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A probabilistic approach of Liouville field theory

Une approche probabiliste de la théorie des champs de Liouville

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Abstract. In this article, we present the Liouville field theory, which was introduced in the eighties in physics by Polyakov as a model for fluctuating metrics in 2D quantum gravity, and outline recent mathematical progress in its study. In particular, we explain the probabilistic construction of this theory carried out by David–Kupiainen–Rhodes–Vargas in [1] and how this construction connects to the modern and general approach of Conformal Field Theories in physics, called conformal bootstrap and based on representation theory.

Résumé. Dans cet article, nous présentons la théorie des champs de Liouville, qui fut introduite en physique dans les années 80 par Polyakov comme modèle de métriques aléatoires dans le cadre de la gravité quantique 2D, et donnons un aperçu de la construction probabiliste de cette théorie proposée par David–Kupiainen–Rhodes–Vargas dans [1]. Nous expliquons comment cette construction se relie à l'approche moderne des théories conformes de champs en physique appelée conformal bootstrap et basée sur la théorie des représentations.

Keywords. 2D quantum gravity, path integral, Liouville field theory, conformal bootstrap.

2020 Mathematics Subject Classification. 60D05, 81T40, 81T20.

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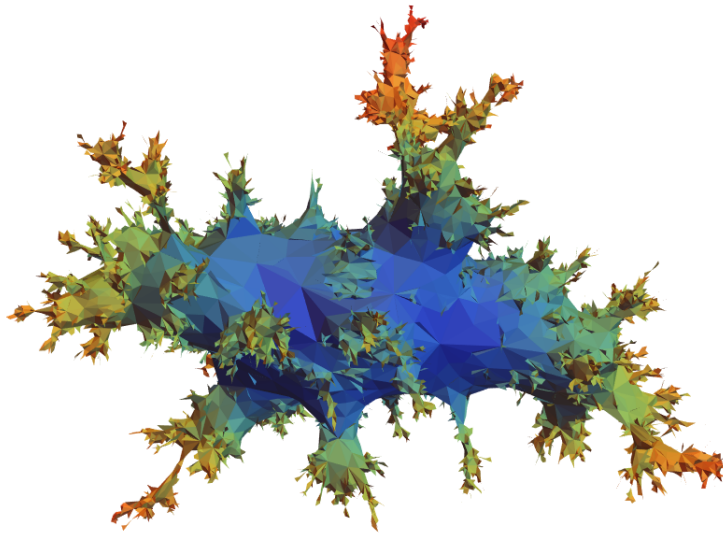


Figure 1. Typical sample of a random triangulation of the sphere (Wolfram Mathematica® code by T. Budd).

1. Introduction

One of the simplest and at the same time most intriguing 2d Conformal Field Theory (CFT) is Liouville field theory. It first appeared in 1981 in Polyakov's formulation of bosonic string theory [2] and, since then, was developed in physics as a model for Euclidean¹ two-dimensional quantum gravity, namely as a way of summing over all possible geometries of a fixed two-dimensional manifold. Ten years ago, this model was also connected to four dimensional gauge theories via the AGT conjecture [3], thus experiencing a considerable renewal of interest.

We will outline below the ideas that have shaped this rich theory, ranging from original Polyakov's path integral formulation to the modern approach of conformal bootstrap.

1.1. Polyakov's path integral

Roughly speaking and in physics, gravity is described by a Riemannian metric g on a fixed Riemann surface \mathcal{M} (namely a two dimensional manifold with holomorphic charts) and quantum gravity can be seen as a way to sum over the space of Riemannian metrics g on \mathcal{M} (seen up to the action of diffeomorphisms), which we will call $\mathbf{R}(\mathcal{M})$. Thus, quantum gravity aims to construct (probability) measures on $\mathbf{R}(\mathcal{M})$, which is a smooth infinite dimensional manifold. If it were finite-dimensional, the natural procedure in (classical) differential geometry would be to define a metric on $\mathbf{R}(\mathcal{M})$ and then consider the attached volume form as a measure on $\mathbf{R}(\mathcal{M})$. The natural metric on $\mathbf{R}(\mathcal{M})$ is the L^2 -metric, which is also called the DeWitt metric, so that the measure on $\mathbf{R}(\mathcal{M})$ relevant to quantum gravity should take on the form²

$$F \mapsto \int_{\mathbf{R}(\mathcal{M})} F(g) e^{-S_{\text{EH}}(g)} Dg \quad (1)$$

¹Time is seen as another space variable via an analytic continuation argument called Wick rotation.

²For the sake of simplicity, we skip here the possibility of coupling gravity to conformal matter fields.

where F is any arbitrary test function on $\mathbf{R}(\mathcal{M})$, S_{EH} is the Einstein-Hilbert functional³ and Dg the putative volume form associated to the DeWitt metric. The obvious caveat is that $\mathbf{R}(\mathcal{M})$ is infinite-dimensional and there is no mathematically grounded way to define the volume form Dg associated to the DeWitt metric. Yet, this could be faced by taking limits of appropriate finite dimensional approximations. More subtle is the fact that defining Dg involves serious nonlinear problems coming from the fact that, even in the finite-dimensional situation, defining volume forms involves computations of determinants. This problem can be regularized on the lattice: this gives rise to the random planar map approach that we discuss below, but Polyakov's tour de force in his founding paper [2] was to understand how to make sense of the measure (1) directly in the continuum. His key argument, which we specify now to the Riemann sphere (identified with the complex plane \mathbb{C} by stereographic projection), was a change of variables in the measure (1) to write each metric g under the form $g = \psi^*(e^{\varphi(x)}|dx|^2)$, where ψ is a diffeomorphism and ψ^* its pushforward, thus reducing the study of the random metric g to the study of its log-conformal factor $\varphi : \mathbb{C} \rightarrow \mathbb{R}$. Performing this change of variables, Polyakov argued that making sense of the integration measure (1) boils down to making sense of the following functional measure

$$\langle F \rangle_{\gamma, \mu} := \int_{\varphi: \mathbb{C} \rightarrow \mathbb{R}} F(\varphi) e^{-S_L(\varphi)} D\varphi, \quad (2)$$

where $D\varphi$ stands for the formal Lebesgue volume form (volume form of the L^2 -metric) over the space of maps $\varphi : \mathbb{C} \rightarrow \mathbb{R}$ with boundary condition $\varphi(x) \sim -2Q \ln|x|$ as $|x| \rightarrow \infty$ for some parameter $Q > 0$, F is any bounded reasonable test function and S_L is the action functional:

$$S_L(\varphi) := \frac{1}{4\pi} \int_{\mathbb{C}} (|\nabla \varphi(x)|^2 + 4\pi\mu e^{\gamma\varphi(x)}) dx. \quad (3)$$

Here the most important parameter is $\gamma \in (0, 2)$ while $\mu > 0$ is less relevant as the theory obeys a scaling relation in μ so that theories with different μ (and same γ) are essentially the same. Finally Q is fixed by conformal symmetry to the value $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

This path integral was subsequently dubbed Liouville field theory due to the fact that critical points φ_c of the functional S_L are solutions to the Liouville equation

$$\Delta \varphi_c = 2\pi\mu\gamma e^{\gamma\varphi_c}, \quad (4)$$

hence provides metrics $e^{\gamma\varphi}|dx|^2$ with uniformized Ricci curvature $R_c = -2\pi\mu\gamma^2$, which was instrumental in Poincaré's approach of uniformization of Riemann surfaces. In this respect, Liouville field theory can be seen as the natural probabilistic theory of uniformization of Riemann surfaces.

The gain in trading (1) for (2) is that, although the problem is still infinite-dimensional, the formal definition of the volume form $D\varphi$ is now linear, which highly increases the possibility of giving a proper construction. In spite of this drastic simplification, this path integral has not been fully understood in physics and has mostly served to heuristically justify inputs in another approach, more algebraic, of Liouville field theory called the conformal bootstrap (more later). It is only very recently that a mathematical definition of this path integral has been achieved by David–Kupiainen–Rhodes–Vargas [1, 4], which we will review subsequently.

1.2. Discrete random geometries

Random planar maps have been introduced as a discretization of Polyakov's path integral (1). They can be seen as a way to pick at random discrete geometries on a fixed Riemann surface.

³Its exact definition is not important for our discussion but it involves geometrical quantities related to the metric g such as its volume or mean curvature.

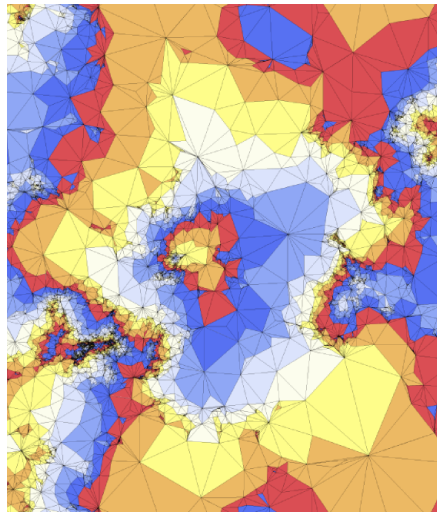


Figure 2. Zooming in the local structure of random triangulations embedded into the sphere. Each triangle carries a unit mass. (Wolfram Mathematica® code by T. Budd).

In this manuscript, we will focus on the case of the Riemann sphere and the situation where the discretization consists in triangulating the surface, a standard procedure in Riemannian geometry. One obtains a triangulation of the sphere by gluing equilateral triangles along their edges in order to obtain a surface with the topology of the sphere (see Figure 1). There are finitely many such triangulations of the sphere built out of a given number of triangles up to orientation preserving homeomorphisms. One can thus pick uniformly at random such a triangulation. Also, each triangulation carries naturally a complex structure (holomorphic charts, which means that the gluing can be done holomorphically) and thus each triangulation can be conformally embedded into the sphere via the Riemann mapping theorem. Thus we have a way to pick at random a discretized geometry on the Riemann sphere so that this procedure is sometimes dubbed discrete quantum gravity.

The question is then to determine the scaling limit of such random geometries when the number of triangles is sent to infinity. In physics, it was soon understood [5] that the scaling limit possesses a rich multifractal structure, as illustrated by Figure 2, and conjectured that this scaling limit should correspond to Polyakov's path integral formulation of Liouville field theory with parameter $\gamma = \sqrt{8/3}$. Also, and instead of considering the uniform probability measure among all possible triangulations with a fixed number of triangles, one could pick at random triangulations according to the partition function of some conformal matter field⁴ put on the triangulation, e.g. the critical Ising model: this will modify the value of the parameter γ of the Liouville theory ($\gamma = \sqrt{3}$ for Ising, see [6] for further pedestrian explanations).

Yet, in spite of a huge activity and important recent progress (see [7]), the connection between random triangulations and the measure (1) is not fully understood at the mathematical level.

⁴Say a model of statistical mechanics on the triangulation with parameters tuned at their critical value so that the model acquires conformal symmetry in the scaling limit.

2. Probabilistic construction

Now we describe the construction of the path integral (2) carried out in [1, 4]. In the same way that the standard Lebesgue measure serves as the reference measure on finite dimensional spaces, Gaussian measures are often the starting point for constructing measures in infinite-dimensional spaces. The probabilistic construction of [1] relies on a Gaussian random field, called Gaussian Free Field (GFF), associated to the squared gradient in (3)

$$\langle F \rangle_{\text{GFF}} := \int_{\varphi: \mathbb{C} \rightarrow \mathbb{R}} F(\varphi) e^{-\frac{1}{4\pi} \int_{\mathbb{C}} |\nabla \varphi(x)|^2 dx} D\varphi, \quad (5)$$

where integration still runs over maps with boundary condition $\varphi(x) \sim -2Q \ln|x|$ as $|x| \rightarrow \infty$, in such a way that the volume form (2) is defined by a reweighting of the Gaussian measure by the exponential potential

$$\langle F \rangle_{\gamma, \mu} = \left\langle F e^{-\mu \int_{\mathbb{C}} e^{\gamma \varphi(x)} dx} \right\rangle_{\text{GFF}}. \quad (6)$$

The Gaussian volume form (5) is well understood, both in mathematics and physics. One of its specific feature is that it produces random outputs $(x, \omega) \mapsto \varphi(x, \omega)$ (ω stands for randomness) such that the paths $x \mapsto \varphi(x, \omega)$ (for fixed ω) are too wild to make sense as pointwise defined functions; instead for all ω the “function” $x \mapsto \varphi(x, \omega)$ lives in some space of Schwartz distributions (also called generalized functions). The outcome of this observation is that the field φ cannot be exponentiated straightforwardly to make sense of (6) and this term has to be renormalized: one has to consider a regularization of the field φ , call it φ_ϵ , which stands for a smoothing of the field either with an ultraviolet frequency cut-off or a mollification at scale ϵ , in such a way that φ_ϵ converges to φ as $\epsilon \rightarrow 0$. Since the field $x \mapsto \varphi_\epsilon(x)$ is smooth, it can be exponentiated to obtain a regularized version of the potential $\int_{\mathbb{C}} e^{\gamma \varphi_\epsilon(x)} dx$. This term diverges as $\epsilon \rightarrow 0$ and Gaussian computations enable to show that the rate of divergency is of the order $\epsilon^{-\gamma^2/2}$ if the variance of the field behaves like $\text{Var}(\varphi_\epsilon(x)) \sim \ln \frac{1}{\epsilon}$. One can then define the regularized potential as the following limit

$$\int_{\mathbb{C}} e^{\gamma \varphi(x)} dx := \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}} e^{\gamma \varphi_\epsilon(x) - \frac{\gamma^2}{2} \text{Var}(\varphi_\epsilon(x))} dx \quad (7)$$

where the renormalization by the variance enables to remove the first order divergence. This renormalization procedure is standard in physics and is called Wick renormalization but in the context of the exponential potential it was implemented rigorously in the eighties by the mathematician Kahane [8] (see also [9]) who derived the optimal criterion for the convergence of (7). His theory goes under the name of Gaussian multiplicative chaos and establishes that the limit is indeed non trivial for $\gamma \in (0, 2)$ ⁵, hence justifying the existence of (6) for $\mu > 0$ and $\gamma \in (0, 2)$.

One can then define the correlation functions (observables) of Liouville theory in a similar way. First, for $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, one defines the vertex operator

$$V_\alpha(z) := e^{\alpha \varphi(z)}.$$

Correlation functions (with n points) are obtained by choosing arbitrary points of the complex plane $z_1, \dots, z_n \in \mathbb{C}$ with respective arbitrary weights $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and by integrating product of vertex operators with respect to the functional integral (2), namely

$$\langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n) \rangle_{\gamma, \mu}. \quad (8)$$

Again, this definition requires a renormalization procedure since the field φ can not be evaluated pointwise. The main result of [1] establishes existence and non triviality of the correlation functions using the aforementioned procedure provided that $\alpha_i < Q$ for all $i = 1, \dots, n$ and

⁵A phase transition occurs for this model at $\gamma = 2$, reminiscent of the freezing transition of some disordered systems or spin glasses [10].

$\sum_i \alpha_i > 2Q$, which implies in particular that $n \geq 3$. Moreover, the work [1] gives a new and *explicit* probabilistic formula for the correlations based on the GFF and its exponential (7), i.e. Gaussian multiplicative chaos (it is beyond the scope of this article to write these explicit formulas here). It is proved in [1] that these correlation functions satisfy the axioms of 2d CFT, among which the fact that they are *conformally covariant*. More precisely, if z_1, \dots, z_n are n distinct points in \mathbb{C} then for a Möbius map $\psi(z) = \frac{az+b}{cz+d}$ (with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$)

$$\left\langle \prod_{i=1}^n V_{\alpha_i}(\psi(z_i)) \right\rangle_{\gamma, \mu} = \prod_{i=1}^n |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \left\langle \prod_{k=1}^n V_{\alpha_i}(z_i) \right\rangle_{\gamma, \mu} \quad (9)$$

where Δ_α is called the *conformal weight* of V_α

$$\Delta_\alpha = \frac{\alpha}{2} \left(Q - \frac{\alpha}{2} \right), \quad \alpha \in \mathbb{C}. \quad (10)$$

The conformal covariance of the correlation functions (9) is a generic fact of CFTs. Therefore the 3 point correlation functions of a given CFT, here $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle_{\gamma, \mu}$, are completely determined by the knowledge of $\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\gamma, \mu}$, which are the so-called structure constants.

3. Conformal bootstrap

It is certainly fair to say that the understanding of Liouville theory via the path integral approach has remained unclear until very recently; instead, most progress (in physics) has been carried out through another approach called the conformal bootstrap, which will be discussed in the case of the Riemann sphere but can also be carried out on other Riemann surfaces (with some non trivial modifications). This concept is born in the eighties following the attempt by Belavin–Polyakov–Zamolodchikov (BPZ, see [11]) to compute the correlation functions of Liouville theory. At that time, BPZ understood that the conformal symmetries that govern 2d CFTs give strong constraints on the theory; in particular they realized that the conformal symmetries of a CFT can be interpreted in terms of a representation of the Lie algebra generated by the symmetries, here called the Virasoro algebra⁶, taking values in the space of operators acting on a Hilbert space. This way, they connected to the classification of Lie algebras in representation theory. They argued that one could parametrize 2d CFTs by a unique parameter c called the central charge and that conformal symmetries can be translated in terms of PDEs (conformal Ward identities), which can be used to obtain expressions for correlation functions in terms of special functions from representation theory. In a nutshell, dilations are symmetries of any CFT and, via the representation of the algebra of symmetries, they give rise to a semigroup of operators acting on some Hilbert space. The generator of this semigroup can be diagonalized to obtain decomposition of the Hilbert space akin to the Plancherel formula in harmonic analysis. With this decomposition, they expressed recursively n point correlation functions into sums of m point correlation functions with $3 \leq m < n$. In other words, n point correlation functions can be computed recursively starting from the structure constants for $n = 3$, hence the name *conformal bootstrap*.

They solved this way many 2d CFTs for certain rational values of c when the CFT has discrete spectrum (minimal models, e.g. the critical Ising model) but fell short of solving Liouville theory, which has a continuous spectrum. It is only 10 years later that Dorn–Otto [12] and the

⁶The infinite-dimensional Lie group of local conformal transformations does not exist mathematically speaking. But if it would, then the Virasoro algebra could be seen as its Lie algebra.

Zamolodchikov brothers [13] proposed the following formula, the celebrated DOZZ formula, for the 3 point structure constants of Liouville theory

$$\begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\gamma, \mu} \\ &= \frac{1}{2} \left(\pi \mu \ell \left(\frac{\gamma^2}{4} \right) \left(\frac{\gamma}{2} \right)^{2 - \frac{\gamma^2}{2}} \right)^{\frac{2Q - \bar{\alpha}}{\gamma}} \frac{Y'_{\frac{\gamma}{2}}(0) Y_{\frac{\gamma}{2}}(\alpha_1) Y_{\frac{\gamma}{2}}(\alpha_2) Y_{\frac{\gamma}{2}}(\alpha_3)}{Y_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - Q) Y_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_1) Y_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_2) Y_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_3)} \end{aligned} \quad (11)$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$, $\ell(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$ with Γ the standard Gamma function and $Y_{\frac{\gamma}{2}}(z)$ is Zamolodchikov's special holomorphic function defined as the analytic continuation to \mathbb{C} of the following expression for $0 < \operatorname{Re}(z) < Q$

$$\ln Y_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z \right)^2 e^{-t} - \frac{\left(\sinh \left(\left(\frac{Q}{2} - z \right) \frac{t}{2} \right) \right)^2}{\sinh \left(\frac{t\gamma}{4} \right) \sinh \left(\frac{t}{\gamma} \right)} \right) \frac{dt}{t}. \quad (12)$$

The bootstrap formalism then states that higher correlation functions ($n \geq 4$) can be recovered from these 3 point structure constants (which can be meromorphically continued to complex values of the parameters $\alpha_1, \alpha_2, \alpha_3$). Of special interest is the case $n = 4$ in which case the bootstrap formula reads

$$\begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} := \frac{1}{8\pi} \\ & \int_0^\infty \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{Q-iP}(\infty) \rangle_{\gamma, \mu} \langle V_{Q-iP}(0) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \end{aligned} \quad (13)$$

where \mathcal{F}_P are holomorphic functions in z called (spherical) *conformal blocks* and have strong representation theoretical content. The conformal blocks are universal in the sense that they only depend on the conformal weights $\Delta_{\alpha_i} = \frac{\alpha_i}{2} (Q - \frac{\alpha_i}{2})$ and the central charge of Liouville theory $c_L = 1 + 6Q^2$.

4. Perspectives

Following the work [1], the present authors initiated with A. Kupiainen a program whose goal is to show that the probabilistic construction of [1] on the Riemann sphere coincides with the bootstrap construction used in physics. At first sight, the two approaches seem very distant, as one is based on exact probabilistic expressions involving Gaussian multiplicative chaos and the other one is based on representation theory. In order to unify both perspectives, the present authors developed with A. Kupiainen a mapping between probability theory and representation theory of the Virasoro algebra. This has led to a proof of the DOZZ formula [14, 15] and a proof of the conformal bootstrap [16] on the Riemann sphere.

As mentioned briefly in the introduction, Liouville theory is expected to be equivalent to a specific 4d gauge theory called $\mathcal{N} = 2$ SUSY Yang–Mills: this is the celebrated AGT conjecture [3]. Essentially, this conjecture states that the conformal blocks of Liouville theory equal the Nekrasov partition functions (which appear as building blocks of $\mathcal{N} = 2$ SUSY Yang–Mills) for certain values of the parameters. The proof of this conjecture on all Riemann surfaces remains open at the level of mathematics though a version of this conjecture on the torus is a consequence of recent works by Maulik–Okounkov [17] and Schiffman–Vasserot [18] (in these works the conformal blocks are seen as formal power series in the moduli parameters and convergence issues are not addressed). Following the recent probabilistic constructions of Liouville field theory on all Riemann surfaces [1, 4], one can hope to show a strong version of this conjecture on any Riemann surface. As an exciting output of this program, Ghosal–Remy–Sun–Sun [19] discovered a

probabilistic expression involving Gaussian multiplicative chaos for the (toric) conformal blocks which play a similar role on the torus to the (spherical) conformal blocks mentioned in the previous paragraph. In particular, the work [19] establishes convergence of toric conformal blocks. A generalization of their formulas to other surfaces would be a magnificent achievement.

Another challenging perspective is to investigate the links with geometric flows and “stochastic uniformization”. Recall that one of the greatest achievements of the past century was to characterize metrics with constant curvature on a given Riemannian manifold (\mathcal{M}, g) via a vast research program led by Emile Picard, Felix Klein or Henri Poincaré. In the eighties, Richard Hamilton brought his own perspective on the topic by introducing a dynamic on metrics, called the Ricci flow (here on compact surfaces)

$$\partial_t g_{ij} = -2R_{ij}(g) + R_{\text{avr}}(g)g_{ij} \quad (14)$$

where g is a time dependent metric on \mathcal{M} , $R_{ij}(g)$ stands for the Ricci tensor and $R_{\text{avr}}(g)$ the mean curvature. As time t goes to ∞ , g converges towards the constant curvature metric. Dubédat and Shen [20] have recently introduced the stochastic version of this flow, called the stochastic Ricci flow, which consists in adding an isotropic white noise in the space of metrics to the right-hand side of (14). They have shown that Liouville theory is an invariant measure for this flow. This opens new perspectives related not only to a rigorous quantization of gravity in two dimensions but also to connections between Gaussian multiplicative chaos, random geometry and uniformization of 2d surfaces.

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