a regime in which the currents flow in opposite directions in the two layers. If now we tune the magnetic field to reach $\nu_{\rm tot} = 1$ electrons and holes are locked together as in Equation (25) and thus form a neutral entity that does not feel the Lorenz force. So the Hall voltage in both layers goes to zero right at $\nu_{\rm tot} = 1$ a striking evidence of ILC [12].

4. Graphene bilayers in the modern era

It has been feasible recently to build devices with stacking of two atomically thin monolayers of graphene. Such systems can reach a regime where $d/\ell \approx 0.1$ which was out of reach of previous set-ups. Combined with high electronic mobility experiments have revealed numerous FQHE states [13, 14] that do not match the fillings of one-component states discussed in Section 2.3. One can first ask whether there is an extension of the CF wavefunctions giving trial wavefunctions for these states. Another more intriguing question is whether one can observe ILC involving as basic building blocks CF instead of electrons.

4.1. ${}^{2}_{1}CF$ *wavefunctions*

The Halperin family of states Equation (22) suggests a simple way to construct CF states [5]. We take a CF state in each layer, make a product and add a correlation factor between the two layers:

$$\Psi = \Psi_{\tilde{\nu}}(z_i) \times \Psi_{\tilde{\nu}}(w_k) \times \prod_{i,k}(z_i - w_k)^m. \tag{28}$$

Now the exponent m can be even or odd since there is no restriction from Fermi statistics. One may expect that the relevant values of m are smaller for larger separations between layers. This state has filling factors $\nu_{1,2}$ for each layer:

$$\frac{1}{\nu_{1,2}} = \frac{1}{\tilde{\nu}} + m = \frac{1}{\nu^*} + 2p + m \tag{29}$$

The CF that are formed in such states are called ${}^{2p}_{m}CF$ where there are $2p$ vortices attached to the electron in the same layer and m interlayer vortices, according to the interpretation of Jastrow factor in the CF language.

By increasing the distance d between the layers we expect a weakening of correlations and transitions [5] between competing states at fixed filling factor. For example at $\nu_{\rm tot} = 1/3$ one may have a ground state described by the (333) state at small separations which is a state of ${}^{2}_{3}CF$. For larger separation we construct the (551) at the same filling which is now a state of ${}^{4}_{1}CF$. For very large separations there should should no interlayer repulsion and so each layer should form a Fermi sea of ${}^{6}CF$ (note that at such a low filling factor $\nu = 1/6$ a Wigner crystal may form instead of a FQHE state). By considering compressible states one may enlarge the familty of trial states. For example there is the Fermi sea of ${}^{4}_{2}CF$ which is also competing at $\nu_{\rm tot} = 1/3$ that would lead to a compressible intermediate phase sandwiched between (333) and (551) states if we guess that the ordering of phase follow the power of the interlayer repulsion factor. At filling factor $\nu_{\rm tot} = 1/2$ one may consider the Halperin state (331) which may be realized in GaAs/GaAlAs devices [15, 16].

By similar reasoning we expect that for $\nu_{\rm tot} = 2/5$ by increasing the separation we find the (332) state which is a ${}^{2}_{2}CF$, then a Fermi sea of ${}^{4}_{1}CF$, then the state (550) which is a product of two Laughlin states at filling $1/5$, an incompressible state of ${}^{4}CF$.

It is also plausible [17] that other non-FQHE state enter the competition like crystal states of electrons or of composite fermions. Only partial theoretical analysis of this complicated situtation is available.

4.2. *Observed states*

By analogy with monolayer electronic gases one expects to find a prominent series of FQHE states involving ${}^{2}_{1}CF$ [5] which means $p = 1$ and $m = 1$ hence $\nu_{1,2} = n/(3n+1)$ where n is the number of filled pseudo-Landau levels of CFs. Indeed the states $\nu_{1,2} = 1/4$ ($n = 1$), $2/7$ ($n = 2$), $3/10$ ($n = 3$) are observed as incompressible states. The negative-flux series is also observed for $\nu_{1,2} = 1/2$ ($n = -1$), $2/5$ ($n = -2$), $3/8$ ($n = -3$). These two series of states should converge to a compressible state for $\nu_{1,2} = 1/3$. Some of these fractions have simple wavefunctions: this is the case of the $1/4$ state which involves two Laughlin states with one Jastrow power between the layers. The wavefunction is the (331) state in the Halperin family Equation (22). But these series of states do not exhaust the observations made in graphene bilayers.

Some additional fractions beyond the principal $n/(3n \pm 1)$ series include states with fractional n and odd denominator such as $\nu_{1,2} = 3/14$ corresponding to $n = 3/5$ or $\nu_{1,2} = 2/9$ with $n = 2/3$. It may be that such corresponds to FQHE states of composite fermions as observed in one-component systems for fractions like $\nu = 4/11$. They may fall in the general hierarchical scheme of FQHE.

Another set of states appear for $\nu_{1,2} = 1/3$ and $2/3$. These states do not manifest any drag Hall resistance so they are decoupled FQHE states likely (330) for $1/3$ and its particle-hole partner for $2/3$. The state observed at $\nu_{1,2} = 1/6$ may be a Halperin state (333) provided interlayer correlations are strong enough which is not completely consistent with uncorrelated states nearby at $1/3$. As we observed in the case of multicomponent states at a given filling factor there are several competing states when we vary the strength of interlayer coupling.

Finally there is evidence for states with half-integer filling of ${}^2_1CF$. This includes $\nu_{1,2} = 1/5\,(n = 1/2), 3/11$ $(n = 3/2), 5/17$ $(n = 5/2)$ and with negative flux $\nu_{1,2} = 3/7$ $(n = -3/2), 5/3$ $(n = -5/2)$. If we take into account the ILC phenomenology as well as the CF construction it is natural to conjecture that these states may display ILC of ${}^2_1CF$.

It is now a theoretical challenge to write down explicit wavefunctions capturing these states that combine the formation of emerging quasiparticles (the CFs) and their condensation in a broken symmetry state.

5. Conclusions

Recent experiments have given evidence for a fascinating interplay of topological order with the creation of a new type of composite fermions, the ${}^2_1CF$ entities. These CF quasiparticles may form Landau levels leading to incompressible states that are a generalization of the series of FQHE states already observed in single-layer one-component systems. However these experiments also revealed the appearence of other FQHE states whose description may involve interlayer phase coherence of composite fermions. Such states await detailed theoretical explanation. The technological advances in the manipulation of layered structures has opened a whole new field of investigation of correlated quantum states and the transitions between them.

Of course there are many more examples of multicomponent quantum Hall systems that we have not addressed in this short review. The experiments we have briefly discussed in this review [13, 14] have a thin insulating barrier of hexagonal boron nitride between two graphene monolayers. The barrier is thin enough to allow strong Coulomb interactions between the two electron gases but nevertheless thick enough to suppress tunneling. It is feasible to fabricate samples with nonzero tunneling. This leads a new set of physical properties. Notably the spin texture of charge excitations is changed and can be manipulated by tilting the applied magnetic field away from the direction perpendicular to the layers. The physics of this situation has been investigated in samples involving two semiconductor quantum wells coupled by tunnel effect. Each one-body quantum state in a well is then combined with its partner in the opposite well giving rise to eigenstates that are symmetric–antisymmetric (SAS) doublets with tunnel energy splitting $\Delta_{\rm SAS}$. It is thus feasible to investigate such a system as a function of the applied magnetic field and $\Delta_{\rm SAS}$. Such experiments have revealed the competition between one-component and two-component quantum Hall states as well as between single-layer and bilayer Wigner crystals [18, 19].

In the graphene world one can also study pure graphene bilayers where there is chemical bonding between two monolayers. The case of Bernal stacking (AB) has been investigated in detail in the quantum Hall regime. This is a very special case since the central Landau level has an orbital degeneracy between levels with $N = 0$ and $N = 1$ character where N is the Landau level index. This is in addition to the spin and valley degeneracies which are also present as in the case of monolayer graphene. The central level orbital degeneracy can be adjusted by applying an electric bias between the two layers leading to a tunable quantum Hall system which can interpolate between $N = 0$ and $N = 1$ FQHE physics. One expects that some if not all FQHE states can be described by appropriately generalized CF wavefunctions. Recent investigations [20] have revealed many FQHE states with *even* denominators that are outside the

scope of CF wavefunctions but likely to belong to the Pfaffian family of FQHE states or its particle-hole partner dubbed the "AntiPfaffian".

It is likely that progress in sample creation or fabrication will lead to more insights into these remarkable states of matter and more guidance for in-depth theoretical studies.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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