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Research article

On the hydrodynamic interpretation of non-relativistic quantum mechanics

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Abstract. The Madelung equations express the Schrödinger equation as a continuity equation and modified Hamilton–Jacobi equation. These equations are equivalent to the Euler equations for a compressible, potential flow, when classical pressure per unit density is replaced by the quantum potential per unit mass. We extend this hydrodynamic interpretation by quantising a single, spinless, non-relativistic particle constrained to a surface wave with small slope. The wave is distinct from the wave function and, in order to reproduce the Schrödinger equation, it must satisfy the kinematic boundary condition for a free surface advected by twice the Madelung velocity field.

Keywords. Madelung hydrodynamics, free surface quantisation, constrained quantum particle, pilot-wave theory, wave mechanics.

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1. Introduction

A hydrodynamic form of quantum mechanics for a spinless non-relativistic particle was first derived by Madelung in 1927 [1]. The equations permit a quasi-classical interpretation of quantum mechanics, wherein a velocity field characterises the motion of real particle trajectories. This ontological point of view has been adopted in de Broglie–Bohm theory [2–4] and Nelson's stochastic mechanics [5–7]. Provided that the ensemble of particle positions is initially distributed according to the Born rule, statistical predictions agree with standard quantum mechanics.

Several difficulties arise if Madelung's formalism is taken to describe a physically real field implied by Schrödinger's equation. The velocity field prescribed by Madelung is independent of the particle; that is, the particle neither generates nor modifies the field that governs its motion. As a consequence, no new testable predictions are made. Moreover, for the wave function implied by the velocity field to be single-valued, the circulation of the field must satisfy an ad hoc quantisation condition along closed paths, as noticed by Takabayasi [8]. This requirement is restrictive and reminiscent of the Bohr–Sommerfeld condition from old quantum theory. Finally, the Madelung equations can fail to yield a unique time-evolution from initial data [9].

Nelson avoids the first issue by proposing that the Schrödinger equation emerges from an underlying diffusion process. However, the current velocity must still satisfy Takabayasi's condition, and the uniqueness of this velocity field is not generally guaranteed [9,10]. Bohm's theory posits that a real wave field guides deterministic particle paths via the quantum potential. Although fully causal, this framework inherits the aforementioned problems.

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The general lines of a causal theory were first presented by de Broglie [11], wherein quantum particle trajectories are deterministic and guided by an underlying pilot-wave. Bohm's formulation of de Broglie's ideas omitted the double solution program, which was a hypothesis that the particle is associated with a wave distinct from the pilot-wave guiding it.

By vertically vibrating a silicone oil bath, Couder et al. [12] demonstrated that displaced millimetric droplets can walk horizontally on the bath's surface, with a motion similar to skipping stones, by interacting with self-generated waves. Theoretical modelling of the system typically treats the magnitude of the wave slope at each droplet bounce as small, since the bath is not significantly deformed in a neighbourhood of the droplet impact when the droplet-bath contact time is short [13,14]. Pilot-wave hydrodynamics explores walking droplets as a macroscopic realisation of de Broglie's theory; see Bush [15] for a detailed review.

Treating the Madelung equations as a mathematical transformation, rather than a reinterpretation of the Schrödinger equation, does not require the phase gradient of the wave function to be single-valued or real particle positions and fields to be invoked. We adopt this perspective and propose a method of quantisation that is motivated by the pilot-wave hydrodynamical view. By quantising a classical particle constrained to a moving surface, which has small slope, we show that the Hamiltonian is separable into the free particle Hamiltonian and a geometric correction. By requiring that the correction vanishes, after acting on the wave function, and writing the wave function in polar form, we arrive at a pair of constraints. They permit interpretation as a kinematic boundary condition and a conservation equation.

As opposed to Nelson's stochastic mechanics and de Broglie–Bohm theory, the surface wave is not introduced as a physical field, nor as a hidden variable. The surface parametrises a family of constrained Hamiltonians, with reduced dynamics on $\mathbb{R}^2$, whose quantisation is compatible with standard Schrödinger evolution on $L^2(\mathbb{R}^2)$. By recovering the kinematic boundary condition for a free surface advected by twice the Madelung velocity, we corroborate Madelung's hydrodynamic formulation while leaving the ontology and empirical content of standard quantum mechanics unchanged.

2. Wave evolution equations implied by quantisation of a constrained classical model

Denote ordinary two-dimensional space by $\mathbf{x} = (x, y)$. We also introduce the vertical coordinate w as an additional degree of freedom; all gradients are taken with respect to $\mathbf{x}$. Now, consider a classical particle with position $\mathbf{X}(t) \in \mathbb{R}^2$ and mass m. It remains in continuous contact with a surface defined by the Monge patch $w = \eta(\mathbf{x}, t)$ so that the holonomic constraint is

$$f(\mathbf{q}, t) := W - \eta(\mathbf{X}, t) = 0,$$

where $\mathbf{q} = (\mathbf{X}, W) \in \mathbb{R}^3$ is the particle's configuration. We are distinguishing between the particle's three-dimensional configuration $\mathbf{q}$ and its two-dimensional position $\mathbf{x}$ to avoid coupling the quantum state to the surface, which leads to non-uniqueness in the quantisation approach [16]. Formally, the quantum state $|\psi(t)\rangle \in L^2(\mathbb{R}^2)$ is defined on a flat plane $\mathbb{R}^2$ and is not confined by a potential to the embedded surface $\{(\mathbf{x}, w) \mid w = \eta(\mathbf{x}, t)\} \subset \mathbb{R}^3$. In the $\mathbf{x}$-representation, it is given by the wave function $\psi(\mathbf{x}, t) = \langle \mathbf{x} | \psi(t) \rangle$. The η wave guides the planar motion of ψ, since the effective dynamics of ψ are obtained from quantising the constrained classical system that follows.

Assume the presence of an external Schrödinger potential $V(\mathbf{x}, t)$ so that the Lagrangian is

$$\mathscr{L} = \frac{m}{2}\left(|\dot{\mathbf{X}}|^2 + \dot{W}^2\right) - V(\mathbf{X}, t) + \lambda\left(W - \eta(\mathbf{X}, t)\right), \tag{1}$$

and the Euler–Lagrange equations give

$$m\ddot{\mathbf{X}} = -\nabla V - \lambda \nabla\eta, \tag{2}$$

$$m\ddot{W} = \lambda, \tag{3}$$

$$W = \eta(\mathbf{X}, t), \tag{4}$$

as necessary conditions to minimise the action along the particle's path $\mathbf{x} = \mathbf{X}(t)$. Twice differentiation of (4) with respect to time, and substitution of (3) into (2), gives

$$m\ddot{\mathbf{X}} = -\nabla V\big|_{\mathbf{x}=\mathbf{X}(t)} - \frac{\nabla\eta}{1+|\nabla\eta|^2}\left(m\left(\frac{\partial^2\eta}{\partial t^2} + 2\nabla(\partial_t\eta)\cdot\dot{\mathbf{X}} + \dot{\mathbf{X}}^\top(\nabla\nabla\eta)\dot{\mathbf{X}}\right) - \nabla\eta\cdot\nabla V\right)\bigg|_{\mathbf{x}=\mathbf{X}(t)} \tag{5}$$

as the classical Newtonian equation of motion.

Differentiating (4), we can express (1) in the reduced form

$$\begin{aligned}\mathscr{L} &= \frac{m}{2}\left(|\dot{\mathbf{X}}|^2 + (\partial_t\eta + \nabla\eta\cdot\dot{\mathbf{X}})^2\right) - V(\mathbf{X}, t)\\ &= \frac{m}{2}\dot{\mathbf{X}}^\top g\dot{\mathbf{X}} + m\partial_t\eta(\nabla\eta\cdot\dot{\mathbf{X}}) + \frac{m}{2}(\partial_t\eta)^2 - V(\mathbf{X}, t),\end{aligned}$$

where $g = I + \nabla\eta\nabla\eta^\top$. A Legendre transform gives the Hamiltonian

$$\begin{aligned}H_\eta &= \dot{\mathbf{X}}\cdot\frac{\partial\mathscr{L}}{\partial\dot{\mathbf{X}}} - \mathscr{L}\\ &= \frac{m}{2}\dot{\mathbf{X}}^\top g\dot{\mathbf{X}} - \frac{m}{2}(\partial_t\eta)^2 + V\\ &= \frac{m}{2}\left(\frac{\mathbf{p}}{m} - (\partial_t\eta)\nabla\eta\right)^\top g^{-1}\left(\frac{\mathbf{p}}{m} - (\partial_t\eta)\nabla\eta\right) - \frac{m}{2}(\partial_t\eta)^2 + V\\ &= \frac{1}{2m}\mathbf{p}^\top g^{-1}\mathbf{p} - \frac{\partial_t\eta}{1+|\nabla\eta|^2}(\mathbf{p}\cdot\nabla\eta) - \frac{m}{2}\frac{(\partial_t\eta)^2}{1+|\nabla\eta|^2} + V,\end{aligned}$$

where canonical momentum is given by

$$\mathbf{p} = \frac{\partial\mathscr{L}}{\partial\dot{\mathbf{X}}} = m\big(g\dot{\mathbf{X}} + (\partial_t\eta)\nabla\eta\big),$$

the particle velocity is

$$\dot{\mathbf{X}} = g^{-1}\left(\frac{\mathbf{p}}{m} - (\partial_t\eta)\nabla\eta\right),$$

and

$$g^{-1} = I - \frac{\nabla\eta\nabla\eta^\top}{1+|\nabla\eta|^2}$$

is the inverse of our g metric using the Sherman–Morrison formula. Taking $|\nabla\eta| \ll 1$ so that $g^{-1} \approx I$,

$$H_\eta \approx \frac{|\mathbf{p}|^2}{2m} - (\partial_t\eta)(\mathbf{p}\cdot\nabla\eta) - \frac{m}{2}(\partial_t\eta)^2 + V = H_0 + H_c,$$

where we have defined

$$H_0 = \frac{|\mathbf{p}|^2}{2m} + V$$

as the standard Hamiltonian and

$$H_c = -(\partial_t\eta)(\mathbf{p}\cdot\nabla\eta) - \frac{m}{2}(\partial_t\eta)^2$$

as the geometric correction, which retains terms up to first-order in $\nabla\eta$. Quantising with $\mathbf{p} \to -i\hbar\nabla$ yields

$$\begin{aligned}\widehat{H}_c &= -\frac{1}{2}\Big((\partial_t\eta)(\nabla\eta\cdot\widehat{\mathbf{p}}) + (\widehat{\mathbf{p}}\cdot\nabla\eta)(\partial_t\eta)\Big) - \frac{m}{2}(\partial_t\eta)^2\\ &\equiv \frac{i\hbar}{2}\Big(2(\partial_t\eta)\nabla\eta\cdot\nabla + \nabla\cdot\big((\partial_t\eta)\nabla\eta\big)\Big) - \frac{m}{2}(\partial_t\eta)^2,\end{aligned}$$

where we have imposed symmetric ordering to ensure that our model is Hermitian, in accordance with Dirac's general prescription [16,17].

The quantum dynamics are given by $i\hbar\partial_t\psi = (\widehat{H}_0 + \widehat{H}_c)\psi$ in $\mathbf{x}$-configuration space. To ensure consistency with Schrödinger evolution $i\hbar\partial_t\psi = \widehat{H}_0\psi$, we impose $\widehat{H}_c\psi = 0$ as a sufficient condition so that

$$i\hbar\Big(2(\partial_t\eta)\nabla\eta\cdot\nabla\ln\psi + \nabla\cdot\big((\partial_t\eta)\nabla\eta\big)\Big) - m(\partial_t\eta)^2 = 0.$$

Writing $\psi = |\psi|e^{i\theta/\hbar}$ in polar form, and then separating real and imaginary parts, we obtain

$$\frac{\partial\eta}{\partial t} = -\frac{2}{m}\nabla\eta\cdot\nabla\theta, \qquad 2\frac{\partial\eta}{\partial t}\nabla\eta\cdot\nabla\ln|\psi| + \nabla\cdot\left(\frac{\partial\eta}{\partial t}\nabla\eta\right) = 0,$$

if we discard the $\partial_t\eta = 0$ solution. Given the first real part equation, the imaginary part reduces to

$$0 = \nabla\cdot\big(|\psi|^2(\nabla\eta\cdot\nabla\theta)\nabla\eta\big).$$

Defining $\mathbf{u} = \nabla\theta/m$ as the Madelung velocity allows us to write a kinematic boundary condition for the free surface

$$\frac{\partial\eta}{\partial t} + 2\mathbf{u}\cdot\nabla\eta = 0 \tag{6}$$

and a conservation condition

$$\oint_C |\psi|^2(\mathbf{u}\cdot\nabla\eta)\nabla\eta\cdot\hat{\mathbf{n}}\,\mathrm{d}l = 0$$

along any closed path C with unit normal $\hat{\mathbf{n}}$. Level sets of the free surface are advected by the velocity field $2\mathbf{u}$, and time-variation in the free surface, weighted by the probability density and directed along the surface-height gradient $\nabla\eta$, has zero net outward flux across C.

The factor of two in (6) originates from the prefactor $m/2$ in the quadratic time-derivative term in the geometric correction H_c. This inertial term is retained upon quantisation and effectively doubles the advection velocity of η. The kinematic condition may be rewritten

$$\frac{\partial\eta}{\partial t} + \mathbf{u}\cdot\nabla\eta = -\mathbf{u}\cdot\nabla\eta.$$

If the horizontal velocity of fluid elements is identified as $\mathbf{u}$, then the vertical velocity required to keep material points on the moving free surface is $-\mathbf{u}\cdot\nabla\eta$.

The hydrodynamic formulation above is valid only away from zeros of $|\psi|$. In the general case,

$$i\hbar(\partial_t\eta)\nabla\eta\cdot\nabla\psi + \frac{i\hbar}{2}\big(\nabla\cdot\big((\partial_t\eta)\nabla\eta\big)\big)\psi - \frac{m}{2}(\partial_t\eta)^2\psi = 0$$

and so multiplying both sides by the complex conjugate ψ^* gives

$$\frac{i\hbar}{2}\Big((\partial_t\eta)\nabla\eta\cdot\nabla\rho + \big(\nabla\cdot\big((\partial_t\eta)\nabla\eta\big)\big)\rho\Big) - m\left((\partial_t\eta)\nabla\eta\cdot\mathbf{j} + \frac{1}{2}(\partial_t\eta)^2\rho\right) = 0$$

using the identity

$$\psi^*\nabla\psi = \frac{1}{2}\nabla\rho + i\frac{m}{\hbar}\mathbf{j}$$

where $\mathbf{j} = \hbar\Im(\psi^*\nabla\psi)/m$ is the probability current and $\rho = |\psi|^2$. Separating real and imaginary parts,

$$(\partial_t\eta)\nabla\eta\cdot\mathbf{j} + \frac{1}{2}(\partial_t\eta)^2\rho = 0, \qquad \nabla\cdot\big(\rho(\partial_t\eta)\nabla\eta\big) = 0$$

are the generic constraints, which are meaningful at nodes where $\psi = 0$. For the real part equation, the trivial solution $\partial_t\eta = 0$ can be discarded so that

$$\rho\frac{\partial\eta}{\partial t} + 2\mathbf{j}\cdot\nabla\eta = 0, \qquad \nabla\cdot\big((\mathbf{j}\cdot\nabla\eta)\nabla\eta\big) = 0.$$

3. Linearised dynamics for the Madelung–free-surface system

In terms of hydrodynamic variables defined by $\rho = |\psi|^2$ and $\phi = \theta/m$ when $\psi = \sqrt{\rho}e^{im\phi/\hbar}$, our system is

$$\partial_t\eta + 2\nabla\phi\cdot\nabla\eta = 0, \qquad \nabla\cdot\big(\rho(\nabla\eta\cdot\nabla\phi)\nabla\eta\big) = 0. \tag{7}$$

The Madelung equations for this choice of ψ are

$$\partial_t\rho + \nabla\cdot(\rho\nabla\phi) = 0, \qquad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 - \frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{V}{m}. \tag{8}$$

The first equation admits interpretation as the mass conservation equation for a compressible flow with mass density $m\rho \propto \rho$ and velocity $\nabla\phi$. The second equation is a Bernoulli equation evaluated on the unperturbed surface $\mathbb{R}^2$, in the presence of a potential V/m, and with an effective pressure originating from the quantum potential.

To linearise the full system, start with (8). Consider points $\mathbf{x}\in S$ in a region $S\subset\mathbb{R}^2$ where, for $\epsilon \ll 1$,

$$\rho(\mathbf{x},t) = \rho_0 + \epsilon r(\mathbf{x},t), \qquad \phi(\mathbf{x},t) = \epsilon\varphi(\mathbf{x},t), \qquad V(\mathbf{x},t) = \epsilon v(\mathbf{x},t)$$

so that the leading-order ψ has effectively constant amplitude and zero phase in the absence of a potential. The mass conservation equation becomes

$$\partial_t r + \rho_0\nabla^2\varphi = 0$$

and the momentum equation implies

$$\partial_t\varphi - \frac{\hbar^2}{4m^2\rho_0}\nabla^2 r + \frac{v}{m} = 0.$$

Hence,

$$\partial_t^2\varphi + \frac{\hbar^2}{4m^2}\nabla^4\varphi = -\frac{1}{m}\partial_t v. \tag{9}$$

Initial conditions are determined by $\psi(\mathbf{x},t_0) = \sqrt{\rho_i(\mathbf{x})}e^{im\phi_i(\mathbf{x})/\hbar}$ so that $\varphi(\mathbf{x},t_0) = \varphi_i(\mathbf{x})$ and

$$\partial_t\varphi(\mathbf{x},t_0) = \frac{\hbar^2}{4m^2\rho_0}\nabla^2 r_i - \frac{1}{m}v(\mathbf{x},t_0)$$

where

$$\varphi_i(\mathbf{x}) = \frac{\phi_i(\mathbf{x})}{\epsilon}, \qquad r_i(\mathbf{x}) = \frac{\rho_i(\mathbf{x}) - \rho_0}{\epsilon}.$$

In the same region, enclosing a point $\mathbf{x}_0\in S$, take $\eta(\mathbf{x},t) = \eta(\mathbf{x},t_0) + \epsilon\xi(\mathbf{x},t)$ where

$$\eta(\mathbf{x},t_0) \approx \eta(\mathbf{x}_0,t_0) + \nabla\eta(\mathbf{x}_0,t_0)\cdot(\mathbf{x}-\mathbf{x}_0).$$

Without loss of generality, we can discard irrelevant constants and consider

$$\eta(\mathbf{x},t) = \mathbf{q}\cdot\mathbf{x} + \epsilon\xi(\mathbf{x},t)$$

where $\mathbf{q} = \nabla\eta(\mathbf{x}_0,t_0)$. The kinematic condition becomes

$$\partial_t\xi + 2\mathbf{q}\cdot\nabla\varphi = 0$$

and the conservation constraint implies

$$(\mathbf{q}\cdot\nabla)^2\varphi = 0.$$

Let $\widehat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$ so $s = \widehat{\mathbf{q}}\cdot\mathbf{x}$ and $z = \widehat{\mathbf{q}}^{\perp}\cdot\mathbf{x}$ where $\widehat{\mathbf{q}}\cdot\widehat{\mathbf{q}}^{\perp} = 0$. Now, $\mathbf{q}\cdot\nabla = |\mathbf{q}|\partial_s$ allows us to solve $(\mathbf{q}\cdot\nabla)^2\varphi = |\mathbf{q}|^2\partial_s^2\varphi = 0$ in the transformed coordinates. With $\varphi(s,z,t) = A(z,t)s + B(z,t)$, the kinematic condition gives

$$A(z,t) = -\frac{1}{2|\mathbf{q}|}\partial_t\xi.$$

Substituting into (9),

$$\left(\partial_t^2 A + \frac{\hbar^2}{4m^2}\partial_z^4 A\right)s + \left(\partial_t^2 B + \frac{\hbar^2}{4m^2}\partial_z^4 B\right) = -\frac{1}{m}\partial_t v(s,z,t).$$

Take $v(s,z,t) = v_A(z,t)s + v_B(z,t)$ so that

$$\partial_t^3 \xi + \frac{\hbar^2}{4m^2}\partial_t \partial_z^4 \xi = \frac{2|\mathbf{q}|}{m}\partial_t v_A.$$

Integrating once in time,

$$\partial_t^2 \xi + \frac{\hbar^2}{4m^2}\partial_z^4 \xi = \frac{2|\mathbf{q}|}{m} v_A(z,t) + C(z).$$

In the absence of an s-directed potential, when $v_A = 0$, we have

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\hbar^2}{4m^2}(\widehat{\mathbf{q}}^\perp \cdot \nabla)^4 \xi = C(\widehat{\mathbf{q}}^\perp \cdot \mathbf{x}),$$

subject to initial conditions $\xi(\mathbf{x}, t_0) = 0$ and $\partial_t \xi(\mathbf{x}, t_0) = -2\mathbf{q}\cdot\nabla\varphi(\mathbf{x}, t_0)$. If $\partial_t^2 \xi(\mathbf{x}, t_0) = 0$, the surface is not initially accelerating and we can identify $C(z) = 0$. This is equivalent to the requirement that $\partial_t \varphi(\mathbf{x}, t_0) = 0$ and so $\nabla^2 r_i = 0$ implies r_i must be harmonic when the z-directed potential $v_B = 0$ is also absent.

By imposing leading order behaviour of η as a tilted plane in the region of interest S, the conservation condition in (7) places nontrivial restrictions on the $O(\epsilon)$ behaviour of ρ and ϕ. Since the condition restricts admissible η, given a form of ψ, it also restricts admissible ψ, given a form of η.

Nonetheless, in $\mathbf{q}$-aligned coordinates, we are free to choose

$$A(z, t_0) \propto \left(\frac{1}{L^2} - \frac{z^2}{L^4}\right)\exp\left(-\frac{z^2}{2L^2}\right) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\exp\left(-\frac{z^2}{2L^2}\right)$$

for instance. Taking $t_0 = 0$, the solution to

$$\partial_t^2 \xi + \alpha^2 \partial_z^4 \xi = 0, \qquad \xi(z,0) = 0, \qquad \partial_t \xi(z,0) = -|\mathbf{q}|(1/L^2 - z^2/L^4)e^{-z^2/2L^2},$$

is

$$\begin{aligned}\xi(z,t) &= -\frac{|\mathbf{q}|L}{\alpha\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikz-(Lk)^2/2}\sin(\alpha t k^2)\,\mathrm{d}k \\ &= -\frac{|\mathbf{q}|L}{\alpha}\left(L^4 + (2\alpha t)^2\right)^{-\frac{1}{4}}\exp\left(-\frac{(zL)^2}{2(L^4+(2\alpha t)^2)}\right)\sin\left(\frac{1}{2}\arctan\left(\frac{2\alpha t}{L^2}\right) - \frac{\alpha t}{L^4+(2\alpha t)^2}z^2\right)\end{aligned}$$

where $\alpha = \hbar/2m$ has units of diffusivity.

In this linearised setting, ξ evolves according to the equation for an Euler–Bernoulli beam with a restoring force that acts along the transverse z coordinate. The restoring force is analogous to flexural rigidity in the theory of elasticity and α sets the strength of bending-wave dispersion.

4. Discussion

We have extended the hydrodynamic interpretation introduced by Madelung. Contrary to de Broglie–Bohm theory, neither the quantum particle nor the guiding wave must be real. Instead, we suppose that standard quantum mechanics can be recovered from quantisation of a particle guided by a small-slope wave. There are intuitive mathematical consequences of this assumption: the guiding wave should obey the usual kinematic boundary condition from hydrodynamics if the horizontal fluid velocity is identified with the Madelung velocity, and a conservation condition is satisfied to ensure consistency with the Schrödinger equation.

The classical model for a particle constrained to a free surface specifies a reduced Hamiltonian on $\mathbb{R}^2$. Upon quantisation, the quantum state is defined on the flat configuration space $L^2(\mathbb{R}^2)$.

Accordingly, the kinematic condition for η should not be interpreted as transporting probability mass on a physical surface in $\mathbb{R}^3$. It is a compatibility condition that determines admissible surface dynamics, which are consistent with the reduced Hamiltonian and preserve standard Schrödinger evolution on the plane.

Our results can be generalised to arbitrary dimensions. The analysis would also benefit from relaxing the small-slope approximation by using a Laplace–Beltrami operator for the Hamiltonian. We would expect sources and sinks to be introduced to the kinematic boundary condition and a cumbersome expression for the conservation condition. For an outline of this method, refer to the conservative constraint section in Ikegami et al. [16].

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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