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Luc Dettwiller

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Polarizing Šolc filters: a pedagogical derivation in the complex plane

Luc Dettwiller [a]

[a] Higher education section of Lycée Blaise Pascal, 36 avenue Carnot, 63037 Clermont-Ferrand Cedex, France
E-mail: dettwiller.luc@gmail.com

Abstract. For the theoretical study of Šolc filters, used in particular in solar observatories or coronagraphs, we use the complex plane representation of polarization, which had apparently never been done before in this context. This avoids the cumbersome matrix calculus, for the two models of basic Šolc filters. That technique seems promising for further studies on the important question of the apodisation of these filters.

Keywords. Polarization, Birefringent filters, Šolc filters, Apodisation, Complex representation (of polarization), Malus' law (in complex representation).

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1. Introduction

Since the end of the xixth century, the interest for very narrow-band optical filters (100 pm or less) appeared in solar observations, to study prominences, the chromosphere and the corona. Bernard Lyot conceived such a filter in 1927, and began to realize it in 1933 [1], but could not complete it because it used calcite polarizers, and at that time calcite was under military embargo in France, England, Germany, etc: high-grade optical calcite was used for gun sights, specifically in anti-aircraft weaponry and bombsights. Lyot's filter was the first birefringent filter: in its simplest form, it consists in a succession of typically 6 to 9 birefringent plates of a uniaxial material, cut parallel to its optic axis, whose thickness is in geometric progression with common ratio 2, each plate being placed between parallel polarizers. With such a device, a passband of 12 pm tunable over ± 1.6 nm has been achieved [2]. Apart from astronomy, and more recently, Lyot-type filters have found applications to snapshot spectral imaging (used for monitoring combustion dynamics, or for retinal imaging) [3], to telecommunications [4], to laser systems [5–7], to sensors [8], etc.

In 1953, Ivan Šolc proposed another birefringent filter [9], simpler and more luminous because it uses a set of N identical plates placed between only one polarizer before the stack and one analyser at the end. Both filters have a spectral transmission curve showing a sequence of very fine peaks. However, in their basic forms, Šolc filters suffer from secondary maxima (at relative level 11%) three times higher than those of Lyot filters just next to the spectral transmission peaks; hence the compelling need of an apodisation, i.e. a reduction of the secondary maxima—a classical task in image formation and in spectroscopy [10].

While Šolc [11] had computed the spectral properties of his filter with help of a general formula deduced from the Jones matrices calculus [12–14] for a system of N birefringent plates [15], Evans [16] used this calculus to directly derive a compact formula describing the overall properties of the basic models of Šolc filters. Some interesting realizations of these filters with astronomical applications have been described first by Valnicek [17], then by Evans [18], Fredga [19], and lately by Berger *et al.* [20] for the NASA IRIS solar physics mission. Birefringent Šolc-type filters are also of current interest for routing wavelength multiplexed fiber-optic signals [21], for sensing gas concentration [22], for amplifying light in Raman devices [23], etc.

The analytical theory of Šolc filters is computationally much heavier than that of Lyot filters. Unfortunately, most of further theoretical developments in the synthesis of lossless birefringent filters do not apply to Šolc filters: for example, Ammann and Chang [24] have considered filters containing one crystal, one compensator and one polarizer per stage (with the Lyot filter as a peculiar case) and Ammann has given general theorems which apply to these lossless birefringent filters [25].

In order to go beyond the basic study of polarization problems, several approaches are open [26–28] in the case of non-depolarizing and linear devices: (i) a matrix approach, where polarization is represented by a vector, and the action of an optical component by multiplication by a Jones matrix, which for non-dichroic devices is proportional to a matrix of the SU(2) unitary group; (ii) a geometrical approach, where a polarization is represented by a point of the Poincaré sphere (Σ), and the action of a retarder plate by a rotation of (Σ) on itself, or by an element of the rotation group $SO_3(\mathbb{R})$; (iii) an algebraic approach, where a perfect polarization is represented by a point of the complex plane, and the action of a linear non-depolarizing device by a complex homography. Although in practice these three approaches lead to very different calculations, their respective fecundities are fundamentally linked, because of the group morphisms of SU(2) towards $SO_3(\mathbb{R})$, and of $SO_3(\mathbb{R})$ towards the complex homography group—the latter resulting in the stereographic projection of (Σ), the pole of this projection being the point of (Σ) representing the linear polarization parallel to Oy [27].

The algebraic approach is rarely used. With it however, Azzam and Bashara [28] derived elegantly the fundamental equation of the nulling scheme ellipsometry—a fruitful technique to study a material sample by measuring the ratio of its reflection coefficients for components of the electric field respectively parallel and perpendicular to the incidence plane. But for the study of Šolc filters, it seems that the algebraic approach has never been used. The aim of this article is to show that this technique is a good alternative to study theoretically Šolc filters. We consider their two basic models—the fan filter and the folded filter [9, 16, 29–31]. Both are made up of N identical retardation plates, normal to Oz, each producing a retardation (phase shift) 2γ of the electric field component (of a monochromatic beam passing through the filter) on its slow axis relative to the component on its fast axis.

For each of these filters, we consider the effect of light propagating through them in the direction of Oz; the axes to be considered for all the plates are parallel to the xOy plane. In order to obtain the expression of the spectral transmittance T of the filter (which depends on the orientations of the polarizer and of the analyser), according to the third approach we use the correspondence [26–28] between, on the one hand, the polarization of a spectral component (proportional to $e^{-i\omega t}$) of the electric field, which is written $\underline{\boldsymbol{E}} = \underline{E}_x\,\boldsymbol{u_x} + \underline{E}_y\,\boldsymbol{u_y}$ using the direct orthonormal basis $(\mathscr{B}) := (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$ (the symbol := indicating a notation or a definition), and on the other hand a point of affix $Z := \underline{E}_y/\underline{E}_x$ in the complex plane, extended (i.e. compactified by adding infinity, in order to be able to take into account the linear polarization parallel to Oy).

Firstly, to study the transmission by the output analyser (whatever the device that precedes it), we introduce a generalized form of the Malus law in the complex plane. Then, in a second and a third step, we shall revisit, using the complex plane, the study of the fan Šolc filter, then the folded Šolc filter, in order to obtain the rigorous expression of their transmittance T in these classical cases.

2. Expression of Malus' law in the complex plane

Let us consider a light of perfect polarization—characterised by Z in the orthonormal basis $(\mathscr{B})$—and of complex electric field $\underline{\boldsymbol{E}}$, incident on a perfect analyser A which transmits without absorption the light whose complex electric field is proportional to $\underline{\boldsymbol{a}}$ of components $(\underline{a}_{\boldsymbol{x}}, \underline{a}_{\boldsymbol{y}}, 0)$ in $(\mathscr{B})$; the perfect polarization of the latter being characterised by $Z_A := \underline{a}_y / \underline{a}_x$. The intensity of the light incident on A is proportional to $\|\underline{\boldsymbol{E}}\|^2 = |\underline{E}_x|^2(1+|Z|^2)$, while that of the transmitted light is proportional to $|\underline{\boldsymbol{E}}\cdot\underline{\boldsymbol{a}}^*|^2/\|\underline{\boldsymbol{a}}\|^2 = |\underline{E}_x\,\underline{a}_x^*|^2|1+Z\,Z_A^*|^2/\|\underline{\boldsymbol{a}}\|^2$ where $\|\underline{\boldsymbol{a}}\| := \sqrt{|\underline{a}_{\boldsymbol{x}}|^2+|\underline{a}_y|^2} = |\underline{a}_{\boldsymbol{x}}|\sqrt{1+|Z_A|^2}$. So

$$|\underline{\boldsymbol{E}}\cdot\underline{\boldsymbol{a}}^*|^2 = |\underline{E}_x|^2|\underline{a}_x^*|^2|1+Z\,Z_A^*|^2 = |\underline{E}_x|^2\|\underline{\boldsymbol{a}}\|^2|1+Z\,Z_A^*|^2/(1+|Z_A|^2) \tag{1}$$

and the spectral transmittance is written as follows:

$$T = \frac{|\underline{\boldsymbol{E}}\cdot\underline{\boldsymbol{a}}^*|^2}{\|\underline{\boldsymbol{E}}\|^2\|\underline{\boldsymbol{a}}\|^2} = \frac{|1+ZZ_A^*|^2}{(1+|Z|^2)(1+|Z_A|^2)}. \tag{2}$$

This result is clearly related to the fact that the polarizations (not necessarily linear) associated with Z and Z_A are orthogonal if and only if $Z\,Z_A^* = -1$ [26–28]. On the other hand, $T \le 1$ according to the Cauchy–Schwarz inequality.

Example

If the incident light has a linear polarization of azimuth $\alpha = (\boldsymbol{u_x}, \underline{\boldsymbol{E}})$, and if the axis of the analyser has the azimuth $\beta = (\boldsymbol{u_x}, \underline{\boldsymbol{a}})$, then $Z = \tan\alpha$, $Z_A = \tan\beta$ and

$$\begin{aligned} T &= \frac{|1+\tan\alpha\tan\beta|^2}{(1+\tan^2\alpha)(1+\tan^2\beta)} = \left[\cos\alpha\cos\beta(1+\tan\alpha\tan\beta)\right]^2 \\ &= \left(\cos\alpha\cos\beta+\sin\alpha\sin\beta\right)^2 = \cos^2(\beta-\alpha) \end{aligned} \tag{3}$$

in accordance with ordinary Malus' law.

3. Basic fan Šolc filter

In this filter, the azimuth (with respect to Ox) of the slow axis L_n of the retardation plate number n is $\theta_n = n\theta$ defined only modulo π (see Figure 1).

3.1. *Relationship between incident and emergent polarizations for the plate stack*

To begin with, we look at the effect of the passage of light through L_1, of incident polarization characterised by Z with the basis $(\mathscr{B})$. For this purpose, we introduce the direct orthonormal basis $(\mathscr{B}_1)$ such that its first and second vector are respectively directed by the slow and fast axes of L_1. For light incident on L_1, the change of basis from $(\mathscr{B})$ to $(\mathscr{B}_1) := (\boldsymbol{u}'_{\boldsymbol{x}}, \boldsymbol{u}'_{\boldsymbol{y}}, \boldsymbol{u_z})$, where $\underline{\boldsymbol{E}} = \underline{E}'_x\boldsymbol{u}'_{\boldsymbol{x}} + \underline{E}'_y\boldsymbol{u}'_{\boldsymbol{y}}$, changes Z into

$$Z' := \frac{\underline{E}'_y}{\underline{E}'_x} = \frac{-\underline{E}_x\sin\theta+\underline{E}_y\cos\theta}{\underline{E}_x\cos\theta+\underline{E}_y\sin\theta} = \frac{Z-\tan\theta}{1+Z\tan\theta}, \tag{4}$$

which must then be multiplied by $u := \mathrm{e}^{-\mathrm{i}2\gamma}$, to account for the passage of the light through L_1.

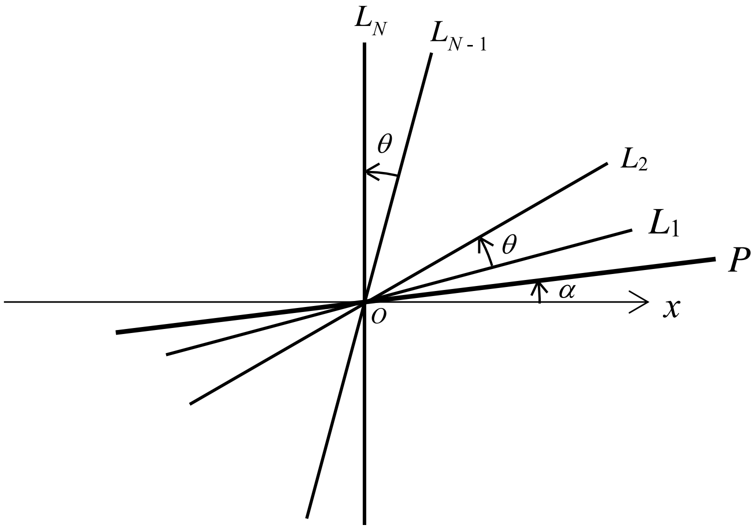

Figure 1. Orientations of the slow axes L_n of the N plates of the fan filter and of the polarizer axis P. In the usual configuration the axes of the polarizer and of the analyser A are parallel to each other ($\beta = \alpha \bmod \pi$), which explains why A is not represented here.

Then, with a fan filter, the complex number Z_n characterising the perfect polarization of the light emerging from plate number n is therefore—in the direct orthonormal basis $(\mathscr{B}_n)$ such that its first and second vectors are respectively directed by the slow and fast axes of L_n—given by the recurrence relation

$$Z_n = u\frac{Z_{n-1} - \tan\theta}{1 + Z_{n-1}\tan\theta}. \tag{5}$$

The rigorous expression of Z_n as a function of Z_0 (which corresponds, in $(\mathscr{B})$, to the perfect polarization of the light incident on L_1) requires to solve a homographic sequence [32]. This is the object of the Appendix A.

3.2. *Transmittance in the usual configuration*

With a basic fan Šolc filter, the axis direction of its analyser A being given by $\beta = (\boldsymbol{u_x}, \underline{\boldsymbol{n}})$, and when the linear polarization delivered by its polarizer is characterized by $Z_0 = \tan\alpha$, the transmittance T is obtained using the expression (A13) for Z—with $n = N$—and the expression (2) where we express Z_A using the same basis as for Z_N, so the basis $(\mathscr{B}_N)$; then $Z_A = \tan(\beta - N\theta)$. In Appendix B, with the variable

$$\chi := \arccos(\cos\gamma\cos\theta) \tag{6}$$

we show that this method gives the classical expression [16, 31]

$$T = \left[\frac{\tan\theta}{\tan\chi}\sin(N\chi)\right]^2, \tag{7}$$

in the usual configuration where $\alpha = \theta/2 = \beta$ and $N\theta = \pi/2 + m\pi$ ($m \in \mathbb{Z}$); in this case $Z_0 = \tan\theta/2$, and $Z_A = \tan(\theta/2 - \pi/2 - m\pi) = -\cot\theta/2$.

When γ tends to $p\pi$ ($p \in \mathbb{Z}$), $(\tan\theta/\tan\chi)^2$ tends to 1, and $[\sin(N\chi)]^2$ tends to $\sin^2(N\theta) = \sin^2(\pi/2 + m\pi) = 1$: that corresponds to the transmission peaks of the filter—for $\gamma = p\pi$, the plates do not change the light polarization, and $T = 1$ because $\alpha = \beta$. Their width is inversely proportional to N for $N \gg 1$. There is indeed an analogy between the fan Šolc filters and the twisted nematic displays. In such displays, the optical axis of the nematic liquid crystal turns regularly along the thickness of this medium, and when the incident light is linearly polarized parallel to the optical axis on the nematic input face, its polarization turns with the optical axis throughout the crystal. That situation, as soon as γ (mod π) is of the order of θ or more, is the infinite N limit of a fan Šolc filter, where $\alpha = \theta/2 = \beta$ tends to zero because $N\theta$ is fixed; then, with $N\theta = \pi/2 + m\pi$, the emerging polarization is crossed with the analyser.

4. Basic folded Šolc filter with an even number of plates

In this filter model, the angle (defined only modulo π) from Ox towards the slow axis of the retardation plate number n is $\theta_n = (-1)^{n-1}\theta/2$. Orientations of the successive plates are therefore alternating, as they are successively symmetrical with respect to Ox (Figure 2), hence the name "folded Šolc filter".

Let us characterize the polarization incident on L_1 by the number Z_0 defined with the components of $\underline{\boldsymbol{E}}$ in $(\mathscr{B}_2)$, and not in $(\mathscr{B})$ as in Section 3. This gives here $Z_0 = \tan(\alpha + \theta/2)$. To find the polarization of the light emerging from L_{2Q}, let us iterate Q times the following sequence of calculations:

- transform Z by the change of basis from $(\mathscr{B}_2)$ to $(\mathscr{B}_1)$, and multiply by u (in order to know in $(\mathscr{B}_1)$ the characteristics of the polarization of the light emerging from a plate of odd number), which gives Z';
- transform Z' by the change of basis from $(\mathscr{B}_1)$ to $(\mathscr{B}_2)$, and multiply by u (in order to know in $(\mathscr{B}_2)$ the characteristics of the polarization of the light emerging from the next plate), which gives Z''.

In $(\mathscr{B}_2)$, the characteristics of the polarization of the light emerging from the last plate of the stack are therefore given by Z_Q deduced from a recurrence homographic relation.

For the usual configuration $\alpha = 0$ and $\beta = \pi/2$, therefore $Z_0 = \tan\theta/2$, and, still with the basis $(\mathscr{B}_2)$, $Z_A = -\cot\theta/2$. When $N = 2Q$ and $N\theta = \pi/2 + m\pi$, the result for T corresponds (by changing γ into $\gamma + \pi/2$ and χ into $\chi + \pi/2$) [31] to the result (7) obtained for the fan filter:

$$T = \left[\frac{\tan\theta}{\cot\kappa}\sin(N\kappa)\right]^2, \tag{8}$$

with

$$\kappa := \arcsin(\sin\gamma\cos\theta). \tag{9}$$

When γ tends to $\pi/2 + m\pi$, $(\tan\theta/\cot\kappa)^2$ tends to 1, and $[\sin(N\kappa)]^2$ tends to $\sin^2[N(\pi/2-\theta)] = \sin^2(Q\pi - N\theta) = \sin^2(N\theta) = \sin^2(\pi/2 + m\pi) = 1$: it corresponds to the transmission peaks of the filter, whose width is inversely proportional to N for $N \gg 1$. But these peaks are just on the middle of the interval between two consecutive peaks of the fan filter [31]. Their interpretation is straightforward: in the particular case when $\gamma = \pi/2$ (mod π), each plate is half-wave and therefore changes the incident polarization into its symmetric with respect to the plate axes, and because the angle between the corresponding axes of two consecutive plates is θ, each pair of these plates induces a rotation of -2θ for the polarization; the effect of the plate stack is a rotation of $Q \times -2\theta = -N\theta$ for the polarization of the light propagating through it. Then $T = 1$ because $\alpha = 0$, $\beta = \pi/2$ and $N\theta = \pi/2 + m\pi$.

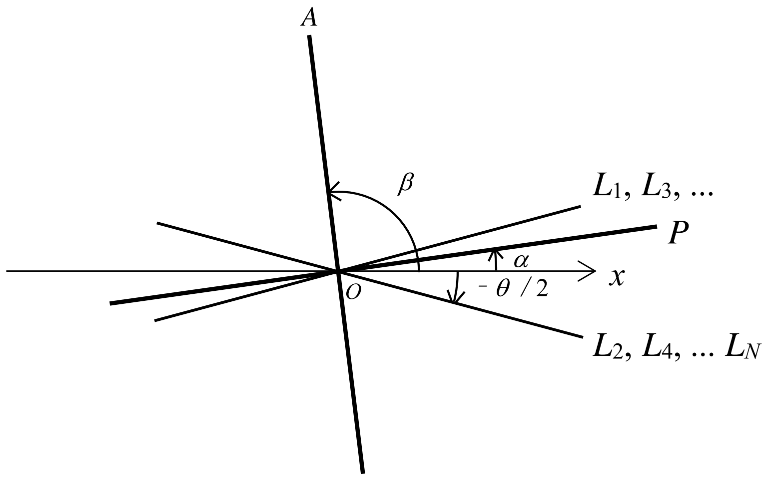

Figure 2. Orientations of the slow axes L_n of the N plates of the folded filter, of the axes of the polarizer P and of the analyser A.

5. Conclusion

A useful mathematical approach to Šolc filters is offered by the representation of polarization in the complex plane. It has already been used by Azzam and Bashara [28] to derive the fundamental equation of the nulling scheme ellipsometry in an elegant way, while other ellipsometry treatises which present this formula give a much more cumbersome derivation using matrix calculus. To study basic Šolc filters, that technique opens a new way that avoids the heavy matrix calculus. Other applications of the technique may appear. In particular, work is underways [33] to shed new light on the important question of the apodisation of the Šolc filters [34–36].

Acknowledgement

The author is grateful to Dr. Pierre Chavel for fruitful discussions.

Appendix A.

In order to study the homographic sequence given by (5), one begins by looking for its fixed points, i.e. the numbers z such that (for sequence Z_n)

$$z = \mathrm{e}^{-\mathrm{i}2\gamma} \frac{z - \tan\theta}{1 + z\tan\theta}. \tag{A1}$$

This is equivalent to a second-degree equation

$$\tan\theta\, z^2 + (1 - \mathrm{e}^{-\mathrm{i}2\gamma})z + \mathrm{e}^{-\mathrm{i}2\gamma}\tan\theta = 0 \tag{A2}$$

whose discriminant reads

$$\begin{aligned}\Delta &= (1 - \mathrm{e}^{-\mathrm{i}2\gamma})^2 - 4\mathrm{e}^{-\mathrm{i}2\gamma}\tan^2\theta = [(\mathrm{e}^{\mathrm{i}\gamma} - \mathrm{e}^{-\mathrm{i}\gamma})^2 - 4\tan^2\theta]\,\mathrm{e}^{-\mathrm{i}2\gamma} \\ &= -4(\sin^2\gamma + \tan^2\theta)\,\mathrm{e}^{-\mathrm{i}2\gamma}\end{aligned} \tag{A3}$$

hence the two fixed points are:

$$z_{1,2} = \frac{-(1 - \mathrm{e}^{-\mathrm{i}2\gamma}) \pm 2\mathrm{i}\sqrt{\sin^2\gamma + \tan^2\theta}\,\mathrm{e}^{-\mathrm{i}\gamma}}{2\tan\theta} = \mathrm{i}\frac{-\sin\gamma \pm \sqrt{\sin^2\gamma + \tan^2\theta}}{\mathrm{e}^{\mathrm{i}\gamma}\tan\theta}. \tag{A4}$$

We can simplify their expression a little by using χ—defined by (6)—because, by multiplying their numerator and denominator by $\cos\theta$, it comes

$$z_{1,2} = \mathrm{i}\frac{-\sin\gamma\cos\theta \pm \sqrt{(1-\cos^2\gamma)\cos^2\theta + \sin^2\theta}}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta} = \mathrm{i}\frac{-\sin\gamma\cos\theta \pm \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}. \tag{A5}$$

For the Poincaré sphere (Σ), the rotation of axes considered in Section 3.1, followed by the passage of the light through of a retardation plate, is described by the composition of two rotations of (Σ) onto itself, which leaves invariant two diametrically opposite points, associated with two orthogonal polarizations that are generally elliptical [26]. We verify that this is the case for the polarizations represented by the roots of (A2), because

$$\begin{aligned} z_1 z_2^* &= \frac{-\sin\gamma\cos\theta + \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}\,\frac{-\sin\gamma\cos\theta - \sin\chi}{\mathrm{e}^{-\mathrm{i}\gamma}\sin\theta} \\ &= \frac{(1-\cos^2\gamma)\cos^2\theta - (1-\cos^2\chi)}{\sin^2\theta} = \frac{\cos^2\theta - 1}{\sin^2\theta} = -1. \end{aligned} \tag{A6}$$

As $\Delta \neq 0$, Z_n is given by the classical relationship [32]

$$\frac{Z_n - z_1}{Z_n - z_2} = s^n \frac{Z_0 - z_1}{Z_0 - z_2} \tag{A7}$$

with s given by the following expressions, where the use of the variable χ is very fruitful:

$$\begin{aligned} s &= \frac{\mathrm{e}^{-\mathrm{i}2\gamma} - z_1\tan\theta}{\mathrm{e}^{-\mathrm{i}2\gamma} - z_2\tan\theta} = \frac{\mathrm{e}^{-\mathrm{i}\gamma}\cos\theta - z_1\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}{\mathrm{e}^{-\mathrm{i}\gamma}\cos\theta - z_2\mathrm{e}^{\mathrm{i}\gamma}\sin\theta} \\ &= \frac{\mathrm{e}^{-\mathrm{i}\gamma}\cos\theta - \mathrm{i}(-\sin\gamma\cos\theta + \sin\chi)}{\mathrm{e}^{-\mathrm{i}\gamma}\cos\theta - \mathrm{i}(-\sin\gamma\cos\theta - \sin\chi)} = \frac{\cos\chi - \mathrm{i}\sin\chi}{\cos\chi + \mathrm{i}\sin\chi} = \mathrm{e}^{-\mathrm{i}2\chi}. \end{aligned} \tag{A8}$$

Then

$$Z_n = \frac{z_1(Z_0 - z_2)\mathrm{e}^{\mathrm{i}n\chi} - z_2(Z_0 - z_1)\mathrm{e}^{-\mathrm{i}n\chi}}{(Z_0 - z_2)\mathrm{e}^{\mathrm{i}n\chi} - (Z_0 - z_1)\mathrm{e}^{-\mathrm{i}n\chi}} = \frac{Z_0(z_1\mathrm{e}^{\mathrm{i}n\chi} - z_2\mathrm{e}^{-\mathrm{i}n\chi}) - z_1 z_2 2\mathrm{i}\sin(n\chi)}{(z_1\mathrm{e}^{-\mathrm{i}n\chi} - z_2\mathrm{e}^{\mathrm{i}n\chi}) + Z_0 2\mathrm{i}\sin(n\chi)} \tag{A9}$$

but

$$\begin{aligned} z_1\mathrm{e}^{\mathrm{i}n\chi} - z_2\mathrm{e}^{-\mathrm{i}n\chi} &= \mathrm{i}\frac{-\sin\gamma\cos\theta + \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}\mathrm{e}^{\mathrm{i}n\chi} - \mathrm{i}\frac{-\sin\gamma\cos\theta - \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}\mathrm{e}^{-\mathrm{i}n\chi} \\ &= 2\mathrm{i}\frac{\sin\chi\cos(n\chi) - (\sin\gamma\cos\theta)\mathrm{i}\sin(n\chi)}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta} \end{aligned} \tag{A10}$$

$$\begin{aligned} z_1\mathrm{e}^{-\mathrm{i}n\chi} - z_2\mathrm{e}^{\mathrm{i}n\chi} &= \mathrm{i}\frac{-\sin\gamma\cos\theta + \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}\mathrm{e}^{-\mathrm{i}n\chi} - \mathrm{i}\frac{-\sin\gamma\cos\theta - \sin\chi}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}\mathrm{e}^{\mathrm{i}n\chi} \\ &= 2\mathrm{i}\frac{\sin\chi\cos(n\chi) + (\sin\gamma\cos\theta)\mathrm{i}\sin(n\chi)}{\mathrm{e}^{\mathrm{i}\gamma}\sin\theta} \end{aligned} \tag{A11}$$

and the product of the roots of (A2) is

$$z_1 z_2 = \mathrm{e}^{-\mathrm{i}2\gamma} \tag{A12}$$

so finally

$$Z_n = \frac{Z_0\left[\sin\chi\cos(n\chi) - (\sin\gamma\cos\theta)\mathrm{i}\sin(n\chi)\right] - \sin(n\chi)\mathrm{e}^{-\mathrm{i}\gamma}\sin\theta}{\left[\sin\chi\cos(n\chi) + (\sin\gamma\cos\theta)\mathrm{i}\sin(n\chi)\right] + Z_0\sin(n\chi)\mathrm{e}^{\mathrm{i}\gamma}\sin\theta}. \tag{A13}$$

Appendix B.

In order to obtain T by our method, we have to consider Z_N, given by the expression (A13) for the fan filter. In the usual configuration, we have $Z_0 = \tan\theta/2 = -1/Z_A$; this yields

$$\begin{aligned} Z_N &= \frac{\sin\chi\cos(N\chi) - (\sin\gamma\cos\theta)\mathrm{i}\sin(N\chi) - 2\sin(N\chi)\mathrm{e}^{-\mathrm{i}\gamma}\cos^2\frac{\theta}{2}}{\left[\sin\chi\cos(N\chi) + (\sin\gamma\cos\theta)\mathrm{i}\sin(N\chi)\right] + \sin(N\chi)\mathrm{e}^{\mathrm{i}\gamma}\sin\theta\tan\frac{\theta}{2}}\tan\frac{\theta}{2} \\ &= \frac{\sin\chi\cos(N\chi) - \sin(N\chi)\cos\gamma(1+\cos\theta) + \mathrm{i}\sin\gamma\sin(N\chi)}{\sin\chi\cos(N\chi) + \sin(N\chi)\cos\gamma(1-\cos\theta) + \mathrm{i}\sin\gamma\sin(N\chi)}\tan\frac{\theta}{2}. \end{aligned} \tag{B1}$$

With

$$a := \sin^2\chi\cos^2(N\chi) + \sin^2(N\chi)(1+\cos^2\chi) - \sin\chi\sin(2N\chi)\cos\gamma\cos\theta \tag{B2}$$

and

$$b := 2\sin^2(N\chi)\cos^2\gamma\cos\theta - \sin\chi\sin(2N\chi)\cos\gamma, \tag{B3}$$

we deduce

$$\begin{aligned}
|Z_N|^2 &= \frac{(\sin^2\chi)\cos^2(N\chi) + \sin^2(N\chi)[1+\cos^2\gamma\cos\theta(\cos\theta+2)] - (\sin\chi)\sin(2N\chi)\cos\gamma(1+\cos\theta)}{(\sin^2\chi)\cos^2(N\chi) + \sin^2(N\chi)[1+\cos^2\gamma\cos\theta(\cos\theta-2)] + (\sin\chi)\sin(2N\chi)\cos\gamma(1-\cos\theta)}\tan^2\frac{\theta}{2} \\
&= \frac{a+b}{a-b}\tan^2\frac{\theta}{2}
\end{aligned} \tag{B4}$$

and

$$\begin{aligned}
(a-b)(1+|Z_N|^2) &= a\left(1+\tan^2\frac{\theta}{2}\right) - b\left(1-\tan^2\frac{\theta}{2}\right) = \frac{a-b\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}} \\
&= \frac{\sin^2\chi\cos^2(N\chi) + \sin^2(N\chi)(1+\cos^2\chi) - \sin\chi\sin(2N\chi)\cos\gamma\cos\theta}{\cos^2\frac{\theta}{2}} \\
&\quad - \frac{2\sin^2(N\chi)\cos^2\gamma\cos\theta - \sin\chi\sin(2N\chi)\cos\gamma}{\cos^2\frac{\theta}{2}}\cos\theta \\
&= \frac{\sin^2\chi\cos^2(N\chi) + \sin^2(N\chi)(1-\cos^2\chi)}{\cos^2\frac{\theta}{2}} \\
&= \frac{2\sin^2\chi}{1+\cos\theta}
\end{aligned} \tag{B5}$$

and also

$$1 + Z_N Z_A^* = \frac{2\sin(N\chi)\cos\gamma}{\sin\chi\cos(N\chi) + \sin(N\chi)\cos\gamma(1-\cos\theta) + \mathrm{i}\sin\gamma\sin(N\chi)}. \tag{B6}$$

Always in the usual configuration,

$$1 + |Z_A|^2 = 1/\sin^2\frac{\theta}{2} = \frac{2}{1-\cos\theta}. \tag{B7}$$

Finally

$$\begin{aligned}
T &= \frac{|1+Z_N Z_A^*|^2}{(1+|Z_N|^2)(1+|Z_A|^2)} \\
&= \frac{[2\sin(N\chi)\cos\gamma]^2}{a-b}\,\frac{(a-b)(1+\cos\theta)}{2\sin^2\chi}\,\frac{1-\cos\theta}{2} \\
&= \frac{\sin^2(N\chi)\cos^2\gamma\sin^2\theta}{\sin^2\chi} = \frac{\sin^2(N\chi)\cos^2\chi\tan^2\theta}{\sin^2\chi}
\end{aligned} \tag{B8}$$

and we get the classic result given by (7).

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